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In her Ph.D. Thesis, Czarneski began a preliminary study of the coefficients of the reciprocal of the Ihara zeta function of a finite graph. We give a survey of the results in this area and then give a complete characterization of the coefficients. As an application, we give a (very poor) bound on the number of Eulerian circuits in a graph. We also use these ideas to compute the zeta function of graphs which are cycles with a single chord. We conclude by posing several questions for future work.

1. Introduction

Ihara wrote two papers [1966a; 1966b] in which he set forth the framework to define the Ihara zeta function of a finite k-regular graph. Then Bass [1992] gave an expression for the zeta function that applied to all graphs, regardless of the regularity. Since then a great deal of work has been done on this function. We refer the reader to the series [Stark and Terras 1996; 2000; Terras and Stark 2007] for a very comprehensive overview. In general, the zeta function of a graph is the reciprocal of a polynomial and can be computed in polynomial time. The aim of this paper is to study the coefficients of this polynomial with an eye towards relating each coefficient to a specific structure in the graph.

Answering this question opens the door to some very interesting questions for future study. By understanding the polynomial, we have a solid ground to investigate families of graphs which are uniquely determined by their zeta functions. This type of question is addressed in a survey by Noy [2003] for several other important polynomial invariants. In addition, the roots of this polynomial connect to the Ramanujan condition on a graph [Bass 1992; Stark and Terras 1996; Kotani and Sunada 2000], and it would be very interesting to be able to construct polynomials

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which are reciprocals of Ihara zeta functions and then to find which graph gives rise to it. We pose some of these questions at the end of the paper.

For the rest of this section, we give a definition of the Ihara zeta function and then survey the work that has been done on the coefficients. We also present our main result at the end of this section. In Section 2, we give Kotani and Sunada's "oriented line graph" construction [2000], which will allow us to write the zeta function as

$$\det(I - uT)^{-1},$$

where T is the adjacency operator on the oriented line graph. Our results come from analyzing this determinant expression, much as Biggs [1994] analyzed the coefficients of the characteristic polynomial. In Section 2, we explicitly compute the zeta function of graphs which are cycles with a single chord. In addition, we give a rough bound on the number of Eulerian circuits in a graph in Section 3. Finally, we conclude by posing several questions for future work.

We begin by defining graphs, digraphs, and the symmetric digraph associated to a graph. All structures treated here are finite. We refer the reader to the books [Harary 1969b; Chartrand and Lesniak 1986] for a good overview of these structures.

A *graph* $X = (V, E)$ is a finite nonempty set V of *vertices* and a finite multiset E of unordered pairs of vertices, called *edges*. If $\{u, v\} \in E$, we say that u is *adjacent* to v and write

$$u \sim v.$$

A graph X is *simple* if there is no edge of the form $\{v, v\}$ and if there is no repeated edge.

A *directed graph* or *digraph* $D = (V, E)$ is a finite nonempty set V of *vertices* and a finite multiset E of ordered pairs of vertices, called *arcs*. For an arc $e = (u, w)$, we define the *origin* of e to be $o(e) = u$ and the *terminus* of e to be $t(e) = w$. The *inverse arc* of e, written as $\overline{e}$, is the arc formed by switching the origin and terminus of e: $\overline{e} = (w, u)$. In general, the inverse arc of an arc need not be present in the arc set of a digraph.

A digraph D is called *symmetric* if whenever (u, w) is an arc of D, its inverse arc (w, u) is as well. There is a natural one-to-one correspondence between the set of symmetric digraphs and the set of graphs, given by identifying an edge of the graph to an arc and its inverse arc on the digraph on the same vertices. We denote by $D(X)$ the symmetric digraph associated with the graph X. We give an example in Figure 1.

To define the Ihara zeta function, we need several cycle definitions. Let X be a graph and $D(X)$ its symmetric digraph. A *cycle c of length n* in X is a sequence $c = (e_1, \ldots, e_n)$ of n arcs in $D(X)$ such that $t(e_i) = o(e_{i+1})$ for $1 \le i \le n-1$ and $t(e_n) = o(e_1)$. We say that c has *backtracking* if $\overline{e_{i+1}} = e_i$ for some i satisfying

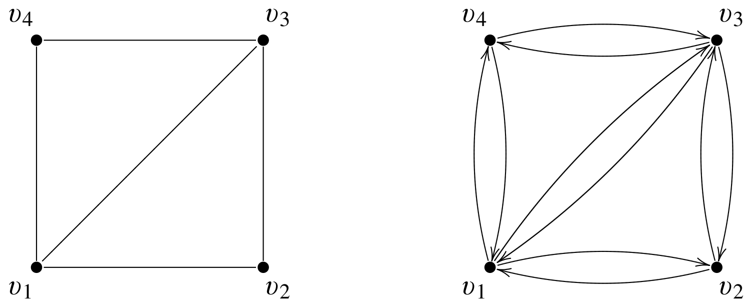

Figure 1. The complete graph minus an edge and its symmetric digraph.

