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Xian Jun Long and Nan Jing Huang

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Gap functions and existence of solutions for generalized vector quasivariational inequalities

Xian Jun Long and Nan Jing Huang

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The gap functions for generalized vector quasivariational inequalities in Hausdorff topological vector spaces are introduced, then using Fan–Knaster–Kuratowski–Mazurkiewicz (FKKM) theorem, some existence theorems for a class of generalized vector quasivariational inequalities under suitable assumptions are established. The obtained results extend and unify corresponding results in the literature.

1. Introduction

The vector variational inequality in a finite-dimensional Euclidean space was first introduced by Giannessi [1980]. It is the vector-valued version of the variational inequality of Hartman and Stampacchia [1966]. Later on, many authors have extensively studied various types of vector variational inequalities in abstract space (see, for example, [Ansari 1995; Chen 1992; Chen et al. 1997; Chen et al. 2005; Ding and Tarafdar 2000; Giannessi 2000; Göpfert et al. 2003; Huang and Fang 2005; Huang and Gao 2003; Huang and Li 2006; Khanh and Luu 2004; Konnov and Yao 1997; Lee and Lee 2000; Lee et al. 1996; Li and He 2005; Siddiqi et al. 1997; Yang 2003; Yang and Yao 2002; Yu and Yao 1996] and the references therein).

The gap function approach is an important research method in the study of variational inequalities. One advantage of the gap function for the variational inequality is that the variational inequality can be transformed into the optimization problem. Thus, powerful optimization solution methods and algorithms can be applied to find solutions of variational inequalities. Recently, many authors have investigated the gap functions for vector variational inequalities. Chen et al. [1997] introduced

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two set-valued functions as the gap functions for two classes of vector variational inequalities. Yang and Yao [2002] introduced the gap function for the multivalued vector variational inequality. Li and He [2005] generalized the results of Yang and Yao [2002] to the generalized vector variational inequality. They introduced a gap function for a class of generalized vector variational inequalities and proved the existence of some solutions for such problems. For some related works, we refer to [Li and Mastroeni 2008] and [Yang 2003].

Inspired and motivated by the reseach mentioned above, we introduce in this paper some new gap functions for generalized vector quasivariational inequalities in Hausdorff topological vector spaces. By using FKKM theorem, we prove a number of existence theorems for a class of generalized vector quasivariational inequalities under certain assumptions. The results presented in this paper extend, improve and unify some corresponding results in the literature.

2. Gap functions for generalized vector quasivariational inequalities

Let X and Y be two real Hausdorff topological vector spaces and E a nonempty subset of X. Let $L(X, Y)$ be the space of all the continuous linear operators from X into Y and σ is the family of bounded subsets of X whose union is total in X, that is, the linear hull of $\cup\{S : S \in \sigma\}$ is dense in X. Let $\mathbf{B}$ be a neighborhood base of 0 in Y. When S runs through σ, V through $\mathbf{B}$, the family

$$M(S, V) = \{t \in L(X, Y) : \cup_{x \in S} \langle t, x\rangle \subset V\}$$

is a neighborhood base of 0 in $L(X, Y)$ for a unique translation-invariant topology, called the topology of uniform convergence on the sets $S \in \sigma$, or, briefly the σ-topology where $\langle t, x\rangle$ denotes the valuation of the linear operator $t \in L(X, Y)$ at $x \in X$ (see, [Schaefer 1971]). By the corollary of Schaefer [1971], $L(X, Y)$ becomes a locally convex topological vector space under the σ-topology, where Y is assumed a locally convex topological vector space.

Lemma 2.1 ([Ding and Tarafdar 2000]). *Let X and Y be two real Hausdorff topological vector spaces and $L(X, Y)$ be the topological vector space under the σ-topology. Then the bilinear mapping*

$$\langle \cdot, \cdot\rangle : L(X, Y) \times X \to Y$$

is continuous in $L(X, Y) \times X$.

