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Divisor concepts for mosaics of integers

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The mosaic of the integer n is the array of prime numbers resulting from iterating the Fundamental Theorem of Arithmetic on n and on any resulting composite exponents. In this paper, we generalize several number theoretic functions to the mosaic of n, first based on the primes of the mosaic, second by examining several possible definitions of a divisor in terms of mosaics. Having done so, we examine properties of these functions.

1. Introduction

Mosaics. Mullin, in a series of papers [1964, 1965, 1967a, 1967b], introduced the number theoretic concept of the mosaic of n and explored several ideas related to it.

Definition 1.1. The *mosaic of the integer n* is the array of prime numbers resulting from iterating the Fundamental Theorem of Arithmetic (FTA) on n and on any resulting composite exponents.

The following example illustrates this definition.

$$\begin{aligned} n &= 1{,}024{,}000{,}000 && \leftarrow\text{-- use the FTA to find the prime factorization of } n\\ &= 2^{16}\cdot 5^{6} && \leftarrow\text{-- apply FTA to composite exponents 16 and 6}\\ &= 2^{2^4}\cdot 5^{2\cdot 3} && \leftarrow\text{-- apply FTA again to composite number 4}\\ &= 2^{2^{2^2}}\cdot 5^{2\cdot 3}. && \leftarrow\text{-- the mosaic of the integer; only primes remain}\end{aligned}$$

Mullin introduced several functions on the mosaic, the first of which was $\psi(n)$, the product of all of the primes in the mosaic of n. As an example, we have

$$\psi\left(2^{17^3}\cdot 3^{5^5}\right) = 2\cdot 17\cdot 3\cdot 3\cdot 5\cdot 5 = 7650.$$

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In number theory, a function is multiplicative if and only if $f(mn) = f(m)f(n)$ whenever m and n are relatively prime. Mullin extended this concept to mosaics by saying that f is *generalized multiplicative* if and only if $f(mn) = f(m)f(n)$ whenever the mosaics of m and n have no primes in common. He showed that $\psi(n)$ is generalized multiplicative and that any multiplicative function is also generalized multiplicative. He also generalized the Möbius function $\mu(n)$ to the *generalized Möbius function*:

$$\mu^*(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if the mosaic of } n > 1 \text{ has any prime number repeated,} \\ (-1)^k & \text{if the mosaic of } n > 1 \text{ has no prime repeated, where } k \text{ is} \\ & \quad \text{the number of \textit{distinct} primes in the mosaic of } n. \end{cases}$$

Similarly, Mullin generalized the concept of additivity: a function f is *generalized additive* if and only if $f(mn) = f(m) + f(n)$ whenever the mosaics of m and n have no primes in common. He defined $\psi^*(n)$ as the sum of the primes in the mosaic of n and showed that this function was generalized additive. As an example,

$$\psi^*\left(5^{2^3\cdot 7} \cdot 11^{13^{19}}\right) = 5+2+3+7+11+13+19 = \psi^*\left(5^{2^3\cdot 7}\right) + \psi^*\left(11^{13^{19}}\right).$$

Levels of the mosaic of n. Following Mullin's work, Gillman [1990, 1992] defined new functions on the mosaic of n. He used the concept of *levels* of the mosaic to describe the different tiers of exponentiation.

Suppose

$$n = 2^{3\cdot 5^7} \cdot 17^{23^{19}\cdot 29^{13^{11}\cdot 89}},$$

then the 2 and 17 are on the first level, the 3, 5, 23 and 29 are on the second level, the 7, 19, 13, and 89 are on the third level, and the 11 is on the fourth level of the mosaic.

Using this idea, Gillman generalized the Möbius function as follows:

$$\mu_i(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if the mosaic of } n \text{ has duplicate primes in the first } i \text{ levels} \\ & \quad \text{(including multiplicities at the } i\text{-th level),} \\ (-1)^k & \text{if the mosaic of } n \text{ consists of } k \text{ distinct primes in the first} \\ & \quad i \text{ levels.} \end{cases}$$

With this definition,

$$\mu_\infty = \mu^* \quad \text{and} \quad \mu_1 = \mu.$$

Gillman also extended Mullin's concept of generalized multiplicative to include the levels of the mosaic.

Definition 1.2. A function f is *i-multiplicative* if and only if $f(mn) = f(m)f(n)$ when m and n have no primes in common in the first i levels of their mosaic.