$1 \le i \le n-1$. Also, c has a *tail* if $e_1 = \overline{e_n}$. We will primarily be interested in cycles with no backtracking or tail.

The *r-multiple* of the cycle c is the cycle c^r formed by going r times around c. We say a cycle is *primitive* if it is not the r-multiple of some other cycle b for $r \ge 2$. We impose an equivalence relation on cycles via cyclic permutation; that is, two cycles $b = (e_1, \dots, e_n)$ and $c = (f_1, \dots, f_n)$ are *equivalent* if there is a fixed $\alpha \in \mathbb{Z}/n\mathbb{Z}$ such that $e_i = f_{i+\alpha}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$ (all indices are considered modulo n). Note that the direction of travel does matter so traversing a cycle in the opposite direction does not give a cycle equivalent to the original one. A *prime cycle* is the equivalence class of primitive cycles which have no backtracking or tail, written as $[c]$.

The *Ihara zeta function* of a graph X is defined as a function of $u \in \mathbb{C}$ for $|u|$ sufficiently small by

$$Z_X(u) = \prod_{[c]} (1 - u^{l(c)})^{-1},$$

where the product is over the prime cycles in X and $l(c)$ is the *length* of the cycle c. Typically, this is an infinite product; however, the function is always rational. In fact, $Z_X(u)$ is always the reciprocal of a polynomial of maximum degree $2|E|$.

For a graph X, we let $n = |V|$ and $m = |E|$. We write

$$\frac{1}{Z_X(u)} = Z_X(u)^{-1} = c_0 + c_1 u + c_2 u^2 + c_3 u^3 + \cdots + c_{2m} u^{2m}.$$

We are concerned with determining the coefficients c_i in terms of structure in the graph X. We cite the known results and then give our main result.

From the definition of $Z_X(u)$, it is immediate that $c_0 = 1$. The first result in this area was given by Kotani and Sunada [2000], which is an expression for c_{2m}.

Theorem 1. *Let X be a graph and $Z_X(u)$ its Ihara zeta function as written above. We take $n = |V|$ and $m = |E|$. We denote by $d(v)$ the* degree *of vertex v which is* *the number of edges to which v is incident. Then,*

$$c_{2m} = (-1)^{m-n} \prod_{v_i \in V} (d(v_i) - 1).$$

Czarneski computed c_1 in her dissertation [2005]:

Theorem 2. *Let X be a graph and $Z_X(u)$ its Ihara zeta function as written above. Then the coefficient c_1 is the negative of twice the number of loops in X.*

In his dissertation, Storm [2007] computed c_3 from the number of triangles in X. The method used in the next section for an arbitrary coefficient is an extension of the one used for this theorem. We will look at it in more detail in the next section.

Theorem 3. *Let X be a simple graph and $Z_X(u)$ its Ihara zeta function as written above. Then the coefficient c_3 is the negative of twice the number of triangles in X.*

The final result in this area comes from Horton's dissertation [2006]. It encompasses Theorem 3; however, it is harder to generalize to realize the other coefficients. He shows that the girth of X can be recovered from the zeta function and relates a coefficient of the zeta function to this. We give two definitions and then his theorem.

Definition 4. Let X be a graph. The *girth* of X is the length of the shortest cycle in X. A *k-gon* in X is a subgraph of X which is isomorphic to the cycle graph C_k–C_k is the connected graph on k vertices such that the degree of every vertex is 2.

Theorem 5. *Let g be the girth of a simple connected graph X with zeta function $Z_X(u)$ written as above. Then, $c_k = 0$ for $1 \leq k < g$. Moreover, c_g is the negative of twice the number of g-gons in X.*

To state our more general result, we need a few more digraph definitions. The *indegree* of a vertex v, $in(v)$, in a digraph D is the number of arcs with terminus v. Similarly, the *outdegree* of v, $out(v)$, is the number of arcs with origin v. A *subgraph* of a digraph D is a digraph having all of its vertices and arcs in D. A *spanning subgraph* is a subgraph containing all of the vertices of D. Finally, a *linear subgraph of a digraph D* is a spanning subgraph in which each vertex has indegree one and outdegree one. A linear subgraph is thus a disjoint spanning collection of directed cycles.