Let E be a nonempty compact subset of X, and $C \subseteq Y$ be a closed, convex, pointed cone in Y with apex at the origin and $\operatorname{int} C \neq \varnothing$. Assume that $K : E \to 2^E$ is a lower semicontinuous with compact-valued mapping, and $T : E \times E \to 2^{L(X,Y)}$ is set-valued mapping such that $T(x, x)$ is compact for any $x \in E$. Assume that $\eta : E \times E \to E$ and $h : E \times E \to Y$ are two continuous functions with respect to the first argument. Let $\eta(x, x) = 0$ and $h(x, x) = 0$ for any $x \in E$. In this section, we consider the following three generalized vector quasivariational inequalities (for short, GVQVIs):

(I) find $x^* \in E$ and $t^* \in T(x^*, x^*)$ such that

$$x^* \in K(x^*) \quad \textit{and} \quad \langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} C, \qquad \text{for all } y \in K(x^*) \tag{1}$$

(II) find $x^* \in E$ and $t^* \in T(x^*, x^*)$ such that

$$x^* \in K(x^*) \quad \textit{and} \quad \langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -C\backslash\{0\}, \qquad \text{for all } y \in K(x^*) \tag{2}$$

Remark 2.1. It is clear that any solution of GVQVI (2) is a solution of GVQVI (1). But the converse is not true in general.

Remark 2.2. If $T(x, x) = T(x)$ and $K = I$ (where I is the identity mapping) for any $x \in E$, then GVQVI (1) and (2) reduce to the following generalized vector variational inequalities (for short, GVVI), respectively:

(I) find $x^* \in E$ and $t^* \in T(x^*)$ such that

$$\langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} C, \qquad \text{for all } y \in E; \tag{3}$$

(II) find $x^* \in E$ and $t^* \in T(x^*)$ such that

$$\langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -C\backslash\{0\}, \qquad \text{for all } y \in E. \tag{4}$$

GVVI (3) and (4) were studied by Li and He [2005].

Remark 2.3. If $\eta(x, y) = x - y$ and $h(x, y) = 0$ for any $x, y \in E$, then GVVI (3) and (4) reduce to the following multivalued vector variational inequalities (for short, MVVI), respectively:

(I) find $x^* \in E$ and $t^* \in T(x^*)$ such that

$$\langle t^*, y - x^*\rangle \notin -\operatorname{int} C, \qquad \text{for all } y \in E; \tag{5}$$

(II) find $x^* \in E$ and $t^* \in T(x^*)$ such that

$$\langle t^*, y - x^*\rangle \notin -C\backslash\{0\}, \qquad \text{for all } y \in E. \tag{6}$$

MVVI (5) and (6) were studied by Yang and Yao [2002].

In the rest of this section, let R^l be an l-dimensional vector space, and let

$$R^l_+ = \{(r_1, \ldots, r_l) \in R^l \mid r_i \geq 0, i = 1, 2, \ldots, l\}$$

be the nonnegative orthant of R^l. Let $Y = R^l$ and $C = R^l_+$. Now we introduce some gap functions for GVQVI (1) and (2). Set

$$S = \{x \in E \mid x \in K(x)\}.$$

Definition 2.1. $\phi : S \to R$ is said to be a gap function for GVQVI (1) (resp. (2)) if it satisfies the following properties:

(i) $\phi(x) \leq 0$ for all $x \in S$;

(ii) $\phi(x^*) = 0$ if and only if x^* solves GVQVI (1) (resp. (2)).