Gillman proved μ_i is i-multiplicative for all i. He then extended Mullin's work on $\psi(n)$ by generalizing it to depend on the levels of the mosaic.

Definition 1.3. For a fixed i and j such that $j > i$, the function $\psi_{j,i}(n)$ is computed as follows: Expand n through the first j levels of its mosaic; for each prime p on the i-th level of this expansion, multiply p by the product of the primes in the $(i+1)$-th through j-th levels above p, including multiplicities of the primes at the j-th level.

The following examples illustrate these computations:

$$\psi_{6,3}\left(2^{3^{5^{7^{11\cdot 13^2}}}} \cdot 3^{2\cdot 5^{3\cdot 7}}\right) = 2^{3^{5\cdot 7\cdot 11\cdot 13\cdot 2}} \cdot 3^{2\cdot 5^{3\cdot 7}},$$

$$\psi_{\infty,1}(n) = \text{product of all primes in the mosaic} = \psi(n).$$

Gillman also introduced the concept of i-relatively prime mosaics. That is, two integers, m and n, are *i-relatively prime* when they have no primes in common in the first i levels of their mosaics. Thus, the integers with mosaics 2^{3^5} and 7^{11^3} are 2-relatively prime, but *not* 3-relatively prime.

Motivation. In this paper, we will introduce new families of functions on the mosaic of n and determine which of these are i-multiplicative or i-additive. In Section 2, the functions will depend only on the primes present in the first i levels and their multiplicities. In Section 3, we evaluate previous attempts to generalize the concept of a divisor to the mosaic, and in Section 4 we introduce a new definition of a mosaic divisor that we believe will be more useful.

2. Mosaic functions

The functions Ω, ω, and λ are number theoretic functions that can be easily generalized to the mosaic of n. We discuss their generalizations because they are either i-multiplicative or i-additive. We also introduce a new function, ψ^*, which is interesting since it is either i-multiplicative or i-additive depending on the value of i.

The functions Ω_i and ω_i. $\Omega(n)$ is the total number of primes in the factorization of n, including repetitions. We generalize this idea with the function $\Omega_i(n)$, the total number of primes in the first i levels of the mosaic of n, including multiplicities on the i-th level. Thus, as an example,

$$\Omega_2\big(2^{3^7}\cdot 5^{11^6}\big) = \Omega_2(2^{3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3}\cdot 5^{11\cdot 11\cdot 11\cdot 11\cdot 11\cdot 11}) = 15,$$
$$\Omega_3\Big(7^{5^{7^5}\cdot 19}\cdot 19^{13^{7\cdot 11^3}}\Big) = \Omega_3\big(7^{5^{7\cdot 7\cdot 7\cdot 7\cdot 7}\cdot 19}\cdot 19^{13^{7\cdot 11\cdot 11\cdot 11}}\big) = 14.$$

Gillman [1990] proved that if f is i-multiplicative then f is j-multiplicative for all $j \geq i$. This is because if there are no primes in common in the first j levels of m and n, then there will clearly be no primes in common in the first i levels of m and n when $j \geq i$. This property is used to prove the following theorem.

Theorem 2.1. *For all i, Ω_i is j-additive for all j.*

Proof. Let m and n be 1-relatively prime. Then $\Omega_i(mn)$ is summing the number of prime divisors of the product mn and the number of primes in levels two though i of the mosaic of mn, including multiplicities at the i-th level. Since m and n are 1-relatively prime, the first term of this sum can be written as the number of prime divisors of m plus the number of prime divisors of n. Similarly, the second term can be written as the number of primes in levels two through i of the mosaic of m plus the number of primes in levels two through i of the mosaic of n, including multiplicities at the i-th level in each of these sums. Rearranging these sums results in $\Omega_i(m)+\Omega_i(n)$. Thus Ω_i is 1-additive and therefore j-additive for all j. □

The following two examples illustrate that it is necessary and sufficient that the first level of the mosaics have distinct primes in order that Ω_i be i-additive, as suggested by the previous proof:

$$\Omega_2\big(2^{3^5}\cdot 3^3\big) = 8 = 6+2 = \Omega_2\big(2^{3^5}\big)+\Omega_2(3^3),$$
$$\Omega_2\big(3^{3^5}\cdot 3^3\big) = \Omega_2(3^{2\cdot 3\cdot 41}) = 4 \neq 8 = 6+2 = \Omega_2\big(3^{3^5}\big)+\Omega_2(3^3).$$

Similarly, $\omega(n)$, the number of distinct primes in the prime factorization of n, can be generalized as $\omega_i(n)$, the number of *distinct* primes in the first i levels of the mosaic of n.

