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Equidissections of kite-shaped quadrilaterals

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Let $Q(a)$ be the convex kite-shaped quadrilateral with vertices $(0,0)$, $(1,0)$, $(0,1)$, and (a,a), where $a > 1/2$. We wish to dissect $Q(a)$ into triangles of equal areas. What numbers of triangles are possible? Since $Q(a)$ is symmetric about the line $y = x$, $Q(a)$ admits such a dissection into any even number of triangles. In this article, we prove four results describing $Q(a)$ that can be dissected into certain odd numbers of triangles.

1. Introduction

We wish to dissect a convex polygon K into triangles of equal areas. A dissection of K into m triangles of equal areas is called an *m-equidissection*. The *spectrum* of K, denoted $S(K)$, is the set of integers m for which K has an m-equidissection. Note that if m is in $S(K)$, then so is km for all $k > 0$. If $S(K)$ consists of precisely the positive multiples of m, we write $S(K) = \langle m \rangle$ and call $S(K)$ *principal*.

Quite a bit is known about the spectrum of the trapezoid $T(a)$ with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(a,1)$ for $a > 0$. For example, if a is rational with $a = r/s$, where r and s are relatively prime positive integers, then $S(T(a)) = \langle r+s \rangle$; if a is transcendental, then $S(T(a))$ is the empty set. See [Kasimatis and Stein 1990] or [Stein and Szabó 1994]. In addition, $S(T(a))$ is known for many irrational algebraic numbers a, particularly a satisfying a quadratic polynomial. See [Jepsen 1996; Jepsen and Monsky 2008; Monsky 1996]. For instance, if $a = (2r-1)+r\sqrt{3}$ where r is an integer ≥ 8, then $S(T(a)) = \{4r, 5r, 6r, \dots\}$.

Less is known about the spectrum of the kite-shaped quadrilateral $Q(a)$ with vertices $(0,0)$, $(1,0)$, $(0,1)$, (a,a) for $a > 1/2$. Here certainly $S(Q(a))$ contains 2 and hence all even positive integers. If $a = 1$, then $Q(a)$ is a square, and in this case $S(Q(a)) = \langle 2 \rangle$. See [Monsky 1970].) For other values of a, the question is, What odd numbers, if any, are in $S(Q(a))$? In Section 2, we prove four theorems that answer this question for certain a. In Section 3, we pose some questions that remain open.

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2. Main results

As in the introduction, $Q(a)$ denotes the quadrilateral with vertices $(0,0)$, $(1,0)$, $(0,1)$, and (a,a) for $a > 1/2$. The following two results about $Q(a)$ are shown in [Kasimatis and Stein 1990, pages 290 and 291]:

(i) Let ϕ_2 be an extension to $\mathbb{R}$ of the 2-adic valuation on $\mathbb{Q}$. (See [Stein and Szabó 1994] for a discussion of valuations.) If $\phi_2(a) > -1$, then $S(Q(a)) = \langle 2 \rangle$. In particular, if a is transcendental, then $S(Q(a)) = \langle 2 \rangle$.

(ii) Let $a > 1/2$ be a rational number such that $\phi_2(a) \leq -1$. That is, $a = r/(2s)$, where r and s are relatively prime positive integers, r is odd, and $r > s$. Then $S(Q(a))$ contains all odd integers of the form $r + 2sk$ for $k \geq 0$.

[Kasimatis and Stein 1990] and [Stein and Szabó 1994] raise two questions:

- Are there rational numbers a with $\phi_2(a) \leq -1$ for which $S(Q(a))$ contains odd numbers less than r?
- Are there irrational algebraic numbers a with $\phi_2(a) \leq -1$ for which $S(Q(a))$ contains odd numbers? In particular, does $S(Q(\sqrt{3}/2))$ contain odd numbers?

We answer these questions in the affirmative. First we present a slight strengthening of statement (ii) above.