Let $x \in S$, $y \in K(x)$ and $t \in T(x, x)$. Denote

$$\langle t, \eta(y, x)\rangle + h(y, x) = ([\langle t, \eta(y, x)\rangle + h(y, x)]_1, \dots, [\langle t, \eta(y, x)\rangle + h(y, x)]_l).$$

Now, we introduce the mappings $\varphi_1 : S \times L(X, R^l) \to R$ and $\varphi : S \to R$ as follows:

$$\varphi_1(x, t) = \min_{y \in K(x)} \max_{1 \leq i \leq l} (\langle t, \eta(y, x)\rangle + h(y, x))_i$$

and

$$\varphi(x) = \max\{\varphi_1(x, t) \mid t \in T(x, x)\}. \tag{7}$$

Since $K(x)$ is compact, η is continuous and h is continuous with respect to the first argument respectively, $\varphi_1(x, t)$ is well-defined. By Lemma 2.1, $\varphi(x)$ is well-defined. For any $x \in S$ and $t \in T(x, x)$, it is easy to see that

$$\varphi_1(x, t) = \min_{y \in K(x)} \max_{1 \leq i \leq l} (\langle t, \eta(y, x)\rangle + h(y, x))_i \leq 0.$$

Theorem 2.1. *The function $\varphi(x)$ defined by Equation* (7) *is a gap function for GVQVI* (1).

Proof. Since

$$\varphi_1(x, t) \leq 0, \qquad \text{for all } x \in S, \quad t \in T(x, x), \tag{8}$$

it follows that

$$\varphi(x) = \max\{\varphi_1(x, t) \mid t \in T(x, x)\} \leq 0, \qquad \text{for all } x \in S.$$

If $\varphi(x^*) = 0$, then there exists a $t^* \in T(x^*, x^*)$ such that $\varphi_1(x^*, t^*) = 0$. Thus,

$$\min_{y \in K(x^*)} \max_{1 \leq i \leq l} (\langle t^*, \eta(y, x^*)\rangle + h(y, x^*))_i = 0.$$

From which it follows that, for any $y \in K(x^*)$,

$$\max_{1 \leq i \leq l} (\langle t^*, \eta(y, x^*)\rangle + h(y, x^*))_i \geq 0,$$

which implies that for any $y \in K(x^*)$,

$$\langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} R^l_+,$$

that is, x^* is a solution of GVQVI (1).

Conversely, if x^* is a solution of GVQVI (1), then there exists a $t^* \in T(x^*, x^*)$ such that

$$x^* \in K(x^*) \quad \textit{and} \quad \langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} R^l_+, \qquad \text{for all } y \in K(x^*).$$

It follows that for any $y \in K(x^*)$,

$$\max_{1\le i\le l} (\langle t^*, \eta(y, x^*)\rangle + h(y, x^*))_i \ge 0.$$

Hence, we have

$$\varphi_1(x^*, t^*) = \min_{y\in K(x^*)} \max_{1\le i\le l} (\langle t^*, \eta(y, x^*)\rangle + h(y, x^*))_i \ge 0. \tag{9}$$

It follows from (8) and (9) that $\varphi_1(x^*, t^*) = 0$. Again, from (8), we obtain

$$\varphi_1(x^*, t) \le 0, \quad t \in T(x^*, x^*).$$

Therefore, $\varphi(x^*) = 0$. This completes the proof. □

From Remark 2.1 and Theorem 2.1, it is easy to see that the following result holds.

Corollary 2.1. *If x^* is a solution of GVQVI* (2), *then $\varphi(x^*) = 0$.*

3. Existence theorems for generalized vector quasivariational inequalities

Let X and Y be two Hausdorff topological vector spaces and E be a nonempty subset of X. Let $L(X, Y)$ be a set of all the continuous linear operators from X into Y. Let $C : E \to 2^Y$ be a set-valued mapping such that for any $x \in E$, $C(x)$ is a point, closed and convex cone in Y with $\operatorname{int} C(x) \ne \varnothing$. Assume that $K : E \to 2^E$ and $T : E \times E \to 2^{L(X,Y)}$ are two set-valued mappings, $\eta : E \times E \to E$ and $h : E \times E \to Y$ are two vector-valued functions. In this section, we consider GVQVI with moving cone $C(x)$: find $x^* \in E$ and $t^* \in T(x^*, x^*)$ such that $x^* \in K(x^*)$ and

$$\langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} C(x^*), \qquad \text{for all } y \in K(x^*). \tag{10}$$

The following problems are special cases of GVQVI (10).