Since

$$\omega_4\big(7^{11\cdot 13^{5^{3^2}}}\cdot 29^{17^{89^2}}\big) = 9 = 5+4 = \omega_4\big(7^{11\cdot 13^{5^{3^2}}}\big)+\omega_4\big(29^{17^{89^2}}\big)$$

and

$$\omega_4\big(11^{5\cdot 11}\cdot 7^{19^3\cdot 53^{5^7}}\big) = 6 \neq 2+5 = \omega_4(11^{5\cdot 11})+\omega_4\big(7^{19^3\cdot 53^{5^7}}\big)$$

as counterexamples, ω_i is not 1-additive and therefore not j-additive for all j. Rather, as the following theorem demonstrates, it is j-additive for $j \geq i$.

Theorem 2.2. *For all i, ω_i is j-additive for $j \geq i$.*

Proof. Let m and n be i-relatively prime. Then $\omega_i(mn)$ is summing the number of prime divisors of the product mn and the distinct primes in levels two though i of the mosaic of mn (which must also be distinct from the prime divisors). Since m and n are relatively prime, the first term of this sum can be written as the prime divisors of m plus the prime divisors of n. Similarly, the second term can be written as the number of distinct primes in levels two through i of the mosaic of m plus the number of distinct primes in levels two through i of the mosaic of n. Thus $\omega_i(mn) = \omega_i(m) + \omega_i(n)$ and therefore ω_i is i-additive. This implies that ω_i is j-additive for $j \geq i$. □

***The function* $\boldsymbol{\lambda_i}$.** The Liouville function, $\lambda(n) = (-1)^{\Omega(n)}$, also generalizes easily in the obvious way as $\lambda_i(n) = (-1)^{\Omega_i(n)}$. This leads to the following theorem, again recalling that i-multiplicative implies j-multiplicative for $j \geq i$.

Theorem 2.3. *For all i, λ_i is j-multiplicative for all $j \geq i$.*

Proof. Assume m and n are 1-relatively prime. It follows that

$$\begin{aligned}\lambda_i(mn) &= (-1)^{\Omega_i(mn)} = (-1)^{\Omega_i(m)+\Omega_i(n)} \\ &= (-1)^{\Omega_i(m)}(-1)^{\Omega_i(n)} = \lambda_i(m)\lambda_i(n).\end{aligned}$$

λ_i is 1-multiplicative and therefore j-multiplicative for all j. □

***The function* $\boldsymbol{\psi^*_{j,i}}$.** Mullin defined the function ψ as the product of all primes in a mosaic. Gillman later extended this to the levels of the mosaic by introducing the function $\psi_{j,i}$. Mullin also defined the function ψ^* as the sum of the primes in a mosaic. To generalize this idea to the levels of the mosaic as well, we define the function $\psi^*_{j,i}$.

Definition 2.4. For fixed i and j, such that $j \geq i$, compute $\psi^*_{j,i}(n)$ as follows: Expand n through the first j levels of its mosaic; for each prime p on the i-th level of this expansion, add p to the sum of the primes in the $(i+1)$-th through j-th levels above p, including the multiplicities of the primes at the j-th level, then convert multiplication on the i-th level to addition.

Again, two examples help to illustrate this computation:

$$\begin{aligned}\psi^*_{4,2}\big(17^{11^{3^7}\cdot 19} \cdot 23^{2^{3^5}}\big) &= 17^{11+3+7+19} \cdot 23^{2+3+5}, \\ \psi^*_{4,1}\big(3^{5^{2^3}\cdot 7}\big) &= 3+5+2+3+7 = 20.\end{aligned}$$

Similar to the previous functions, $\psi^*_{j,1}$ is 1-additive, and therefore:

Theorem 2.5. *$\psi^*_{j,1}(n)$ is k-additive for all j and k.*

Proof. Let m and n be integers which are 1-relatively prime. $\psi^*_{j,1}(mn)$ is the sum of primes in the first j levels of the mosaic of mn, including multiplicities on the j-th level. This is equivalent to the sum of prime divisors of mn plus the sum of the primes in levels two through j of the mosaic of mn including multiplicities at the j-th level. Since m and n are relatively prime, the sum of prime divisors of mn can be written as the sum of prime divisors of m plus the sum of prime divisors of n. Similarly, the second term can be written as the sum of primes in levels two through j of the mosaic of m plus the number of primes in levels two through j of the mosaic of n including multiplicities at the j-th level in each. Rearranging these sums results in $\psi^*_{j,1}(m)+\psi^*_{j,1}(n)$. Thus $\psi^*_{j,1}$ is 1-additive and therefore k-additive for all k. □

Interestingly, while $\psi^*_{j,i}$ is k-additive for all j and k when $i=1$, for any $i>1$, $\psi^*_{j,i}$ is 1-multiplicative and therefore k-multiplicative for all j and k.

Theorem 2.6. *For all $i>1$ and $j\geq i$, $\psi^*_{j,i}(n)$ is k-multiplicative for all k.*

Proof. Let m and n be 1-relatively prime integers with prime factorizations

$$p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r} \quad \text{and} \quad q_1^{\beta_1}q_2^{\beta_2}\cdots q_s^{\beta_s},$$

respectively. Because m and n are 1-relatively prime,

$$mn = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}q_1^{\beta_1}q_2^{\beta_2}\cdots q_s^{\beta_s}.$$

Further, because $i>1$, $\psi^*_{j,i}(mn)$ has the same first $(i-1)$ levels as the mosaic of mn and the i-th level is equal to the i-th level of $\psi_{j,i}(mn)$ with multiplication converted to addition. Thus the unchanged first level can be partitioned into the parts that have the same first $(i-1)$ levels as m and n and with the i-th levels equal to the i-th levels of $\psi_{j,i}(m)$ and $\psi_{j,i}(n)$ respectively with multiplication converted to addition. That is,

$$\begin{aligned}\psi^*_{j,i}(mn) &= p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}q_1^{b_1}q_2^{b_2}\cdots q_s^{b_s}\\ &= (p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r})(q_1^{b_1}q_2^{b_2}\cdots q_s^{b_s})\\ &= \psi^*_{j,i}(m)\psi^*_{j,i}(n),\end{aligned}$$

where a_* and b_* are the second through i-th levels of the mosaic, with the $(i+1)$-th to j-th levels brought down to the i-th level with multiplication converted to addition. Thus $\psi^*_{j,i}(mn)$ is 1-multiplicative and therefore k-multiplicative for all k. □

3. Early attempts at mosaic divisors

Many number theoretic functions are defined in terms of the divisors of n, so an analogous concept is needed for the mosaic. In this section, we examine two early attempts at this.

Submosaics. A mosaic can be viewed as a connected graph where the primes in the mosaic are the vertices and there is an edge between vertices if one prime is multiplied by the other or one is an exponent of the other. Mullin [1965] introduced the concept of submosaics as the mosaic corresponding to a connected subgraph of the graph of the full mosaic. Therefore, submosaics seems like a natural candidate for a mosaic divisor.

Mullin tried to show that functions of the form

$$F(n) = \sum f(d),$$

where the sum is over the set of submosaics of n, are generalized multiplicative when f is generalized multiplicative. Unfortunately, this is not true. If we let $C(n)$ be the set of all submosaics of n, Mullin assumed that $C(mn) = C(m) \times C(n)$, but this is not true as shown in the example:

$$2 \in C(13^2), \quad 17 \in C(17^5), \quad 2 \cdot 17 \notin C(13^2 \cdot 17^5).$$

Givisors. The givisor, from Gillman's divisor, was Gillman's attempt to generalize the concept of a divisor for mosaics. We examine this concept and its implications in this subsection and the next.

Definition 3.1. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Each $p_i^{\alpha_i}$ is a *prime* givisor. Then a givisor of n is

(a) 1,

(b) a prime givisor, or

(c) a product of 1-relatively prime givisors from part (b).