Definition 6. Let D be a digraph. We denote by $\mathcal{S}_k(D)$ the set of subgraphs of D which have exactly k vertices. For an element $\tilde{D}$ of $\mathcal{S}_k(D)$, we denote by $\mathcal{E}_k(\tilde{D})$ the number of linear subgraphs of $\tilde{D}$ which consists of an even number of cycles of even length. Similarly we denote by $\mathcal{O}_k(\tilde{D})$ the number of linear subgraphs of $\tilde{D}$ with an odd number of cycles of even length.

We now state our main theorem:

Theorem 7. *Let X be a connected graph with oriented line graph $L^o X$ (defined in the next section) and $Z_X(u)$ its Ihara zeta function as before. We also take the notation of Definition 6 as applied to the digraph $L^o X$. Then for $1 \le k \le 2m$, the coefficient c_k can be realized as*

$$c_k = \sum_{D \in \mathcal{S}_k(L^o X)} (-1)^k (\mathscr{E}_k(D) - \mathbb{O}_k(D)).$$

We prove this theorem in the next section and explore some of its consequences. In particular, we can realize Theorems 3 and 5 as corollaries to this. We will also give a practical list of things the Ihara zeta function must determine about a graph as a consequence of this theorem. In particular, Corollary 14 points out that the Ihara zeta function of a simple graph determines the number of triangles, squares, and pentagons in the graph.

2. Explicit representation of the coefficients

The first step to analyzing the coefficients of the zeta function is to realize the zeta function as a determinant expression. To do this, we construct an oriented line graph, a technique which was first proposed by Kotani and Sunada [2000].

We begin with a graph X and form its symmetric digraph $D(X)$. Hence $D(X)$ has $2|E(X)|$ arcs. Now we construct the oriented line graph $L^o X = (V_L, E_L^o)$ by

$$V_L = E(D(X)),$$
$$E_L^o = \{(e_i, e_j) \in E(D(X)) \times E(D(X));\ \overline{e_i} \neq e_j,\ t(e_i) = o(e_j)\}.$$

We give an example of this construction in Figure 2. The intuitive idea is that we are building a digraph which models all of the "legal" moves we could take to get prime cycles in X. It is for this reason that we disallow going from an arc to its inverse arc. We are particularly concerned with the adjacency matrix of this digraph.

Definition 8. Let D be a digraph with n vertices, written as $\{v_1, \ldots, v_n\}$. The *adjacency matrix* T of D is the $n \times n$ matrix given by setting the (i, j)-entry $T_{i,j}$ to be 1 if there is an arc with origin v_i and terminus v_j, and zero otherwise.

Thus for the oriented line graph, the matrix T is a $2|E(X)| \times 2|E(X)|$ matrix which catalogues whether it is legal for an arc in $D(X)$ to feed into another arc. This matrix is given several different names in the zeta function literature. Stark and Terras [1996] refer to it as an "edge routing matrix". Kotani and Sunada [2000] call it the Perron–Frobenius matrix.

The following proposition, found in [Kotani and Sunada 2000], makes it clear why we are concerned with the oriented line graph and its adjacency matrix.

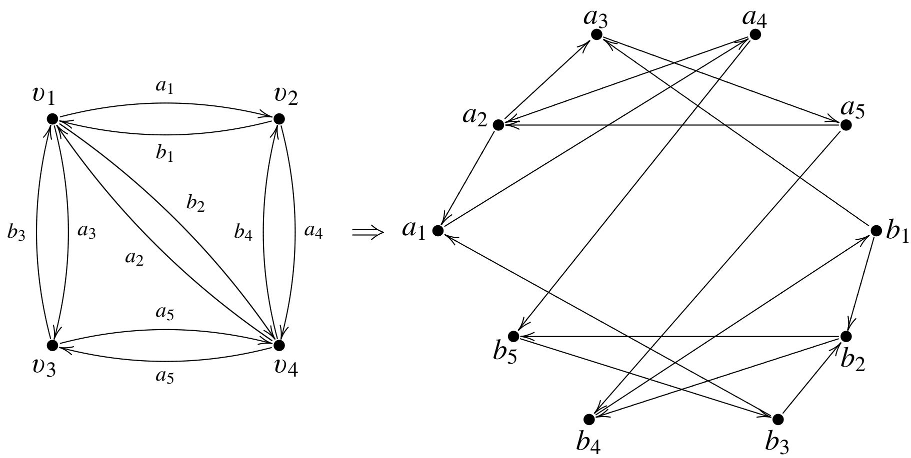

Figure 2. Construction of an oriented line graph of K_4 minus an edge.