(1) If $T(x, x) = T(x)$ and $K = I$ (where I is the identity mapping) for any $x \in E$, then problem (10) reduces to the following problem: find $x^* \in E$ and $t^* \in T(x^*)$ such that

$$\langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} C(x^*), \qquad \text{for all } y \in E, \tag{11}$$

which was considered by Lee and Lee [2000] and Li and He [2005].

(2) If $\eta(y, x) = y - x$ and $h(y, x) = 0$ for any $x, y \in E$, then problem (11) reduces to the following problem: find $x^* \in E$ and $t^* \in T(x^*)$ such that

$$\langle t^*, y - x^*)\rangle \notin -\operatorname{int} C(x^*), \qquad \text{for all } y \in E, \tag{12}$$

which was considered by Konnov and Yao [1997].

(3) If T is a single-valued mapping, then problem (12) reduces the to following problem: find $x^* \in E$ such that

$$\langle T(x), y - x^*) \rangle \notin -\operatorname{int} C(x^*), \quad \text{for all } y \in E, \tag{13}$$

which was considered by Chen [1992] and Yu and Yao [1996].

(4) If T is a single-valued mapping, $\eta(y, x) = y - g(x)$ and $h(y, x) = 0$ for any $x, y \in E$, where $g : E \to E$, then problem (11) reduces to the following problem: find $x^* \in E$ such that

$$\langle T(x), y - g(x) \rangle \notin -\operatorname{int} C(x^*), \quad \text{for all } y \in E, \tag{14}$$

which was considered by Siddiqi et al. [1997].

In order to prove our main results, we need the following definitions and lemma.

Definition 3.1 ([Fan 1960/1961]). A multivalued mapping $G : X \to 2^X$ is called a KKM-mapping if for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of X, $co\{x_1, x_2, \ldots, x_n\}$ is contained in $\bigcup_{i=1}^n G(x_i)$, where coA denotes the convex hull of the set A.

Lemma 3.1 ([Fan 1960/1961]). *Let M be a nonempty subset of a Hausdorff topological vector space X. Let $G : M \to 2^X$ be a KKM-mapping such that $G(x)$ is closed for any $x \in M$ and is compact for at least one $x \in M$. Then $\bigcap_{y \in M} G(y) \neq \varnothing$.*

Definition 3.2. Let $h : E \times E \to Y$ be a vector-valued mapping. Then $h(\cdot, x)$ is said to be $C(x)$-convex on E for a fixed $x \in E$ if, for any $y_1, y_2 \in E$ and $\lambda \in [0, 1]$,

$$h(\lambda y_1 + (1 - \lambda) y_2, x) \in \lambda h(y_1, x) + (1 - \lambda) h(y_2, x) - C(x).$$

Remark 3.1. It is easy to say that $h(\cdot, x)$ is $C(x)$-convex if and only if for any given $x \in E$,

$$h\Big(\sum_{i=1}^{n} \lambda_i y_i, x\Big) \in \sum_{i=1}^{n} \lambda_i h(y_i, x) - C(x),$$

for any $y_i \in E$ and $\lambda_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^n \lambda_i = 1$.

