On graphs for which every planar immersion lifts to a knotted spatial embedding

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We call a graph $G$ intrinsically linkable if there is a way to assign over/under information to any planar immersion of $G$ such that the associated spatial embedding contains a pair of nonsplittably linked cycles. We define intrinsically knottable graphs analogously. We show there exist intrinsically linkable graphs that are not intrinsically linked. (Recall a graph is intrinsically linked if it contains a pair of nonsplittably linked cycles in every spatial embedding.) We also show there are intrinsically knottable graphs that are not intrinsically knotted. In addition, we demonstrate that the property of being intrinsically linkable (knottable) is not preserved by vertex expansion.

1. Introduction

We start with a quick review of some definitions. A graph $G$ consists of a finite nonempty set $V(G)$ of vertices together with a set $E(G)$ of unordered pairs of (usually distinct) vertices, called edges. If $x = (u, v) \in E(G)$, for $u, v \in V(G)$, we say that $u$ and $v$ are adjacent vertices, and that vertex $u$ and edge $x$ are incident with each other, as are $v$ and $x$.

A walk in a graph $G$ is an alternating sequence of vertices and edges $$v_0, x_1, v_1, \ldots, x_{n-1}, v_{n-1}, v_n$$ beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. A cycle is a walk with $n \geq 2$ vertices and with all vertices distinct except $v_0 = v_n$. We say such a cycle has length $n$.  


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Let $G$ be a graph with
\[ V(G) = \{v_1, v_2, \ldots, v_n\} \quad \text{and} \quad E(G) = \{x_1, x_2, \ldots, x_m\}. \]

A spatial embedding of $G$ is a map $f$ of $G$ to a subspace $G(M)$ of $\mathbb{R}^3$ such that
\[ G(M) = \left( \bigcup_{i=1}^{n} v_i(M) \right) \cup \left( \bigcup_{j=1}^{m} x_j(M) \right), \]
where
(i) $v_1(M), v_2(M), \ldots, v_n(M)$ are $n$ distinct points of $\mathbb{R}^3$ with $f(v_i) = v_i(M)$;
(ii) $x_1(M), \ldots, x_m(M)$ are $m$ mutually disjoint open arcs in $\mathbb{R}^3$ with
\[ f(x_j) = x_j(M); \]
(iii) $x_j(M) \cap v_i(M) = \emptyset, i = 1, \ldots, n, j = 1, \ldots, m$;
(iv) if $x_j = (v_{j_1}, v_{j_2})$, then the open arc $x_j(M)$ has $v_{j_1}(M)$ and $v_{j_2}(M)$ as end points for $j = 1, \ldots, m$.

In the above definition, an arc in $\mathbb{R}^3$ is a homeomorphic image of $[0, 1]$; an open arc is an arc less its two end points, the images of 0 and 1. More informally, a spatial embedding is a way to place a given graph in space.

We define a planar immersion of a graph $G$ similar to a spatial embedding of $G$, except the codomain is $\mathbb{R}^2$ instead of $\mathbb{R}^3$, and we allow the image of edges of $G$ to intersect, though we require that no three edges can intersect at the same point and we require the image of our edges to intersect transversely (they intersect locally in only one point, and they are not tangent to each other). We will assume that all embeddings and immersions are tame, that is, can be approximated by a finite collection of line segments. We will often simply use the term immersion instead of planar immersion. We use $\hat{G}$ to denote the image of an immersion of $G$ under the map $\hat{f}$. If $H$ is a subgraph of $G$, we similarly denote by $\hat{H}$ the image of $H$ under $\hat{f}$.

Given an immersion $\hat{f}$ of a graph $G$ with image $\hat{G}$, one can, by assigning over/under information to its double points, lift the immersion into 3-space, thereby creating a well-defined spatial embedding $\hat{f}$ with image $\tilde{G}$. If $\pi$ is the standard projection $\pi(x, y, z) = (x, y)$, and $\hat{f} = \pi \circ \tilde{f}$, we have the commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\hat{f}} & \hat{G} \\
\downarrow & \nearrow \pi & \\
\tilde{G} & \uparrow & \hat{G}
\end{array}
\]
If there exists a lift of the immersion $\hat{f}$ whose image contains a pair of nonsplit-tably linked cycles (in other words, cannot be deformed to have a planar projection with no crossings between strands from two different components), then we say the immersion is linkable. We define the graph $G$ to be intrinsically linkable if every immersion of $G$ is linkable. We define knottable and intrinsically knottable analogously.