We denote the set of all givisors of n by $G(n)$ and, as an example, consider

$$n = 2^{3^5} \cdot 3^{5^{17}} \cdot 5.$$

The givisors of n are

$$G\big(2^{3^5} \cdot 3^{5^{17}} \cdot 5\big) = \big\{1,\ 2^{3^5},\ 3^{5^{17}},\ 5,\ 2^{3^5} \cdot 3^{5^{17}},\ 2^{3^5} \cdot 5,\ 3^{5^{17}} \cdot 5,\ 2^{3^5} \cdot 3^{5^{17}} \cdot 5\big\}.$$

Gillman selected this structure because the mosaic above each prime in the first level is fixed, and the structure of the remaining mosaic does not change when a mosaic is divided by a givisor; we are simply splitting the mosaic into two parts.

In particular, givisors solve the fundamental problem that submosaics have, as we see in the following lemma.

Lemma 3.2. *For all i, $G(mn) = G(m) \times G(n)$ when m and n are i-relatively prime.*

Proof. Let the prime-power factorizations of m and n be

$$p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} \quad \text{and} \quad q_1^{b_1} q_2^{b_2} \cdots q_t^{b_t},$$

respectively. Since m and n are i-relatively prime, the set of primes in the first level of m and the set of primes in the first level of n have no common elements. Therefore, the prime-power factorization of mn is

$$mn = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} q_1^{b_1} q_2^{b_2} \cdots q_t^{b_t}.$$

If $d \in G(mn)$, then

$$d = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} q_1^{f_1} q_2^{f_2} \cdots q_t^{f_t}$$

where e_i is either 0 or a_i for $i = 1, 2, \ldots, s$ and f_j is either 0 or b_j for $j = 1, 2, \ldots, t$. Now let

$$d_1 = \gcd(d, m) \quad \text{and} \quad d_2 = \gcd(d, n).$$

Then

$$d_1 = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} \quad \text{and} \quad d_2 = q_1^{f_1} q_2^{f_2} \cdots q_t^{f_t}.$$

It follows that $d_1 \in G(m)$ and $d_2 \in G(n)$. Since $d = d_1 d_2$, $d \in G(m) \times G(n)$

Similarly, if $d_1 \in G(m)$ and $d_2 \in G(n)$, then $d_1 d_2 \in G(m) \times G(n)$ and $d_1 d_2 \in G(mn)$. Thus the sets are the same. □

Theorem 3.3. *If f is an i-multiplicative function, then*

$$F(n) = \sum_{d \in G(n)} f(d)$$

is also i-multiplicative.

Proof. To show that F is an i-multiplicative function, we must show that when m and n are i-relatively prime, $F(mn) = F(m)F(n)$. So assume that m and n are i-relatively prime. We know

$$F(mn) = \sum_{d \in G(mn)} f(d).$$

By the lemma, each givisor of mn can be written as the product $d = d_1 d_2$ where $d_1 \in G(m)$ and $d_2 \in G(n)$, and d_1 and d_2 are i-relatively prime. So

$$F(mn) = \sum_{d_1 \in G(m), d_2 \in G(n)} f(d_1 d_2).$$

Because f is i-multiplicative, and d_1 and d_2 are i-relatively prime, we see that

$$F(mn) = \sum_{d_1 \in G(m)} \sum_{d_2 \in G(n)} f(d_1) f(d_2) = \sum_{d_1 \in G(m)} f(d_1) \sum_{d_2 \in G(n)} f(d_2) = F(m)F(n). \quad \square$$

Functions defined by givisors. Givisors provide a mechanism for generalizing functions dependent on the concept of a divisor, and in this subsection we generalize three of these: τ — the number of divisors of n, σ — the sum of the positive divisors of n, and ϕ — the number of integers less than n relatively prime to n.

The function ${}_g\tau(n)$ counts the number of givisors of n, and hence can be computed by the formula

$${}_g\tau(n) = \sum_{d \in G(n)} 1.$$

The value of ${}_g\tau$ will always be a power of two with the exponent equal to the number of prime divisors of n. Using Theorem 3.3 with $f(d) = 1$, which is obviously i-multiplicative, we obtain:

Corollary 3.4. *${}_g\tau$ is i-multiplicative for all i.*

Similarly, we define ${}_g\sigma(n)$ as the sum of the givisors of n, and compute it using the formula

$${}_g\sigma(n) = \sum_{d \in G(n)} d.$$

Again using Theorem 3.3, except with $f(d) = d$, which is also i-multiplicative, we have:

Corollary 3.5. *${}_g\sigma$ is i-multiplicative for all i.*

We can generalize the concept of the number theoretic function $\phi(n)$, the number of integers less than n relatively prime to n, but the canonical formula for computing this,

$$\phi(n) = \sum_{d} \mu(d) \frac{n}{d},$$

does not generalize with it. Thus, while we can define $\phi_i(n)$ as the number of integers less than n that are i-relatively prime to n, it is not computed by the obvious generalization of the ϕ function, as given here:

$${}_gh_i(n) = \sum_{d \in G(n)} \mu_i(d) \frac{n}{d}.$$

By letting $n = 2^3$, we can compute ${}_gh_2(n) = 9$, but also determine, by listing the integers, that the number of integers 2-relatively prime to 8 is only 3. In spite of this significant disappointment, we have:

Theorem 3.6. *${}_gh_i$ is i-multiplicative for all i.*

Proof.

$$\begin{aligned}{}_g h_i(mn) &= \sum_{d\in G(mn)} \mu_i(d)\frac{mn}{d} = \sum_{d\in G(m)\times G(n)} \mu_i(d)\frac{mn}{d}\\ &= \sum_{d_1\in G(m)}\sum_{d_2\in G(n)} \mu_i(d_1)\mu_i(d_2)\frac{m}{d_1}\frac{n}{d_2}\\ &= \sum_{d_1\in G(m)} \mu_i(d_1)\frac{m}{d_1}\sum_{d_2\in G(n)} \mu_i(d_2)\frac{n}{d_2} = {}_g h_i(m)\,{}_g h_i(n).\end{aligned}$$

□

It is worth noticing that givisors are defined independently of the levels of the mosaic; that is, each positive integer n has the same set of givisors no matter which level i that we consider. Thus, the values of ${}_g\tau$ and ${}_g\sigma$ do not change as i varies, and ${}_g h_i$ only varies with i because μ_i changes as i varies.

To make these functions more dependent on the level i of the mosaic being considered, we might compose them with $\psi_{i,1}$, and examine functions of the form

$${}_g f_i(n) = {}_g f \circ \psi_{i,1}(n).$$

These functions are also i-multiplicative if ${}_g f$ is, and do vary in value with the choice of i.

If the integer n is squarefree, then

$${}_g h_i \circ \psi_{i,1}(n) = {}_g h_i(n) = \phi(n).$$

Further, the function

$${}_g\tau \circ \psi_{i,1}(n)$$

will always result in a power of two, but in this case the exponent is equal to the number of *distinct* primes in the first i levels of n. Hence, another formula is

$${}_g\tau \circ \psi_{i,1}(n) = 2^{\omega_i(n)}.$$

4. Mivisors

Neither submosaics nor givisors capture the properties of divisors that are desired. Submosaics do not effectively partition mosaics, and givisors are not sensitive to the parameter i representing the levels of the mosaic. With these two concerns in mind, we turn our investigation to a more promising generalization.

Definition of a mivisor. We define *prime* i-mivisors, mosaic divisor, as follows.

Definition 4.1. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. For each p_j, a *prime i-mivisor of n* is $p_j^{\alpha_j}$ expanded through i levels with multiplicities above the i-th level truncated, denoted $P_{j,i}$.

Let

$$P_{j,i}^{(a_{j,1},a_{j,2},\cdots,a_{j,s_j})}$$

denote $p_j^{\alpha_j}$ expanded through i levels including the multiplicities $a_{j,1}, a_{j,2}, \ldots, a_{j,s_j}$ on the $(i+1)$-th level. For example, if

$$n = 2^{2^4 3^7} \cdot 5^{11},$$

then

$$P_{1,2} = 2^{2\cdot 3} \quad \text{and} \quad P_{1,2}^{(a_{1,1},a_{1,2},\cdots,a_{1,s_1})} = P_{1,2}^{(4,7)} = 2^{2^4 3^7}.$$

With this notation, we obtain the following definition.

Definition 4.2. If $n = \prod_{j=1}^{k} P_{j,i}^{(a_{j,1},a_{j,2},\cdots,a_{j,s_j})}$, then an i-mivisor of n is

(a) 1,
(b) $P_{j,i}^{(b_{j,1},b_{j,2},\cdots,b_{j,s_j})}$ where $1 \le b_j \le a_j$, or
(c) a product of 1-relatively prime i-mivisors from part (b).