Theorem 3.1. *Assume that the following conditions hold*:

(i) *E is a compact subset of X and $E \cap K(x)$ is nonempty and convex for any $x \in E$*;

(ii) *K is a closed mapping and $K^{-1}(y)$ is open in E for any $y \in E$*;

(iii) *for any $x \in E$, $\eta(x, x) = h(x, x) = 0$*;

(iv) *for any $x \in E$, the mapping $y \to h(y, x)$ is $C(x)$-convex*;

(v) *for any fixed $x, y \in E$ and each $t \in T(x, x)$, the mapping $y \to \langle t, \eta(y, x) \rangle$ is $C(x)$-convex*;

(vi) *for any* $y \in E$, $\{x \in E : \exists t \in T(x,x), \langle t, \eta(y,x)\rangle + h(y,x) \notin -\operatorname{int} C(x)\}$ *is closed.*

Then GVQVI (10) *has a solution.*

Proof. For any $x, y \in E$, set

$$\begin{aligned} S &= \{x \in E : x \in K(x)\},\\ P(x) &= \{z \in E : \langle T(x,x), \eta(z,x)\rangle + h(z,x) \subset -\operatorname{int} C(x)\},\\ \varphi(x) &= \begin{cases} K(x) \cap P(x), & \text{if} \quad x \in S,\\ E \cap K(x), & \text{if} \quad x \in E\backslash S \end{cases} \end{aligned}$$

and

$$Q(y) = E\backslash \varphi^{-1}(y).$$

First, we show that Q is a KKM-mapping. Indeed, suppose that there exists a finite subset $N = \{y_1, y_2, \dots, y_n\} \subseteq E$ and that $\alpha_i \geq 0$, $i = 1, 2, \dots, n$ with $\sum_{i=1}^n \alpha_i = 1$ such that $x = \sum_{i=1}^n \alpha_i y_i \notin \bigcup_{i=1}^n Q(y_i)$. Then, $x \notin Q(y_i)$; that is, $y_i \in \varphi(x)$ for $i = 1, 2, \dots, n$. If $x \in S$, then

$$\varphi(x) = K(x) \cap P(x).$$

Thus, $y_i \in P(x)$, $\quad i = 1, 2, \dots, n$, which implies that

$$\langle T(x,x), \eta(y_i,x)\rangle + h(y_i,x) \subset -\operatorname{int} C(x).$$

It follows that

$$\sum_{i=1}^n \alpha_i \langle T(x,x), \eta(y_i,x)\rangle + \sum_{i=1}^n \alpha_i h(y_i,x) \subset -\operatorname{int} C(x). \tag{15}$$

By conditions (iii)–(v) of Theorem 3.1 and (15), we have for any $x \in E$ and $t \in T(x,x)$

$$\begin{aligned} 0 &= \langle t, \eta(x,x)\rangle + h(x,x)\\ &\in \sum_{i=1}^n \alpha_i \langle t, \eta(y_i,x)\rangle - C(x) + \sum_{i=1}^n \alpha_i h(y_i,x) - C(x)\\ &\subseteq -\operatorname{int} C(x) - C(x) - C(x)\\ &\subseteq -\operatorname{int} C(x). \end{aligned}$$

Therefore, $0 \in -\operatorname{int} C(x)$, which is a contradiction. So, the only possibility is $x \in E\backslash S$. By the definition of S, $x \notin K(x)$. On the other hand, for $i = 1, 2, \dots, n$

$$y_i \in \varphi(x) = E \cap K(x).$$

Hence,

$$x = \sum_{i=1}^{n} \alpha_i y_i \in K(x),$$

represents another contradiction. Thus, Q is a KKM-mapping.

Next, we show that $Q(y)$ is a closed set for any $y \in E$. In fact, we have

$$\begin{aligned}
\varphi^{-1}(y) &= \{x \in S : y \in K(x) \cap P(x)\} \cup \{x \in E \backslash S : y \in K(x)\} \\
&= \{x \in S : x \in K^{-1}(y) \cap P^{-1}(y)\} \cup \{x \in E \backslash S : x \in K^{-1}(y)\} \\
&= [S \cap K^{-1}(y) \cap P^{-1}(y)] \cup [(E \backslash S) \cap K^{-1}(y)] \\
&= [(E \backslash S) \cup P^{-1}(y)] \cap K^{-1}(y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
Q(y) &= E \backslash \{[(E \backslash S) \cup P^{-1}(y)] \cap K^{-1}(y)\} \\
&= \{E \backslash [(E \backslash S) \cup P^{-1}(y)]\} \cup [E \backslash K^{-1}(y)] \\
&= [S \cap E \backslash P^{-1}(y)] \cup [E \backslash K^{-1}(y)]. \qquad (16)
\end{aligned}$$