The study of intrinsically linkable graphs was inspired by two different ideas: intrinsically linked graphs, and graphs with a knot inevitable projection. The property of having a knot inevitable projection was introduced by Taniyama [1995] and studied by others (for example, Sugiura and Suzuki [2000], and Tamura [2004]). A (planar) graph has a knot inevitable projection if there exists a regular projection (that is, a planar immersion) of the graph such that every choice of over/under-crossings induces a spatial embedding that is knotted (in other words, cannot be deformed to a spatial embedding that has a planar projection without crossings).

The first results concerning intrinsically linked graphs were written up by Conway and Gordon [1983], and by Sachs [1983], who independently showed that every spatial embedding of $K_6$ (the graph on 6 vertices that contains all 15 possible edges between vertices) contains a pair of disjoint cycles that form a nonsplittable link, that is, $K_6$ is intrinsically linked. (See [Adams 2004] for a good background on knot theory in general, and on intrinsically linked and knotted graphs in particular.)

Conway and Gordon [1983] also showed that every spatial embedding of $K_7$ contains a cycle that forms a nontrivial knot, that is, $K_7$ is intrinsically knotted. Robertson et al. [1995] later showed that the collection of minor-minimal intrinsically linked graphs is exactly the Petersen family, that is, the seven graphs obtainable from the classic Petersen graph by repeated $\Delta$-$Y$ and $Y$-$\Delta$ exchanges. No one has yet classified the minor-minimal intrinsically knotted graphs, though they are known to be finite in number [Robertson and Seymour 2004].

Recall that a graph $H$ is a minor of a graph $G$ if $H$ can be obtained from $G$ by a sequence of deletions and/or contractions of edges and/or deletions of vertices. A graph $G$ is minor minimal with respect to a given property if it has the property, but no minor of $G$ has the property. Let $a$, $b$, and $c$ be vertices of a graph $G$ such that edges $(a, b)$, $(a, c)$, and $(b, c)$ exist. Then a $\Delta$-$Y$ exchange on a triangle $(a, b, c)$ of graph $G$ is as follows. Vertex $v$ is added to $G$, edges $(a, b)$, $(a, c)$, and $(b, c)$ are deleted, and edges $(a, v)$, $(b, v)$, and $(c, v)$ are added. A $Y$-$\Delta$ exchange is the reverse operation.

Clearly, an intrinsically linked (knotted) graph is also intrinsically linkable (knottable), but the converse is not true. In this paper, we present several intrinsically
linkable graphs, each of which is a proper minor of some graph in the Petersen family (and hence not intrinsically linked), and several intrinsically knottable graphs, which are all in the Petersen family (and not intrinsically knotted).

Recall that a vertex expansion of a vertex \( v \) in a graph \( G \) is achieved by replacing \( v \) with two vertices \( v' \) and \( v'' \), adding the edge \((v', v'')\) and connecting a subset of the edges that were incident to \( v \) to \( v' \) and the rest of the edges that were incident to \( v \) to \( v'' \). A graph \( G \) is considered to be an expansion of a graph \( H \) if \( G \) can be obtained by vertex expansions of \( H \). It is well known that vertex expansions preserve intrinsic linking and intrinsic knotting; see [Nešetřil and Thomas 1985; Fellows and Langston 1988]. We demonstrate several intrinsically linkable (knottable) graphs for which vertex expansion destroys intrinsic linkability (knottability). We thus conjecture that vertex expansion preserves intrinsic linkability (knottability) only for those graphs that are intrinsically linked (knotted).

2. Intrinsically linkable graphs

We start this section with a quick introduction to the linking number. Recall that given a link of two components, \( L_1 \) and \( L_2 \) (two disjoint circles embedded in space), one computes the linking number of the link by examining a projection (with over and under-crossing information) of the link. Choose an orientation for each component of the link. At each crossing between two components, one of the pictures in Figure 1 will hold. We count +1 for each crossing of the first type (where you can rotate the over-strand counterclockwise to line up with the under-strand) and −1 for each crossing of the second type. To get the linking number, \( \text{lk}(L_1, L_2) \), take the sum of +1’s and −1’s and divide by 2. One can show that the absolute value of the linking number is independent of projection, and of chosen orientations (see [Adams 2004] for further explanation). Note that if \( \text{lk}(L_1, L_2) \neq 0 \), then the associated link is nonsplit. The converse does not hold. That is, there are nonsplit links with linking number 0 (the Whitehead link is a famous example, see again [Adams 2004]).

