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# Equidissections of kite-shaped quadrilaterals

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Let  $Q(a)$  be the convex kite-shaped quadrilateral with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ , and  $(a, a)$ , where  $a > 1/2$ . We wish to dissect  $Q(a)$  into triangles of equal areas. What numbers of triangles are possible? Since  $Q(a)$  is symmetric about the line  $y = x$ ,  $Q(a)$  admits such a dissection into any even number of triangles. In this article, we prove four results describing  $Q(a)$  that can be dissected into certain odd numbers of triangles.

## 1. Introduction

We wish to dissect a convex polygon  $K$  into triangles of equal areas. A dissection of  $K$  into  $m$  triangles of equal areas is called an  $m$ -*equidissection*. The *spectrum* of  $K$ , denoted  $S(K)$ , is the set of integers  $m$  for which  $K$  has an  $m$ -equidissection. Note that if  $m$  is in  $S(K)$ , then so is  $km$  for all  $k > 0$ . If  $S(K)$  consists of precisely the positive multiples of  $m$ , we write  $S(K) = \langle m \rangle$  and call  $S(K)$  *principal*.

Quite a bit is known about the spectrum of the trapezoid  $T(a)$  with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ , and  $(a, 1)$  for  $a > 0$ . For example, if  $a$  is rational with  $a = r/s$ , where  $r$  and  $s$  are relatively prime positive integers, then  $S(T(a)) = \langle r + s \rangle$ ; if  $a$  is transcendental, then  $S(T(a))$  is the empty set. See [Kasimatis and Stein 1990] or [Stein and Szabó 1994]. In addition,  $S(T(a))$  is known for many irrational algebraic numbers  $a$ , particularly  $a$  satisfying a quadratic polynomial. See [Jepsen 1996; Jepsen and Monsky 2008; Monsky 1996]. For instance, if  $a = (2r - 1) + r\sqrt{3}$  where  $r$  is an integer  $\geq 8$ , then  $S(T(a)) = \{4r, 5r, 6r, \dots\}$ .

Less is known about the spectrum of the kite-shaped quadrilateral  $Q(a)$  with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ ,  $(a, a)$  for  $a > 1/2$ . Here certainly  $S(Q(a))$  contains 2 and hence all even positive integers. If  $a = 1$ , then  $Q(a)$  is a square, and in this case  $S(Q(a)) = \langle 2 \rangle$ . See [Monsky 1970].) For other values of  $a$ , the question is, What odd numbers, if any, are in  $S(Q(a))$ ? In Section 2, we prove four theorems that answer this question for certain  $a$ . In Section 3, we pose some questions that remain open.

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## 2. Main results

As in the introduction,  $Q(a)$  denotes the quadrilateral with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ , and  $(a, a)$  for  $a > 1/2$ . The following two results about  $Q(a)$  are shown in [Kasimatis and Stein 1990, pages 290 and 291]:

- (i) Let  $\phi_2$  be an extension to  $\mathbb{R}$  of the 2-adic valuation on  $\mathbb{Q}$ . (See [Stein and Szabó 1994] for a discussion of valuations.) If  $\phi_2(a) > -1$ , then  $S(Q(a)) = \langle 2 \rangle$ . In particular, if  $a$  is transcendental, then  $S(Q(a)) = \langle 2 \rangle$ .
- (ii) Let  $a > 1/2$  be a rational number such that  $\phi_2(a) \leq -1$ . That is,  $a = r/(2s)$ , where  $r$  and  $s$  are relatively prime positive integers,  $r$  is odd, and  $r > s$ . Then  $S(Q(a))$  contains all odd integers of the form  $r + 2sk$  for  $k \geq 0$ .

[Kasimatis and Stein 1990] and [Stein and Szabó 1994] raise two questions:

- Are there rational numbers  $a$  with  $\phi_2(a) \leq -1$  for which  $S(Q(a))$  contains odd numbers less than  $r$ ?
- Are there irrational algebraic numbers  $a$  with  $\phi_2(a) \leq -1$  for which  $S(Q(a))$  contains odd numbers? In particular, does  $S(Q(\sqrt{3}/2))$  contain odd numbers?

We answer these questions in the affirmative. First we present a slight strengthening of statement (ii) above.