We denote the set of all i-mivisors of n by $M_i(n)$ and, as an example, consider

$$n = 2^{3^{5^3\cdot 7}\cdot 5} \cdot 3^{7^{11^2}}.$$

The prime 3-mivisors of n are $2^{3^{5\cdot 7}\cdot 5}$ and $3^{7^{11}}$, and the set $M_3(n)$ is

$$\Big\{1,\ 2^{3^{5\cdot7}\cdot5},\ 2^{3^{5^2\cdot7}\cdot5},\ 2^{3^{5^3\cdot7}\cdot5},\ 3^{7^{11}},\ 3^{7^{11^2}},\ 2^{3^{5\cdot7}\cdot5}\cdot3^{7^{11}},\ 2^{3^{5^2\cdot7}\cdot5}\cdot3^{7^{11}},$$
$$2^{3^{5^3\cdot7}\cdot5}\cdot3^{7^{11}},\ 2^{3^{5\cdot7}\cdot5}\cdot3^{7^{11^2}},\ 2^{3^{5^2\cdot7}\cdot5}\cdot3^{7^{11^2}},\ 2^{3^{5^3\cdot7}\cdot5}\cdot3^{7^{11^2}}\Big\}.$$

We immediately have the following lemma.

Lemma 4.3. *For all i, $M_i(mn) = M_i(m) \times M_i(n)$.*

Proof. Let m and n be i-relatively prime integers such that $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ and $n = q_1^{\beta_1} q_2^{\beta_2} \cdots q_u^{\beta_u}$. After applying the FTA to generate i levels of the mosaics of m and n, let

$$m = \prod_{j=1}^{t} P_{j,i}^{(a_{j,1},a_{j,2},\cdots,a_{j,r_j})} \quad \text{and} \quad n = \prod_{j=1}^{u} Q_{j,i}^{(b_{j,1},b_{j,2},\cdots,b_{j,s_j})}.$$

Since m and n are i-relatively prime,

$$mn = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t} q_1^{\beta_1} q_2^{\beta_2} \cdots q_u^{\beta_u} = \prod_{j=1}^{t} P_{j,i}^{(a_{j,1},a_{j,2},\cdots,a_{j,r_j})} \prod_{j=1}^{u} Q_{j,i}^{(b_{j,1},b_{j,2},\cdots,b_{j,s_j})}.$$

If d is an i-mivisor of mn, then

$$d = \prod_{k=1}^{v} P_{k,i}^{\prime(c_{k,1},c_{k,2},\cdots,c_{k,r_k})} \prod_{k=1}^{w} Q_{k,i}^{\prime(e_{k,1},e_{k,2},\cdots,e_{k,s_k})},$$

where for each k, $P'_{k,i} = P_{j,i}$ for some j and $1 \le c_{k,\ell} \le a_{j,\ell}$ and $Q'_{k,i} = Q_{j,i}$ for some j and $1 \le e_{k,\ell} \le b_{j,\ell}$. Let d_1 be an i-mivisor of m such that

$$d_1 = \prod_{k=1}^{v} P_{k,i}^{\prime(c_{k,1},c_{k,2},\cdots,c_{k,r_k})}.$$

Let d_2 be an i-mivisor of n such that

$$d_2 = \prod_{k=1}^{w} Q_{k,i}^{\prime(e_{k,1},e_{k,2},\cdots,e_{k,s_k})}.$$

It follows that $d_1 \in M_i(m)$ and $d_2 \in M_i(n)$. Then d_1 and d_2 are i-relatively prime and $d = d_1 d_2$, so $d \in M_i(m) \times M_i(n)$.

Similarly, if $d_1 \in M_i(m)$ and $d_2 \in M_i(n)$, then $d_1 d_2 \in M_i(m) \times M_i(n)$ and $d_1 d_2 \in M_i(mn)$. Therefore the sets are the same. □

Theorem 4.4. *If f is an i-multiplicative function, then*

$$F(n) = \sum_{d \in M_i(n)} f(d)$$

is also i-multiplicative.

Proof. Similar to Theorem 3.3. □

The functions $_m\tau_i$ and $_m\sigma_i$. Similar to previous section, we let $_m\tau_i$ count the number of i-mivisors of n, and it is therefore computed as

$$_m\tau_i(n) = \sum_{d \in M_i(n)} 1.$$

By Theorem 4.4, we find

Corollary 4.5. *For all i, $_m\tau_i$ is i-multiplicative.*

$_m\tau_i(n)$ can be computed easily, as we see in the following lemma and theorem.