**Lemma 2.1.** Let a graph \( G \) consist of two disjoint cycles \( A \) and \( B \). A planar immersion \( \hat{f} \) of \( G \) is linkable if and only if \( \hat{A} \) and \( \hat{B} \) intersect.

**Proof.** Suppose there is a planar immersion \( \hat{f} \) with disjoint cycles \( \hat{A} \) and \( \hat{B} \) that intersect. We will construct from \( \hat{f} \) a spatial embedding \( \tilde{f} \) in which the linking number \( \text{lk}(\hat{A}, \hat{B}) \) is nonzero. Arbitrarily choose orientations for \( \hat{A} \) and \( \hat{B} \), and then choose each crossing in \( \hat{G} \) to be positive. It is assumed that \( \hat{A} \) and \( \hat{B} \) intersect, so there exists at least one crossing between them. We now have an induced spatial embedding \( \tilde{f} \) in which \( \text{lk}(\hat{A}, \hat{B}) > 0 \).

The other implication is trivial to prove. \( \square \)

Here, we provide a sufficient condition for a graph to be intrinsically linkable:
**Lemma 2.2.** A graph $G$ is intrinsically linkable if it contains a nonplanar subgraph $H$ such that for any pair $\{e_1, e_2\}$ of nonadjacent edges in $H$, $e_1$ and $e_2$ belong to disjoint cycles in $G$.

**Proof.** Let $\hat{G}$ be any graph that satisfies the above condition and let $\hat{f}$ be any immersion of $\hat{G}$. Since $H$ is nonplanar, there exists in $\hat{H}$ at least one pair $\{e_1, e_2\}$ of nonadjacent edges that intersect. By hypothesis there are disjoint cycles, $C_1$ and $C_2$, that contain $e_1$ and $e_2$ respectively. Since $\hat{C}_1$ and $\hat{C}_2$ intersect, by Lemma 2.1 $\hat{f}$ is linkable. □

**Remark** (A remark on notation). We use the notation $G - e_{m,n}$ to denote the subgraph of $G$ obtained by removing an edge connecting a vertex of degree $m$ to a vertex of degree $n$. This notation is used only when the edge classes of $G$ are uniquely determined by the degree of the incident vertices. If no subscript is present on $e$, then all edges of $G$ belong to the same class. (Recall that the degree of a vertex is the number of edges incident to that vertex.)

We denote the graph in the Petersen family obtained from $K_6$ by a single $\Delta$-$Y$ exchange by $P_7$, and we denote the graph in the Petersen family obtained from $K_{3,3,1}$ by a single $\Delta$-$Y$ exchange by $P_8$. Finally, we denote the graph in the Petersen family obtained from $P_8$ by a single $\Delta$-$Y$ exchange by $P_9$. (Recall that $K_{3,3,1}$ is the graph of 7 vertices with vertices in three classes: $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$ and $\{v_7\}$ and edges between two vertices if and only if they lie in different classes. The graph $K_{4,4}$ is defined similarly on 8 vertices with two vertex classes of size 4.)

**Theorem 2.3.** The following graphs are intrinsically linkable: $K_6 - e$, $K_{3,3,1} - e_{4,6}$, $P_7 - e_{4,5}$, $P_7 - e_{5,5}$, $(K_{4,4} - e) - e_{4,4}$, and $P_8 - e_{4,5}$.

**Proof.** We will show that $G = K_{3,3,1} - e_{4,6}$ is intrinsically linkable. Proofs for the remaining graphs are similar.

Label the vertices as in Figure 2. Notice that in this labeling scheme the vertex classes are $S = \{s_1\}$, $U = \{u_1\}$, $V = \{v_1, v_2, v_3\}$, and $W = \{w_1, w_2\}$. We say that
an edge is in the class $SV$ if it connects a vertex in $S$ with a vertex in $V$. Naming
the other edge classes similarly, we have four edge classes in total: $SV$, $SW$, $UV$, and $VW$.

Take any immersion $\hat{f}$ of $G$. Let $H$ be the subgraph induced by

$\{u_1, v_1, v_2, v_3, w_1, w_2\}$.

Since $H$ is isomorphic to $K_{3,3}$, $H$ is nonplanar and thus $\hat{H}$ has a pair of nonadjacent intersecting edges. There are two cases.