Proposition 9. *There is a one-to-one correspondence between primitive cycles with no backtracking or tail in X and primitive cycles in L^oX. Moreover, if X is a connected graph, the zeta function $Z_X(u)$ can be written as*

$$Z_X(u) = \det(I - uT)^{-1},$$

where T is the adjacency matrix of the oriented line graph L^oX.

Studying this determinant expression will give us insight into the coefficients. For the rest of this section we let $m = |E(X)|$. We first note that the coefficients of the characteristic polynomial of T and those of the reciprocal of the zeta function are intimately related.

Lemma 10. *Let T be the adjacency matrix of the oriented line graph associated with the connected graph X. We write the characteristic polynomial of T as*

$$\chi_T(u) = \det(T - uI) = u^{2m} + c_1u^{2m-1} + \cdots + c_{2m}.$$

Then the reciprocal of the Ihara zeta function of X can be written as

$$\frac{1}{Z_X(u)} = Z_X(u)^{-1} = 1 + c_1u + c_2u^2 + c_3u^3 + \cdots + c_{2m}u^{2m}.$$

Proof. We begin by considering $\chi_T(u) = \det(T - uI)$. We rewrite this as

$$\det(T - uI) = (-u)^{2m} \det\Big(I - \frac{1}{u}T\Big).$$

We now replace u by $1/u$ and the result follows. □

This is very helpful since the coefficients of characteristic polynomials are very well understood as the sum of the principal minors of the matrix involved.

Definition 11. A *principal minor* of a square matrix M is the determinant of a submatrix of M formed by selecting a subset of the matrix's rows and the columns indexed by the same subset.

We make use of a useful linear algebra fact:

Lemma 12. *Let M be an $n \times n$ square matrix with characteristic polynomial*

$$\chi_M(u) = u^n + c_1 u^{n-1} + \cdots + c_n.$$

Then the coefficient c_i is $(-1)^i$ times the sum of all $i \times i$ principal minors of M.

We wish to apply Lemma 12 to the characteristic polynomial of the adjacency matrix T of the oriented line graph of X. This will then give us the information we need about coefficients of the reciprocal of the Ihara zeta function.

How can we interpret a principal minor of the matrix T? We let I be the index set which determines which rows we are keeping when we pass to the principal minor. Each row and the corresponding column represent a vertex in the oriented line graph. These vertices in turn represent arcs in the symmetric digraph $D(X)$. Then by reducing the matrix T to only keeping the rows and columns indexed by I, we are in fact looking at the matrix $\tilde{T}$ we would get by taking the subgraph induced on $D(X)$ by the arcs indexed by I and then forming the submatrix's oriented line graph and adjacency matrix. Thus an $i \times i$ principal minor can be computed by taking the appropriate subgraph of $D(X)$ induced by i edges, forming its $\tilde{T}$ matrix, and then taking the determinant.

This leaves only the question: how can we compute the determinant of the adjacency matrix of a digraph? Fortunately Harary [1962] answers this by

Lemma 13. *Let D be a digraph whose linear subgraphs are D_i, for $i = 1, \ldots, n$, and suppose each D_i has e_i even cycles. Then*

$$\det A = \sum_{i=1}^{n} (-1)^{e_i},$$

where A is the adjacency matrix of D.

Proof of Theorem 7. We consider the coefficient c_k for $2 \le k < 2m$. By Lemma 12, we must consider all of the $k \times k$ principal minors of T. Each such principal minor corresponds to picking k vertices of $L^o X$ and then taking the subdigraph induced by those vertices. Such a subgraph is then a member of $\mathcal{S}_k(L^o X)$. We call this subgraph $\tilde{D}$.

Then the principal minor corresponds to the determinant of the adjacency operator $\tilde{T}$ of $\tilde{D}$. To take this determinant, we use Lemma 13. We let $\tilde{D}_i$ for $i = 1, \ldots, j$

be the linear subgraphs of $\tilde{D}$. Then

$$\det \tilde{T} = \sum_{i=1}^{j} (-1)^{e_i},$$

where e_i is the number of even cycles in $\tilde{D}_i$. Using the notation of Definition 6, we have

$$\det \tilde{T} = \mathscr{E}_k(\tilde{D}) - \mathscr{O}_k(\tilde{D}).$$

We combine this statement with Lemma 12 to get the result

$$c_k = \sum_{\tilde{D} \in \mathscr{S}_k(L^o X)} (-1)^k \big(\mathscr{E}_k(\tilde{D}) - \mathscr{O}_k(\tilde{D})\big). \qquad \square$$