Since K is closed mapping, S is closed set. From the definition of $P(x)$, we have

$$\begin{aligned}
E \backslash P^{-1}(y) &= \{x \in E : y \notin P(x)\} \\
&= \{x \in E : \exists t \in T(x, x), \langle t, \eta(y, x)\rangle + h(y, x) \notin -\operatorname{int} C(x)\},
\end{aligned}$$

which is closed by condition (vi). It follows from condition (ii) and (16) that $Q(y)$ is closed for any $y \in E$. Since E is compact, so is $Q(y)$. Therefore, by Lemma 3.1, we have that there exists $x^* \in E$ such that

$$x^* \in \bigcap_{y \in E} Q(y) = E \backslash \bigcup_{y \in E} \varphi^{-1}(y).$$

Thus, for any $y \in E$, $x^* \notin \varphi^{-1}(y)$; that is, $\varphi(x^*) = \varnothing$. If $x^* \in E \backslash S$, then we have

$$\varphi(x^*) = E \cap K(x^*) = \varnothing,$$

which contradicts condition (i).

If $x^* \in S$, that is, $x^* \in K(x^*)$, then

$$\varnothing = \varphi(x^*) = K(x^*) \cap P(x^*).$$

Thus, for any $y \in K(x^*)$, $y \notin P(x^*)$. It follows that there exists $t \in T(x^*, x^*)$ such that

$$\langle t, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} C(x^*), \qquad \text{for all } y \in K(x^*).$$

This completes the proof. $\square$

Example 3.1. Let $X = Y = R$, $E = [0, 1]$, $C(x) = R_+$,

$$K(x) = [0, \frac{1}{2}(x+1)], \qquad \text{for all } x \in [0, 1],$$

$$T(x, x) = \begin{cases} [0, 2], & \text{if} \quad x = 0.5, \\ [4x, 4], & \text{if} \quad x \neq 0.5, \end{cases}$$

$$\eta(y, x) = \begin{cases} \frac{x-y}{2}, & \text{if} \quad x \geq y, \\ \frac{y-x}{2}, & \text{if} \quad x < y, \end{cases}$$

$$h(y, x) = y^2 - x^2.$$

It is easy to verify that assumptions (i)–(iii) of Theorem 3.1 are fulfilled and for any $y \in E$, $K^{-1}(y)$ is an open set which was shown in [Khanh and Luu 2004]. Since

$$\begin{aligned} \lambda h(y_1, x) + (1-\lambda)h(y_2, x) - h(\lambda y_1 + (1-\lambda)y_2, x) &= \\ &= \lambda(y_1^2 - x^2) + (1-\lambda)(y_2^2 - x^2) - [(\lambda y_1 + (1-\lambda)y_2)^2 - x^2] \\ &= \lambda y_1^2 + (1-\lambda)y_2^2 - x^2 - [(\lambda y_1 + (1-\lambda)y_2)^2 - x^2] \\ &= \lambda(1-\lambda)(y_1 - y_2)^2 \\ &\geq 0, \end{aligned}$$

then condition (iv) of Theorem 3.1 is satisfied.