Case 1: Suppose one edge belongs to $UV$ and the other to $VW$. We may assume the two edges to be $(u_1, v_2)$ and $(v_1, w_1)$. Then the disjoint cycles

$$(s_1, v_1, w_1) \quad \text{and} \quad (u_1, v_2, w_2, v_3)$$

intersect in $\hat{G}$.

Case 2: Suppose both edges belong to $VW$. We may assume the two edges to be $(v_1, w_1)$ and $(v_2, w_2)$. Then the disjoint cycles

$$(s_1, v_1, w_1) \quad \text{and} \quad (u_1, v_2, w_2, v_3)$$

intersect in $\hat{G}$.

Thus in either case we have a pair of disjoint cycles that intersect in $\hat{G}$. By Lemma 2.2, $G$ is intrinsically linkable. $\square$

Figure 2. Vertex classes of $K_{3,3,1} - e_{4,6}$ and the subgraph $H$.

Figure 3. An immersion of $P_8 - e_{3,3}$ with only one crossing.
Since vertex expansion, $\Delta$-$Y$ exchange, and $Y$-$\Delta$ exchange preserve intrinsic linking [Nešetřil and Thomas 1985; Fellows and Langston 1988; Motwani et al. 1988; Robertson et al. 1995], it is natural to ask if these same graph operations preserve intrinsic linkability. In general, this is not the case. For example, $P_8 - e_{4,4}$ can be obtained from $P_7 - e_{4,5}$ by $\Delta$-$Y$ exchange, but $P_8 - e_{4,4}$ is not intrinsically linkable (See Figure 3).

In addition, certain expansions of $K_6 - e$ and $K_{3,3,1} - e_{4,6}$, which are exhibited in Figure 4 (notice that the expanded immersions contain only one crossing), are not intrinsically linkable. Any intrinsically linkable graph for which vertex expansion does preserve linkability, we call strongly linkable. Having found many examples in which expansion kills intrinsic linkability, we conjecture the following:

**Conjecture 2.4.** A graph is strongly linkable if and only if it is intrinsically linked.

Figure 4. Two graphs for which vertex expansion destroys intrinsic linkability.
3. Intrinsically knottable graphs

3A. Introduction. The following lemma about knots is from [Kauffman 1983]. Note that we use $lk_2(L_1, L_2)$ to denote the mod 2 linking number for link components $L_1$ and $L_2$. Recall that a knot is a tame embedding of $S^1$ into $\mathbb{R}^3$.

Lemma 3.1. For a knot $K$, the Arf invariant $\alpha(K)$ is the second coefficient of the Conway polynomial (mod 2). It satisfies the following Skein relation (see Figure 5):

$$\alpha(K_+) = \alpha(K_-) + lk_2(L_1, L_2).$$

Note that if $\alpha(K) \neq 0$, then $K$ is nontrivial. (There are, however, many nontrivial knots with vanishing Arf invariant).

We use the following lemma from [Taniyama and Yasuhara 2001] (see also [Foisy 2002]). This lemma uses the second coefficient of the Conway polynomial of a knot, which is denoted by $a_2(K)$, for a knot $K$ (again, if $a_2(K) \neq 0$, then $K$ is nontrivial). Recall that a Hamiltonian cycle in a graph is a cycle that uses every vertex of the graph.

Figure 5. The neighborhoods involved in Lemma 3.1.

Figure 6. A planar embedding of $D_4$. 
Lemma 3.2. Consider the graph $D_4$, labeled as in Figure 6. Let $f$ be a function embedding $D_4$ in space. Let $S_0$ and $S_1$ be sets of Hamiltonian cycles where

\[ S_0 = \{ (a_i b_j c_k d_l) \mid i + j + k + l \text{ is even} \}, \]

\[ S_1 = \{ (a_i b_j c_k d_l) \mid i + j + k + l \text{ is odd} \}. \]

Let

\[ \lambda(f) = \sum_{C \in S_0} a_2(f(C)) - \sum_{C \in S_1} a_2(f(C)). \]

Then

\[ \lambda(f) = |lk(C_1, C_3) \cdot lk(C_2, C_4)|. \]

In particular, if $\lambda(f)$ is nonzero, one of the Hamiltonian cycles must be knotted.

The following corollary is an immediate consequence; see [Taniyama and Yasuhara 2001; Foisy 2002].

Corollary 3.3. If for a given embedding of $G$, there is an expansion of $D_4$ contained as an embedded subgraph with

\[ lk(C_1, C_3) \cdot lk(C_2, C_4) > 0, \]

then the embedded $G$ contains a knotted cycle.