With this theorem it is fairly easy to compute the coefficients of smaller powers of u. We use Proposition 9 to take information about cycles in the oriented line graph back to information about cycles in the graph. In particular, notice that a linear subgraph of $L^o X$ corresponds to an edge-disjoint collection of backtrack-free, tailless cycles in the symmetric digraph of the original graph. Therefore, each subgraph of the symmetric digraph that has k edges and consists only of edge-disjoint backtrackfree, tailless cycles contributes to the coefficient c_k. This approach, of course, is not a practical way to compute higher powers; fortunately, we can get a great deal of information from the lower powers. We first give a very explicit statement; then, we give a second corollary which is more general.

Corollary 14. *Let X be a connected graph with Ihara zeta function as above.*

(1) *If X has loops, the coefficient c_1 can be computed by Theorem 2.*

(2) *If X does not have loops, then the coefficient c_2 is the negative of twice the number of primitive cycles of length 2 in X. Also, the coefficient c_3 is the negative of twice the number of triangles in X. In addition, c_4 is the number of primitive cycles of length 2 plus twice the number of pairs of primitive cycles of length 2 that share an edge plus four times the number of edge disjoint pairs of primitive cycles of length 2 minus twice the number of squares in X.*

(3) *If X is a simple graph, the coefficients c_3, c_4, and c_5 are the negative of twice the number of triangles, squares, and pentagons in X respectively. Also, c_6 is the negative of twice the number of hexagons in X plus four times the number of pairs of edge disjoint triangles plus twice the number of pairs of triangles with a common edge, while c_7 is the negative of twice the number of heptagons in X plus four times the number of edge disjoint pairs of one triangle and one square plus twice the number of pairs of one triangle and one square that share a common edge.*

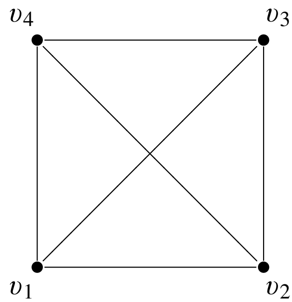

Figure 3. The complete graph K_4.

Proof. We leave the proof as an exercise to the reader. Particular care should be taken to get the coefficient c_4 as detailed in the second statement. The possible ways to orient the smaller cycles show up in the number of subgraphs on 4 vertices of the oriented line graph. □

Corollary 14 provides a very concrete way to compute the coefficients of smaller powers of u. We give a definition and then a more general statement.

Definition 15. Let X be a graph with two cyclic subgraphs C_n and C_m. We call C_n and C_m *compatible* if it is possible to orient the edges of X so that C_n and C_m both become oriented cycles.

Example 16. We consider the complete graph K_4 shown in Figure 3. For our first cycle, we choose the cycle which goes from v_1 to v_2 to v_3 to v_4 and back to v_1. This is a copy of C_4. Now consider the copy of C_3 given by going from v_1 to v_2 to v_3 and back to v_1. These two cycles are compatible. Any orientation which makes our copy of C_4 into an oriented cycle will work so long as we orient the edge $\{v_1, v_3\}$ correctly.

Let's look at an example of some cycles which are not compatible. We keep the same graph and the same initial copy of C_4. Now we choose a second copy of C_4 given by going from v_1 to v_2 to v_4 to v_3 and back to v_1. These two cycles are not compatible. Orient the first cycle so that you get an oriented cycle. Now either the edge $\{v_1, v_2\}$ or $\{v_3, v_4\}$ will not be oriented correctly to make the second cycle into an oriented cycle, irrespective of how the edges $\{v_1, v_3\}$ and $\{v_2, v_4\}$ are oriented.

Compatible cycles play an important role in this analysis since, whenever two cycles are compatible, they give rise to edge-disjoint cycles in the symmetric digraph — simply take one cycle as oriented then reverse the edge orientations for the other cycles so that neither of them ever use an edge in the same direction. These edge-disjoint cycles then show up in the oriented line graph as disjoint unions of cycles, exactly the structures that contribute to the coefficients of the zeta function. Now that we have this connection in general, we can state a more general corollary.

Corollary 17. *Let X be a connected graph with girth g and Ihara zeta function as above.*

(1) *Whenever $0 < i < g$, the coefficient c_i equals* 0.