Lemma 4.6. *Let p be a prime and α be a positive integer. Then*

$$_m\tau_i(p^\alpha) = 1 + \prod a_j,$$

where a_j is an element of the unfactored $(i+1)$ level of $p^\alpha = P_{\cdot,i}^{(a_1,a_2,\ldots,a_k)}$.

Proof. 1 is an i-mivisor of p^α and so is $P_{\cdot,i}^{(b_1,b_2,\ldots,b_k)}$ where $1 \le b_j \le a_j$ for all j. Since there are $\prod a_j$ ways to select the set $\{b_1, b_2, \ldots, b_j\}$, there are $1 + \prod a_j$ i-mivisors of p^α. □

Theorem 4.7. *Let n have the prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$. Then*

$$_m\tau_i(n) = \prod_{k=1}^{s} \Bigl(1 + \prod a_j\Bigr),$$

where a_j is an element of the unfactored $(i+1)$ level of $p_k^{\alpha_k}$.

Proof. Because $_m\tau_i$ is i-multiplicative for all i, we see that

$$_m\tau_i(n) = {_m\tau_i}(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}) = {_m\tau_i}(p_1^{\alpha_1}) {_m\tau_i}(p_2^{\alpha_2}) \cdots {_m\tau_i}(p_s^{\alpha_s}).$$

Inserting the values from Lemma 4.6, we see that

$$_m\tau_i(n) = \prod_{k=1}^{s} \Bigl(1 + \prod a_j\Bigr). \qquad \square$$

Moving forward, we let the function $_m\sigma_i(n)$ sums the i-mivisors of n,

$$_m\sigma_i(n) = \sum_{d \in M_i(n)} d.$$

By using Theorem 4.4 again, we obtain:

Corollary 4.8. *$_m\sigma_i$ is i-multiplicative.*

The function $_m\phi_i$. Using i-mivisors, we can generalize the concept of a common i-mivisor of two integers in the obvious way and more importantly, generalize the notion of a greatest common divisor.

Definition 4.9. A mosaic d is the *greatest common i-mivisor of m and n*, when at least one of them is not 0, if all of these conditions are satisfied:

(a) d is positive;

(b) d is an i-mivisor of a and b;

(c) if c is an i-mivisor of a and b, then c is an i-mivisor of d.

We write the greatest common i-mivisor of m and n as $\mathrm{GCM}_i(m,n)$, and have the following examples:

$$\mathrm{GCM}_3\bigl(2^{3^5},\ 2 \cdot 5^{3^2}\bigr) = 1, \qquad \mathrm{GCM}_3\bigl(2^{3^{5^{20}}} \cdot 3^{7^{11}},\ 2^{3^{5^{10}}} \cdot 7\bigr) = 2^{3^{5^{10}}}.$$

Finally, we say that two integers m and n are GCM_i relatively prime if and only if $\mathrm{GCM}_i(m,n) = 1$.

We can now generalize the ϕ function in a natural way, by letting $_m\phi_i(n)$ be the number of integers GCM_i relatively prime to n that are less than or equal to n. Notice that this is very different from the function ϕ_i, which counts the number of integers less than n that are i-relatively prime to n. The latter function only detects and responds to the presence of primes in the mosaic, whereas the former function is sensitive to both the presence and configuration of the primes in the array. Thus, 2^3 is not 2-relatively prime to 3^2, but they are GCM_2 relatively prime.

Unfortunately, the obvious generalization of the summation formula for ϕ does not compute ${}_m\phi_i$ and, worse still, ${}_m\phi_i(n)$ is *not* an i-multiplicative function, as we see in this final example:

$$ {}_m\phi_2(2) = 1, \quad {}_m\phi_2(3) = 2, \quad {}_m\phi_2(2 \cdot 3) = 3 \neq {}_m\phi_2(2) \cdot {}_m\phi_2(3). $$

5. Conclusion

In conclusion, we have generalized several number theoretic functions in terms of the levels of the mosaic and explored their properties, building on the work of Mullin and Gillman. Further, we refined the notion of a divisor for mosaics so that we could begin to look at a broader class of number theoretic functions and to develop an arithmetic for mosaics. However, there are still significant open problems, and the first among these is to look for ways to compute ϕ_i and ${}_m\phi_i$.

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