3B. Nontrivial examples of intrinsically knottable graphs. We explore the connection between intrinsic linking and intrinsic knottability by looking at the Petersen graphs. We originally conjectured that an intrinsically linked graph would necessarily be intrinsically knottable, but we quickly found counterexamples. It is easy to see that an immersion must have at least three crossings in order to be knottable. There are immersions of $P_9$, $PG$, and $P_8$ that have only two crossings (see, for example Figure 7), so clearly these graphs are not intrinsically knottable.

Theorem 3.4. The graph $K_6$ is intrinsically knottable.

Our proof of this theorem relies heavily on the following lemma which is similar to Lemma 3.2.

Lemma 3.5. Let $D'_4$ be a graph with four vertices, two nonadjacent 2-cycles $C_1$ and $C_2$, and two nonadjacent edges $A_1$ and $A_2$ that connect $C_1$ and $C_2$ (see Figure 8). Given any immersion of $D'_4$, if $C_1$ and $C_2$ cross and $A_1$ and $A_2$ cross, then the immersion is knottable.

Proof. Take any immersion of $D'_4$ such that $C_1$ and $C_2$ cross and $A_1$ and $A_2$ are crossed. Assign over/under information to the crossings of $C_1$ and $C_2$ such that $lk_2(C_1, C_2) = 1$. We will show that there is a way to assign over/under information to the crossings on $A_1$ and $A_2$ such that the resulting embedding contains a knot.
Let \( S \) be the set of all Hamiltonian cycles of \( D'_4 \). Given any embedding of \( D'_4 \), we can define \( \sigma \) as follows:

\[
\sigma = \sum_{C \in S} \alpha(C).
\]

For disjoint arcs \( a_1 \) and \( a_2 \) in an embedding of \( D'_4 \), define \( \omega(a_1, a_2) \in \mathbb{Z}_2 \) to be the number of times mod 2 that \( a_1 \) crosses over \( a_2 \). Note that by definition, for any embedding of \( D'_4 \),

\[
\omega(e_1, e_3) + \omega(e_1, e_4) + \omega(e_2, e_3) + \omega(e_2, e_4) = lk_2(C_1, C_2).
\]

Assign arbitrarily all crossings of \( A_1 \) with \( A_2 \) but one. Consider the crossing that has not been assigned. Let \( D_+ \) denote the embedding of \( D'_4 \) in which \( A_1 \) crosses over \( A_2 \) at that crossing and \( D_- \) denote the embedding of \( D'_4 \) in which \( A_2 \) crosses over \( A_1 \). Consider the change \( \Delta \sigma \) in \( \sigma \) that will result from changing the crossing on \( A_1 \) and \( A_2 \).

**Figure 7.** An immersion of the classic Petersen graph with only two crossings.

**Figure 8.** The graph \( D'_4 \).
Let \( C \) be a Hamiltonian cycle containing \( A_1 \) and \( A_2 \) and \( \epsilon(C) \) be the change in \( \alpha(C) \) induced by the crossing change. Now by Lemma 3.1 above,

\[
\epsilon(C) = \alpha(C_+) + \alpha(C_-) = \text{lk}_2(L_1, L_2) = \sum_{E_1 \in L_1, E_2 \in L_2} \omega(E_1, E_2).
\]

Now, summing up \( \epsilon(C) \) over all Hamiltonian cycles \( C \) gives the change in \( \sigma \). Fortunately most of the terms cancel out and we are left with

\[
\Delta \sigma = \sum_{C \in S} \epsilon(C) = \omega(e_1, e_3) + \omega(e_1, e_4) + \omega(e_2, e_3) + \omega(e_2, e_4) = \text{lk}_2(C_1, C_2) = 1.
\]

This means that either \( D_+ \) or \( D_- \) contains a knot. \( \square \)

**Proof of Theorem 3.4.** Take any lift of any immersion of \( K_6 \). Since \( K_6 \) is intrinsically linked, there is a pair of linked triangles, \( C_1 \) and \( C_2 \), in the resulting embedding.

Suppose that we temporarily ignore the edges of \( C_1 \) and \( C_2 \). We are left with \( K_{3,3} \), which has a crossing in nonadjacent edges, say \( A_1 \) and \( A_2 \). Notice that \( A_1 \) and \( A_2 \) connect the cycles \( C_1 \) and \( C_2 \). The cycles \( C_1 \) and \( C_2 \), along with the edges \( A_1 \) and \( A_2 \), make up a subgraph of \( K_6 \) that is \( D'_4 \) (with some extra degree 2 vertices). Since \( C_1 \) and \( C_2 \) are linkable and \( A_1 \) and \( A_2 \) cross, this subgraph immersion is knottable, by Lemma 3.5. Thus \( K_6 \) is knottable. \( \square \)

Now we show that \( K_{4,4} - e \) is intrinsically knottable. First we need the following lemma.