(2) *Whenever $g \leq i < 2g$, the coefficient c_i is the negative of twice the number of i-gons in X.*

(3) *Whenever $2g \leq i < 3g$, the coefficient c_i is the sum of the following terms*:

- *the negative of twice the number of i-gons in X,*
- *four times the number of edge disjoint pairs of a k-gon and a $(c_i - k)$-gon for $g \leq k < 2g$,*
- *twice the number of pairs of a k-gon and a $(c_i - k)$-gon that share at least one edge and are compatible for $g \leq k < 2g$, and*
- *twice the number of edge disjoint pairs of a k_1-gon and a k_2-gon that have a path of length $\frac{1}{2}(c_i - k_1 - k_2)$ between them and are compatible for $k_1 + k_2 < 3g$.*

Corollary 17 thus encompasses Theorems 3 and 5. It also makes it possible to write down the zeta function of certain graphs, particularly graphs which have very few cycles. The fewer the number of cycles, the easier it is to identify where the linear subgraphs are showing up in the calculations of Theorem 7. We look at the graphs C_n and the graphs which are a cycle with a single chord. The zeta function of C_n is easy to compute directly from the definition, but it is instructive to apply the corollary to these graphs.

Example 18. Consider the graph C_n which is the graph that is a cycle on n vertices. Then its zeta function is given by

$$Z_{C_n}(u)^{-1} = 1 - 2u^n + u^{2n}.$$

The graph C_n has girth n, so all of the coefficients up to c_n are zero. In addition, there is a single n-gon, so the coefficient c_n is given by -2. There are no other k-gons, so all of the rest of the coefficients up to c_{2n} must be zero. Finally, c_{2n} can be computed by Czarneski's result or as a consequence of there being only one n-gon and no primitive cycle of length $2n$.

Cycles which have exactly one chord are a bit more delicate since there are more cycles to consider. With due care, we can still work out the zeta function. We define the graph $CH_{n,k}$ by starting with the cycle graph C_n and adding an additional edge so that the smallest cycle in $CH_{n,k}$ has length $k+1$. We illustrate $CH_{10,4}$ in Figure 4.

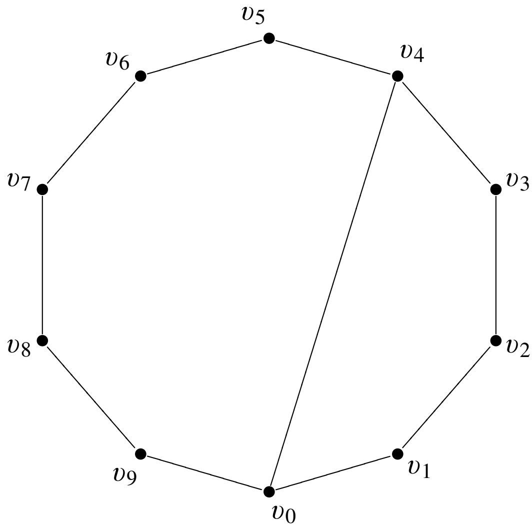

Figure 4. $CH_{10,4}$.

Corollary 19. *The zeta function of $CH_{n,k}$ is given by*

$$Z_{CH_{n,k}}(u)^{-1} = 1 - 2u^{k+1} - 2u^{n-k+1} - 2u^n + 2u^{2n-k+1} \\ +2u^{n+2} + 2u^{n+k+1} + u^{2k+2} + u^{2n-2k+2} + u^{2n} - 4u^{2n+2}.$$

Proof. By inspection, it is easy to see that there are exactly six backtrackfree, tailless cycles in the symmetric digraph $D(CH_{n,k})$. Specifically, $D(CH_{n,k})$ contains clockwise and counterclockwise copies of C_n, C_{n-k+1}, and C_{k+1}. Taken individually, these cycles contribute the second, third and fourth term of $Z_{C_{n,k}}(u)^{-1}$ above. There are nine different subgraphs of the symmetric digraph that consist of exactly two of these cycles; these subgraphs contribute the next six terms. Finally there are four linear subgraphs of the oriented line graph of $CH_{n,k}$, giving us the final u^{2n+2} term. We leave it to the reader to find these four linear subgraphs and verify that they break up into an odd number of cycles. There is no further backtrackfree, tailless cycle in $D(CH_{n,k})$. □

Example 20. We return to the example of $CH_{10,4}$. By direct calculation, using the formula $\det(I - uT)$, we see that

$$Z_{CH_{10,4}}(u)^{-1} = 1 - 2u^5 - 2u^7 - u^{10} + 2u^{12} + u^{14} + 2u^{15} + 2u^{17} + u^{20} - 4u^{22}.$$

Corollary 19 would have us write the function as

$$Z_{CH_{10,4}}(u)^{-1} = 1 - 2u^5 - 2u^7 - 2u^{10} + 2u^{17} + 2u^{12} + 2u^{15} + u^{10} + u^{14} + u^{20} - 4u^{22}.$$

By collecting common powers in the second expression, we see that we do have the same polynomial.