Theorem 1. *Let $a = r/(2s)$, where r and s are relatively prime positive integers, r is odd, and $r > s$. Then $S(Q(a))$ contains all integers of the form $r + 2k$ for $k \geq 0$.*

Proof. Partition $Q(a)$ into three triangles as in Figure 1, left. We want to find nonnegative integers t_1, t_2, t_3 so that the areas A_1, A_2, A_3 of the three triangles satisfy

$$A_1 t = a t_1, \qquad A_2 t = a t_2, \qquad A_3 t = a t_3, \tag{1}$$

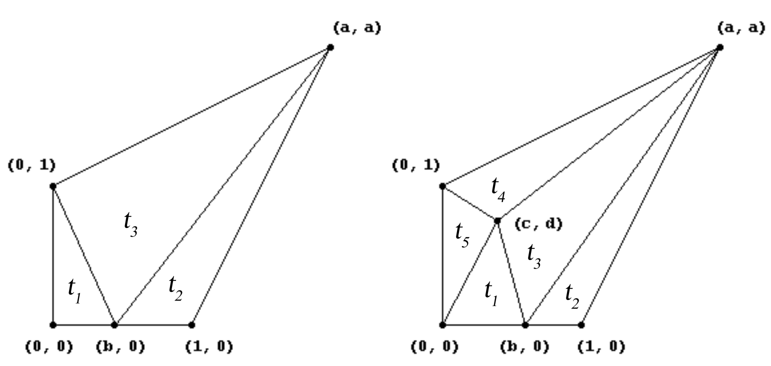

Figure 1

where $t = t_1 + t_2 + t_3$. (Note that the area of $Q(a)$ is a.) Then $Q(a)$ can be further dissected into t triangles each of area a/t. Here $A_1 = \frac{1}{2}b$, $A_2 = \frac{1}{2}a(1-b)$, and $A_3 = \frac{1}{2}(a+ab-b)$. For $k \geq 0$, choose $t_1 = s$, $t_2 = k$, $t_3 = r - s + k$, and $b = r/(r+2k)$. Then $t = r + 2k$, $b = r/t$, and equations (1) are satisfied. Thus $r + 2k \in S(Q(a))$. □

Theorem 2. *Let a be as in Theorem 1, and suppose r is not a prime number. Then $S(Q(a))$ contains odd numbers less than r.*

Proof. We know that $S(Q(a)) = S(Q(a/(2a-1)))$ for any a [Kasimatis and Stein 1990, pages 284 and 285]. If $a = r/(2s)$, then $a/(2a-1) = r/((2(r-s))$. So replacing s by $r-s$ if necessary, we may assume s is odd. Partition $Q(a)$ into five triangles as shown in Figure 1, right. We want the areas A_1, A_2, A_3, A_4, A_5 of the triangles to satisfy

$$A_1 t = at_1, \quad A_2 t = at_2, \quad A_3 t = at_3, \quad A_4 t = at_4, \quad A_5 t = at_5, \tag{2}$$

where $t = t_1 + t_2 + t_3 + t_4 + t_5$. In this case, $A_1 = \frac{1}{2}bd$, $A_2 = \frac{1}{2}a(1-b)$, $A_5 = \frac{1}{2}c$, $A_4 = \frac{1}{2}(c(a-1) - a(d-1))$, and $A_3 = \frac{1}{2}(d(a-b) - a(c-b))$. Since r is an odd composite number, we can write $r = r_1 r_2$, where $3 \leq r_1 \leq r_2$.

Case (i): $s > r_2$. Choose $t_1 = 1$, $t_2 = \frac{1}{2}(s - r_1)$, $t_3 = \frac{1}{2}(r_1 + r_2) - 1$, $t_4 = \frac{1}{2}(s - r_2)$, $t_5 = 0$, $b = r_1/s$, $c = 0$, and $d = r^2/s$. Then $t = s$, and we check that equations (2) are satisfied. Then $s \in S(Q(a))$ and $s < r$.