**Lemma 3.6.** Suppose \( G \) is a graph that contains in every immersion two pairs of linkable cycles, \( C_1 \) and \( C_2 \), \( C_3 \) and \( C_4 \). Suppose the union of the cycles is an expansion of \( D_4 \) with \( C_1 \) and \( C_2 \) opposite each other and \( C_3 \) and \( C_4 \) opposite each other (so \( C_1 \) and \( C_2 \) are disjoint, \( C_3 \) and \( C_4 \) are disjoint, and all other pairs of \( C_i \) and \( C_j \), for \( i \neq j \), intersect in either a vertex, an edge, or a simple path). If there is a way to orient the cycles consistently, then \( G \) is intrinsically knottable.

**Proof.** Orient the cycles in a consistent way, and assign all crossings to be positive. Then \( \text{lk}(C_1, C_2) \) and \( \text{lk}(C_3, C_4) \) are both positive. Since the cycles \( C_1 \), \( C_2 \), \( C_3 \), and \( C_4 \) form a subgraph of \( G \) that is an expansion of \( D_4 \) with the desired linking properties, we can apply Corollary 3.3 and conclude that the resulting embedding contains a knot. \( \square \)

**Theorem 3.7.** The graph \( K_{4,4} - e \) is intrinsically knottable.
Figure 9. Case 1 (left): $C_3$ shares exactly one edge with $C_1$ and one edge with $C_2$. Case 2 (right): $C_3$ shares exactly one edge with $C_1$ and one edge with $C_2$.

Proof. We first label the vertices of $K_{4,4} - e$ as $v_1, \ldots, v_4, w_1, \ldots, w_4$, where every $v_i$ belongs to one partition and every $w_j$ belongs to the other partition. Let $(v_1, w_3)$ be the missing edge.

Take any lift of any immersion of $K_{4,4} - e$. Since $K_{4,4} - e$ is intrinsically linked, there is a pair of nonsplittably linked (thus linkable) 4-cycles in the lift embedding. We again denote these 4-cycles as $C_1$ and $C_2$ where $C_1$ is $(v_1, w_1, v_2, w_2)$ and $C_2$ is $(v_3, w_3, v_4, w_4)$. (Up to symmetry this is the only way to get disjoint 4-cycles.)

Now the subgraph of $K_{4,4} - e$ resulting from the removal of $(v_1, w_1)$ is intrinsically linkable by Theorem 2.3 above. So there is a pair of linkable cycles, $C_3$ and $C_4$ in the subgraph. There are two ways in which $C_3$ and $C_4$ can be related to $C_1$ and $C_2$: $C_3$ shares exactly one edge with $C_1$ and one edge with $C_2$, or $C_3$ shares exactly one edge with $C_1$ and one edge with $C_2$.

In each case, there is a way to orient the cycles $C_1, C_2, C_3$, and $C_4$ consistently. (See Figure 9.) Since the cycles $C_1, C_2, C_3$, and $C_4$ form a subgraph of $K_{4,4} - e$ that is an expansion of $D_4$ with the desired linkability properties, we can apply Lemma 3.6 and conclude that $K_{4,4} - e$ is intrinsically knottable.

The techniques of this proof can also be applied to prove that $K_6, P_7$ and $K_{3,3,1}$ are intrinsically knottable.

3C. Strongly knottable graphs. We say that a graph $G$ is strongly knottable if every expansion of $G$ is intrinsically knottable.

Proposition 3.8. The graphs $K_6, K_{3,3,1}, K_{4,4},$ and $P_7$ are not strongly knottable.

Proof. In Figures 10 and 11, we exhibit immersions of expansions of $K_6$ and $K_{4,4}$, such that each immersion has only two crossings, and thus certainly is not knottable. Similar immersions for $P_7$ and $K_{3,3,1}$ exist.

This leads us to the following conjecture.
Figure 10. An immersion of an expansion of $K_6$ with only two crossings.

Figure 11. An immersion of an expansion of $K_{4,4}$ with only two crossings.

**Conjecture 3.9.** A graph is strongly knottable if and only if it is intrinsically knotted.

**References**


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