Thus the reciprocal of the Ihara zeta function of a graph encodes a great deal of structural information about the graph, particularly about the graph's primitive cycles. We have high hopes that, in general, it encodes enough information to allow us to conclude that certain families of graphs are determined by their zeta functions. We will pose this question and a few others in the next section.

3. Conclusion

In this section, we first explore the question of bounding the number of Eulerian circuits in an Eulerian graph. Recently, the problem of counting the number of Eulerian circuits has been shown to be $\#P$-complete in the class of undirected graphs by Brightwell and Winkler [2004]. There has been very little success at even bounding this number. We give a fairly rough bound, which is very inaccurate.

We let X be an (undirected) Eulerian graph and denote by $\operatorname{eul} X$ the number of Eulerian circuits on X. An *Eulerian circuit* is a cycle which uses every edge of X exactly once. As such, it is a primitive cycle of length m where m is the number of edges in X. When counting Eulerian circuits, we do distinguish direction of travel, so given one circuit, we can get another by traversing the same edge sequence by in reverse order. Thus the cycle graphs C_n satisfy $\operatorname{eul} C_n = 2$.

To state our bound, we need to define the *permanent* of a matrix M.

Definition 21. Let $M = (m_{i,j})$ be an $n \times n$ square matrix. The *permanent* of M the "signless determinant", that is,

$$\operatorname{perm} M = \sum_{\sigma \in S_n} \prod_{i=1}^{n} m_{i,\sigma(i)},$$

where S_n is the symmetric group over the set $\{1, \ldots, n\}$ (the group of permutations of this set).

The permanent shows up in several interesting ways in graph theory. For instance, it gives the number of perfect matchings of a bipartite graph [Harary 1969a]. For a general (0, 1)-square matrix, the same reference gives us a useful expression connecting the permanent to linear subgraphs of a digraph:

Lemma 22. *We use the notation from Definition 6. Let D be a digraph with n vertices and adjacency matrix A. Then the permanent of A is given by*

$$\operatorname{perm} A = \mathscr{E}_n(D) + \mathscr{O}_n(D).$$

In other words, the permanent of A counts the number of linear subgraphs of D.

With this interpretation, it is easier to be persuaded of the validity of the next result:

Theorem 23. *Let X be an Eulerian graph with oriented line graph $L^o X$. We let T be the adjacency matrix of $L^o X$. Then*

$$\operatorname{eul} X \leq \det T + \operatorname{perm} T.$$

We will use the notation $\vartheta(X) = \det T + \operatorname{perm} T$.

Proof. For a particular Eulerian circuit c, we denote by $\bar{c}$ the Eulerian circuit formed by traversing the edges in the opposite direction. Thus the pair of Eulerian circuits c and $\bar{c}$ induce a linear subgraph of the oriented line graph $L^o X$. This linear subgraph is composed of exactly 2 cycles. They are either both even or both odd cycles. In either case, this linear subgraph contributes positive 1 to the computation of $\det T$.

We can think of the determinant of T as the sum of the positive contribution minus the negative contribution. The permanent, however, is the sum of the positive contribution plus the negative contribution. Thus if we take $\det T + \operatorname{perm} T$, we get twice the positive contribution. Since two Eulerian circuits add exactly 1 to the positive contribution, we get the desired result. □

There is a fairly serious flaw with this bound. In general, computing the permanent of a (0, 1)-matrix is a $\#P$-complete problem [Valiant 1979], so we do not seem to have really improved matters. Fortunately, there are polynomial probabilistic algorithms that can compute the permanent within a specific amount of error [Jerrum et al. 2004]. As there is no known polynomial algorithm to even estimate the number of Eulerian circuits in a graph, we have actually managed to say something.

We present in Table 1 the results of computing the number of Eulerian circuits as well as the sum of the determinant and the permanent of the adjacency matrix of the oriented line graph for all connected Eulerian graphs on 6 vertices. We denote by n the number of vertices, by m the number of edges, by $-\chi$ the negative of the *Euler number* (which is $n - m$), by eul the exact number of Eulerian circuits, and by ϑ the given bound. The graphs here are given in graph6 format. We use the program Nauty [McKay 2007] to generate the graphs. All calculations were done in SAGE [Stein 2008]. The exact number of Eulerian circuits was computed using an algorithm that is currently being worked on by Klyve and Storm. The graphs are small enough and the algorithm is developed well enough so we are certain of the calculations presented.