**Theorem 1.** *Let  $a = r/(2s)$ , where  $r$  and  $s$  are relatively prime positive integers,  $r$  is odd, and  $r > s$ . Then  $S(Q(a))$  contains all integers of the form  $r + 2k$  for  $k \geq 0$ .*

*Proof.* Partition  $Q(a)$  into three triangles as in Figure 1, left. We want to find nonnegative integers  $t_1, t_2, t_3$  so that the areas  $A_1, A_2, A_3$  of the three triangles satisfy

$$A_1 t = at_1, \quad A_2 t = at_2, \quad A_3 t = at_3, \quad (1)$$

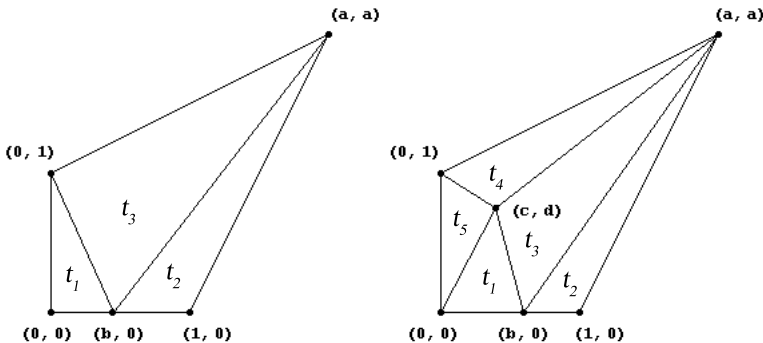


Figure 1

where  $t = t_1 + t_2 + t_3$ . (Note that the area of  $Q(a)$  is  $a$ .) Then  $Q(a)$  can be further dissected into  $t$  triangles each of area  $a/t$ . Here  $A_1 = \frac{1}{2}b$ ,  $A_2 = \frac{1}{2}a(1-b)$ , and  $A_3 = \frac{1}{2}(a+ab-b)$ . For  $k \geq 0$ , choose  $t_1 = s$ ,  $t_2 = k$ ,  $t_3 = r - s + k$ , and  $b = r/(r+2k)$ . Then  $t = r+2k$ ,  $b = r/t$ , and equations (1) are satisfied. Thus  $r+2k \in S(Q(a))$ .  $\square$

**Theorem 2.** *Let  $a$  be as in Theorem 1, and suppose  $r$  is not a prime number. Then  $S(Q(a))$  contains odd numbers less than  $r$ .*

*Proof.* We know that  $S(Q(a)) = S(Q(a/(2a-1)))$  for any  $a$  [Kasimatis and Stein 1990, pages 284 and 285]. If  $a = r/(2s)$ , then  $a/(2a-1) = r/((2(r-s)))$ . So replacing  $s$  by  $r-s$  if necessary, we may assume  $s$  is odd. Partition  $Q(a)$  into five triangles as shown in Figure 1, right. We want the areas  $A_1, A_2, A_3, A_4, A_5$  of the triangles to satisfy

$$A_1t = at_1, \quad A_2t = at_2, \quad A_3t = at_3, \quad A_4t = at_4, \quad A_5t = at_5, \quad (2)$$

where  $t = t_1 + t_2 + t_3 + t_4 + t_5$ . In this case,  $A_1 = \frac{1}{2}bd$ ,  $A_2 = \frac{1}{2}a(1-b)$ ,  $A_5 = \frac{1}{2}c$ ,  $A_4 = \frac{1}{2}(c(a-1) - a(d-1))$ , and  $A_3 = \frac{1}{2}(d(a-b) - a(c-b))$ . Since  $r$  is an odd composite number, we can write  $r = r_1r_2$ , where  $3 \leq r_1 \leq r_2$ .

*Case (i):*  $s > r_2$ . Choose  $t_1 = 1$ ,  $t_2 = \frac{1}{2}(s-r_1)$ ,  $t_3 = \frac{1}{2}(r_1+r_2) - 1$ ,  $t_4 = \frac{1}{2}(s-r_2)$ ,  $t_5 = 0$ ,  $b = r_1/s$ ,  $c = 0$ , and  $d = r^2/s$ . Then  $t = s$ , and we check that equations (2) are satisfied. Then  $s \in S(Q(a))$  and  $s < r$ .