Let $\lambda y_1 + (1-\lambda)y_2 > x$. If $y_1 > x$ and $y_2 > x$, then

$$\lambda\langle t, \eta(y_1, x)\rangle + (1-\lambda)\langle t, \eta(y_2, x)\rangle - \langle t, \eta(\lambda y_1 + (1-\lambda)y_2, x)\rangle = 0.$$

If $y_1 > x$ and $y_2 \leq x$, then we have

$$\lambda\langle t, \eta(y_1, x)\rangle + (1-\lambda)\langle t, \eta(y_2, x)\rangle - \langle t, \eta(\lambda y_1 + (1-\lambda)y_2, x)\rangle \\ = \langle t, \frac{2(1-\lambda)(x-y_2)}{2}\rangle \geq 0.$$

If $y_1 \leq x$ and $y_2 > x$, then

$$\lambda\langle t, \eta(y_1, x)\rangle + (1-\lambda)\langle t, \eta(y_2, x)\rangle - \langle t, \eta(\lambda y_1 + (1-\lambda)y_2, x)\rangle \\ = \langle t, \frac{2\lambda(x-y_1)}{2}\rangle \geq 0.$$

Let $\lambda y_1 + (1-\lambda)y_2 \leq x$. If $y_1 \leq x$ and $y_2 \leq x$, then we have

$$\lambda\langle t, \eta(y_1, x)\rangle + (1-\lambda)\langle t, \eta(y_2, x)\rangle - \langle t, \eta(\lambda y_1 + (1-\lambda)y_2, x)\rangle = 0.$$

If $y_1 > x$ and $y_2 \le x$, then

$$\lambda\langle t, \eta(y_1, x)\rangle + (1-\lambda)\langle t, \eta(y_2, x)\rangle - \langle t, \eta(\lambda y_1 + (1-\lambda)y_2, x)\rangle = \\ = \langle t, \frac{2\lambda(y_1 - x)}{2}\rangle > 0.$$

If $y_1 \le x$ and $y_2 > x$, then we have

$$\lambda\langle t, \eta(y_1, x)\rangle + (1-\lambda)\langle t, \eta(y_2, x)\rangle - \langle t, \eta(\lambda y_1 + (1-\lambda)y_2, x)\rangle = \\ = \langle t, \frac{2(1-\lambda)(y_2 - x)}{2}\rangle > 0.$$

Therefore, condition (v) of Theorem 3.1 is satisfied.

If $x = 0.5$, $x \ge y$ and let $t = 2$, then

$$\langle t, \eta(y, x)\rangle + h(y, x) = \langle 2, \frac{x-y}{2}\rangle + y^2 - \frac{1}{4} = (y - \frac{1}{2})^2 \ge 0.$$

If $x = 0.5$, $x < y$ and let $t = 0$, then we have

$$\langle t, \eta(y, x)\rangle + h(y, x) = \langle 0, \frac{y-x}{2}\rangle + y^2 - \frac{1}{4} = y^2 - \frac{1}{4} > 0.$$

If $x \ne 0.5$, $x \ge y$ and let $t = 4x$, then

$$\langle t, \eta(y, x)\rangle + h(y, x) = \langle 4x, \frac{y-x}{2}\rangle + y^2 - x^2 = (x - y)^2 \ge 0.$$

If $x \ne 0.5$, $x < y$ and let $t = 4$, then we have

$$\langle t, \eta(y, x)\rangle + h(y, x) = \langle 4, \frac{y-x}{2}\rangle + y^2 - x^2 = (y+1)^2 - (x+1)^2 > 0.$$

Thus, for any $y \in E$,

$$\{x \in E : \exists t \in T(x, x), \langle t, \eta(y, x)\rangle + h(y, x) \notin -\operatorname{int} C(x)\} = [0, 1]$$

is a closed set. Therefore, all the assumptions of Theorem 3.1 are satisfied. It is easy to see that $x = 1$ and $t = 4$ is a solution of GVQVI (10).

Remark 3.2. Theorem 3.1 extends and unifies corresponding results of [Chen 1992; Konnov and Yao 1997; Lee and Lee 2000; Lee et al. 1996; Li and He 2005; Siddiqi et al. 1997; Yang 2003; Yang and Yao 2002; Yu and Yao 1996]. Furthermore, our proof is different from the methods used in these papers.