From the data presented, we see that this bound fluctuates wildly in terms of error and would be completely ineffective for a graph with decent size.

This work suggests a great many problems for further research. We present a few of them here, in no particular order of perceived difficulty.

Graph	n	m	$-\chi$	eul	ϑ	error
EqGW	6	6	0	2	2	0
E@ro	6	7	1	4	6	2
E_lw	6	8	2	12	90	78
E?~o	6	8	2	12	90	78
EElw	6	9	3	32	702	670
ET\w	6	10	4	88	6642	6554
Er\w	6	11	5	264	58806	58542
E}lw	6	12	6	744	532170	531426

Table 1. Computations for all connected Eulerian graphs on 6 vertices.

Problem 24 (Graphs determined by their zeta functions). Recently, there has been some good work showing that several infinite families of graphs are determined by their Tutte polynomials [de Mier and Noy 2004]. One of the keys to these proofs is that the Tutte polynomial determines the number of triangles and squares in a graph. We saw in the previous section that the zeta function determines the number of triangles, squares, and pentagons in a simple graph. This gives us some hope that some large families of graphs are determined by their zeta functions. This would be particularly interesting since the Ihara zeta function can be computed in polynomial time.

A reader interested in this problem may want to start with the survey by Stark and Terras [1996] to become familiar with the edge zeta function as this function is necessary to determine if every vertex of a graph has degree greater than or equal to 2 or not. Cooper [2006] also has some preliminary work towards identifying other graph invariants determined by the zeta function that could prove useful.

We conjecture that the wheel graphs W_n defined by taking the cycle C_n and adding a vertex which is adjacency to every other vertex are uniquely determined by the Ihara zeta function among the connected graphs for which every vertex has degree at least 2. Through a computer search, we have verified

Theorem 25. *Within the family of connected graphs such that the degree of every vertex in a graph is at least* 2, *the graphs* W_3, W_4, W_5, W_6, W_7, W_8 *and* W_9 *are determined by their Ihara zeta functions. If, instead, we consider the edge zeta function defined by Stark and Terras [1996], we can remove the condition on the degrees of the vertices.*

In the left half of Table 2, we count the number of connected graphs on n vertices for $n = 4, \ldots, 8$ as well as how many distinct zeta functions, characteristic polynomials, and pairs of zeta function and characteristic polynomial. In the right half, we only count graphs which are "md2"—every vertex has degree at least

Vertices	Graphs	Distinct Zetas	Distinct Spectra	Distinct Pairs	md2 Graphs	Distinct Zetas	Distinct Spectra	Distinct Pairs
4	6	5	6	6	3	3	3	3
5	21	16	21	21	11	11	11	11
6	112	77	111	112	61	61	61	61
7	853	584	821	850	507	507	494	507
8	11117	10423	8025	11106	7442	7441	7064	7442

Table 2. Graph and zeta function counting.

2 — as these are the more natural classes to consider zeta functions. The column referring to “Spectra” is counting the number of unique adjacency matrix spectra which appear. We see that the zeta function does remarkably well at distinguishing graphs, suggesting that there could be a lot of opportunities to show that families are uniquely determined.

Problem 26 (The inverse problem). Though we have given a characterization of the coefficients of the reciprocal of the Ihara zeta function, we have not answered some important questions.

(1) Given a polynomial $p(u)$, determine if it is the reciprocal of the Ihara zeta function of some graph.

(2) Given a polynomial $p(u)$ which is the reciprocal of the Ihara zeta function of a graph, construct an oriented line graph which gives rise to it. This is equivalent to constructing a graph which gives rise to it, as Cooper’s algorithm [2006] recovers the graph from its oriented line graph.

(3) Construct a polynomial which satisfies the graph “Riemann” hypothesis (see [Kotani and Sunada 2000; Stark and Terras 1996] for details) and which is also the reciprocal of the Ihara zeta function of some graph.

Solving the last two questions would provide a new construction of Ramanujan graphs. We also suggest [Horton et al. 2006; Murty 2003] for more information on Ramanujan graphs and their connection to the Ihara zeta function.

Problem 27 (A better Eulerian circuit count bound). It should be possible to give a better bound than the one found in Theorem 23. In our examination of Eulerian circuits, we really only scratched the surface of the structure that the zeta function tells us about. A deeper study may prove fruitful.

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