Case (ii): $s < r_2$. Choose $t_1 = \frac{1}{2}(r_1 - 1)$, $t_2 = \frac{1}{2}(r_1 r_2 - r_1 - 2s)$, $t_3 = \frac{1}{2}(r_2 + 1)$, $t_4 = 0$, and $t_5 = \frac{1}{2}(r - r_2 - 2s)$. The assumption on s implies that the t_i are nonnegative, and their sum t is $r - 2s$. Now let $b = (t - 2t_2)/t = r_1/t$, $c = (2at_5)/t$, and $d = (2at_1)/(bt) = (2at_1)/r_1$. Then $s = tt_1 - r_1 t_5$, and again we check that equations (2) are satisfied. Thus $r - 2s \in S(Q(a))$ and $r - 2s < r$. □

Theorem 3. *Let $a = \sqrt{3}/2$. Then 21 is in $S(Q(a))$.*

Proof. Partition $Q(a)$ into five triangles shown in Figure 2, left. The areas of the five triangles are in the proportion

$$\frac{3}{14\sqrt{3}} : \frac{3}{14\sqrt{3}} : \frac{1}{14\sqrt{3}} : \frac{7}{14\sqrt{3}} : \frac{7}{14\sqrt{3}}$$

or $3:3:1:7:7$. Hence we can further dissect $Q(a)$ into $t = 3+3+1+7+7 = 21$ triangles each of area $1/(14\sqrt{3}) = (1/21)(\sqrt{3}/2)$. □

There are infinitely many radicals besides $\sqrt{3}/2$ that have odd numbers in their spectra. For example, the next theorem says $11 \in S(Q(\sqrt{5}/4))$, $15 \in S(Q(\sqrt{21}/4)$, $17 \in S(Q(\sqrt{33}/4)$, $21 \in S(Q(\sqrt{65}/4)$, and so forth.

Theorem 4. *For $k \geq 1$, let $a = \sqrt{(2k+1)(2k+3)}/(4\sqrt{3})$. Then $2k+9$ lies in $S(Q(a))$.*

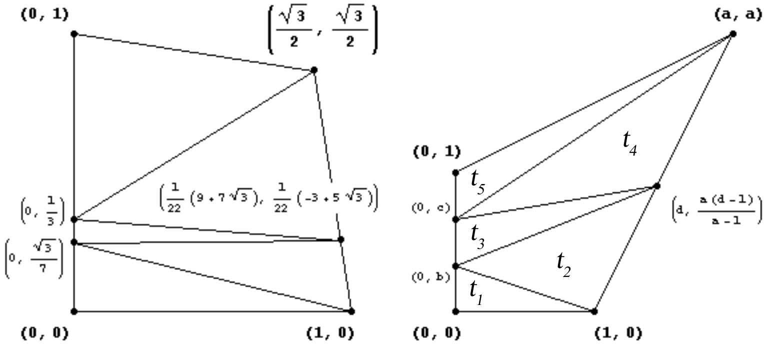

Figure 2

Proof. Partition $Q(a)$ into five triangles as shown in Figure 2, right. As before, we want the areas A_i of the triangles to satisfy equations (2) above. Here $A_1 = \frac{1}{2}b$, $A_3 = \frac{1}{2}(c-b)d$, $A_5 = \frac{1}{2}a(1-c)$,

$$A_2 = \frac{1}{2}\left(\frac{d-1}{a-1}\right)(a+ab-b), \quad \text{and} \quad A_4 = \frac{1}{2}\left(\frac{a-d}{a-1}\right)(a+ac-c).$$

Choose $t_1 = t_2 = t_3 = 2$, $t_5 = 3$, and $t_4 = 2k$, so $t = 2k+9$ and $48a^2 = (t-8)(t-6)$. Now let $b = (4a)/t$, $c = (t-6)/t$, and $d = (4a)/(t-6-4a)$. We show once again that equations (2) are satisfied. Thus $2k+9 \in S(Q(a))$. □

3. Open questions

While we have answered a few questions about odd numbers in $S(Q(a))$, many others remain:

(i) Is the converse of Theorem 2 true? That is, if a is as in Theorem 1 and r is a prime number, is r the smallest odd number in $S(Q(a))$?

(ii) Let a be as in Theorem 2. What is the smallest odd number in $S(Q(a))$? What are all the odd numbers in $S(Q(a))$?

(iii) Let a be an irrational algebraic number with $\phi_2(a) \leq -1$. Does $S(Q(a))$ always contain odd numbers?

(iv) Let a be arbitrary, and let m be an odd number. If m is in $S(Q(a))$, is $m+2$ in $S(Q(a))$? (This is the same as, Is $S(Q(a))$ closed under addition?)

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