Corollary 3.1. *Assume that conditions* (*i*)–(*v*) *of Theorem 3.1 hold and the following assumptions are satisfied*:

(a) *if* $x_\alpha \to x$, $y_\alpha \to y$ *in* E *and if* $t_\alpha \in T(x_\alpha, x_\alpha)$, *then there exists* $t \in T(x, x)$ *and subnets* x_β, y_β *and* $t_\beta \in T(x_\beta, x_\beta)$ *such that* $(t_\beta, y_\beta) \to (t, y)$;

(b) *for any* $y \in E$, *the mappings* $x \to \eta(y, x)$ *and* $x \to h(y, x)$ *are continuous*;

(c) *the mapping* $x \to Y \backslash (-\operatorname{int} C(x))$ *is closed.*

Then GVQVI (10) *has a solution.*

Proof. By Theorem 3.1, it is sufficient to show that for any $y \in E$, the set

$$M = \{x \in E : \exists t \in T(x, x), \langle t, \eta(y, x)\rangle + h(y, x) \notin -\operatorname{int} C(x)\}$$

is closed. Let $\{x_\alpha\} \subset M$ and $x_\alpha \to x^*$. Then, there exists $t_\alpha \in T(x_\alpha, x_\alpha)$ such that

$$\langle t_\alpha, \eta(y, x_\alpha)\rangle + h(y, x_\alpha) \notin -\operatorname{int} C(x_\alpha).$$

By assumptions (a) and (b) of Corollary 3.2, there exists $t^* \in T(x^*, x^*)$ and subnets x_β and $t_\beta \in T(x_\beta, x_\beta)$ such that

$$\langle t_\beta, \eta(y, x_\beta)\rangle + h(y, x_\beta) \to \langle t^*, \eta(y, x^*)\rangle + h(y, x^*).$$

It follows from condition (c) that

$$\langle t^*, \eta(y, x^*)\rangle + h(y, x^*) \notin -\operatorname{int} C(x^*).$$

Therefore, $x^* \in M$. This means M is a closed set. This completes the proof. □

Remark 3.3. Example 2.1 in [Khanh and Luu 2004] illustrates that assumption (a) is satisfied.

Remark 3.4. If $T(x, x) = T(x)$, $\eta(y, x) = y - g(x)$ and $\eta(y, x) = 0$, where $g : E \to E$ is continuous mapping, then Corollary 3.1 reduces Theorem 2.1 in [Khanh and Luu 2004].

Corollary 3.2. *Assume that all conditions in Theorem 3.1 hold, except the assumed compactness of E which is replaced by one of the following conditions*:

(a) *there exists* $y^* \in E$ *such that* $E \backslash K^{-1}(y^*)$ *is compact and there exists a compact subset* $B \subset E$ *such that*

$$\langle T(x, x), \eta(y^*, x)\rangle + h(y^*, x) \subset -\operatorname{int} C(x), \quad \textit{for all } x \in E \backslash B;$$

(b) *there exists* $y^* \in E$ *such that* $E \backslash K^{-1}(y^*)$ *is compact and S is compact.*

Then GVQVI (10) *has a solution.*

Proof. From (16), it is sufficient to verify the compactness of $S \cap E \backslash P^{-1}(y^*)$ so that the FKKM theorem can be applied.

In case (a), we can obtain $E \backslash B \subset P^{-1}(y^*)$. Thus, $E \backslash P^{-1}(y^*) \subset B$. Since $E \backslash P^{-1}(y^*)$ is closed and B is compact, then both $E \backslash P^{-1}(y^*)$ and $S \cap E \backslash P^{-1}(y^*)$ are compact. For case (b), since S is compact and $E \backslash P^{-1}(y^*)$ is closed, then $S \cap E \backslash P^{-1}(y^*)$ is compact. Therefore, the FKKM theorem can be applied in cases (a) and (b). By Theorem 3.1, GVQVI (10) has a solution. This completes the proof. □

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xianjunlong@hotmail.com *Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China*

nanjinghuang@hotmail.com *Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China*