*Case (ii):*  $s < r_2$ . Choose  $t_1 = \frac{1}{2}(r_1-1)$ ,  $t_2 = \frac{1}{2}(r_1r_2-r_1-2s)$ ,  $t_3 = \frac{1}{2}(r_2+1)$ ,  $t_4 = 0$ , and  $t_5 = \frac{1}{2}(r-r_2-2s)$ . The assumption on  $s$  implies that the  $t_i$  are nonnegative, and their sum  $t$  is  $r-2s$ . Now let  $b = (t-2t_2)/t = r_1/t$ ,  $c = (2at_5)/t$ , and  $d = (2at_1)/(bt) = (2at_1)/r_1$ . Then  $s = tt_1 - r_1t_5$ , and again we check that equations (2) are satisfied. Thus  $r-2s \in S(Q(a))$  and  $r-2s < r$ .  $\square$

**Theorem 3.** *Let  $a = \sqrt{3}/2$ . Then 21 is in  $S(Q(a))$ .*

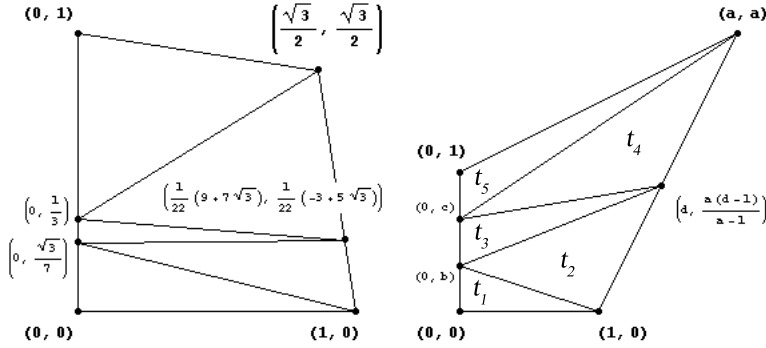
*Proof.* Partition  $Q(a)$  into five triangles shown in Figure 2, left. The areas of the five triangles are in the proportion

$$\frac{3}{14\sqrt{3}} : \frac{3}{14\sqrt{3}} : \frac{1}{14\sqrt{3}} : \frac{7}{14\sqrt{3}} : \frac{7}{14\sqrt{3}}$$

or  $3 : 3 : 1 : 7 : 7$ . Hence we can further dissect  $Q(a)$  into  $t = 3 + 3 + 1 + 7 + 7 = 21$  triangles each of area  $1/(14\sqrt{3}) = (1/21)(\sqrt{3}/2)$ .  $\square$

There are infinitely many radicals besides  $\sqrt{3}/2$  that have odd numbers in their spectra. For example, the next theorem says  $11 \in S(Q(\sqrt{5}/4))$ ,  $15 \in S(Q(\sqrt{21}/4))$ ,  $17 \in S(Q(\sqrt{33}/4))$ ,  $21 \in S(Q(\sqrt{65}/4))$ , and so forth.

**Theorem 4.** *For  $k \geq 1$ , let  $a = \sqrt{(2k+1)(2k+3)}/(4\sqrt{3})$ . Then  $2k+9$  lies in  $S(Q(a))$ .*



**Figure 2**

*Proof.* Partition  $Q(a)$  into five triangles as shown in Figure 2, right. As before, we want the areas  $A_i$  of the triangles to satisfy equations (2) above. Here  $A_1 = \frac{1}{2}b$ ,  $A_3 = \frac{1}{2}(c - b)d$ ,  $A_5 = \frac{1}{2}a(1 - c)$ ,

$$A_2 = \frac{1}{2} \left( \frac{d-1}{a-1} \right) (a + ab - b), \quad \text{and} \quad A_4 = \frac{1}{2} \left( \frac{a-d}{a-1} \right) (a + ac - c).$$

Choose  $t_1 = t_2 = t_3 = 2$ ,  $t_5 = 3$ , and  $t_4 = 2k$ , so  $t = 2k + 9$  and  $48a^2 = (t - 8)(t - 6)$ . Now let  $b = (4a)/t$ ,  $c = (t - 6)/t$ , and  $d = (4a)/(t - 6 - 4a)$ . We show once again that equations (2) are satisfied. Thus  $2k + 9 \in S(Q(a))$ .  $\square$

### 3. Open questions

While we have answered a few questions about odd numbers in  $S(Q(a))$ , many others remain:

- (i) Is the converse of Theorem 2 true? That is, if  $a$  is as in Theorem 1 and  $r$  is a prime number, is  $r$  the smallest odd number in  $S(Q(a))$ ?
- (ii) Let  $a$  be as in Theorem 2. What is the smallest odd number in  $S(Q(a))$ ? What are all the odd numbers in  $S(Q(a))$ ?
- (iii) Let  $a$  be an irrational algebraic number with  $\phi_2(a) \leq -1$ . Does  $S(Q(a))$  always contain odd numbers?
- (iv) Let  $a$  be arbitrary, and let  $m$  be an odd number. If  $m$  is in  $S(Q(a))$ , is  $m + 2$  in  $S(Q(a))$ ? (This is the same as, Is  $S(Q(a))$  closed under addition?)

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