involve
a journal of mathematics

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mathematical sciences publishers

2009 Vol. 2, No. 2

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(Communicated by Michael Dorff)

A Poincaré–Hopf theorem in the spirit of Pugh is proven for compact orbifolds with boundary. The theorem relates the index sum of a smooth vector field in generic contact with the boundary orbifold to the Euler–Satake characteristic of the orbifold and a boundary term. The boundary term is expressed as a sum of Euler characteristics of tangency and exit-region orbifolds. As a corollary, we express the index sum of the vector field induced on the inertia orbifold to the Euler characteristics of the associated underlying topological spaces.

1. Introduction

The Poincaré–Hopf Theorem states that if M is a smooth, compact n-manifold and X is a vector field on M that points outwards everywhere on ∂M, then $\mathfrak{Ind}(X)$, the index of X, is equal to the Euler characteristic $\chi(M)$ of M. Pugh [1968] gave a generalization of this theorem for such manifolds where the vector field X on M has generic contact with ∂M. This means that the subset Γ^1 of ∂M on which X is tangent to ∂M is a codimension-1 submanifold of ∂M, the subset Γ^2 of Γ^1 on which X is tangent to Γ^1 is a codimension-1 submanifold of Γ^1, etc. This generalization bears the elegance of associating the index sum with a sum of Euler characteristics only. Here we show that in the case of a compact orbifold with boundary and a smooth vector field in generic contact with the boundary, Pugh's result extends naturally. A proper introduction to orbifolds and the precise definition we use are available as an appendix in [Chen and Ruan 2002]. Note that this definition of an orbifold requires group actions to have fixed-point sets of codimension at least 2 as opposed to other definitions which do not (see, for example, [Thurston 1978]); we make this requirement as well. By *smooth*, we always mean $\mathscr{C}^\infty$.

MSC2000: 55R91, 57R12, 57R25.

Keywords: orbifold, orbifold with boundary, Euler–Satake characteristic, Poincare–Hopf theorem, vector field, vector field index, Morse index, orbifold double.

The first author was supported by a Kalamazoo College Field Experience Grant. The second author was supported by a Rhodes College Faculty Development Endowment Grant.

The main result we prove is as follows.

Theorem 1.1. *Let Q be an n-dimensional smooth, compact orbifold with boundary. Let Y be a smooth vector field on Q that is in* generic contact *with ∂Q, and then*

$$\mathfrak{Ind}^{\mathrm{orb}}(Y; Q) = \chi_{\mathrm{orb}}(Q, \partial Q) + \sum_{i=1}^{n} \chi_{\mathrm{orb}}(R^i_-, \Gamma^i). \tag{1-1}$$

The expressions $\mathfrak{Ind}^{\mathrm{orb}}$ and χ_{orb} are orbifold analogues of the manifold notions of the topological index of a vector field and the Euler characteristic, respectively. The definitions of both of these, along with R^i_-, Γ^i, and generic contact, are reviewed in Section 2.

In this paper, we follow a procedure resembling Pugh's original technique, and we show that many of the same techniques applicable to manifolds can be applied to orbifolds as well. In Section 2, we explain our notation and review the result of Satake which relates the orbifold index to the Euler–Satake characteristic for closed orbifolds. We give the definition of each of these terms. In Section 3, we show that a neighborhood of the boundary of an orbifold may be decomposed as a product $\partial Q \times [0, \epsilon)$. We then construct the double of Q and charts near the boundary respecting this product structure. This generalizes well-known results and constructions for manifolds with boundary. Section 4 provides elementary results relating the topological index of an orbifold vector field to an orbifold Morse Index. The orbifold Morse Index is defined in terms of the Morse Index on a manifold in a manner analogous to Satake's definition of the topological vector field index. These results generalize corresponding results for manifolds. In Section 5, we use the above constructions to show that a smooth vector field on Q may be perturbed near the boundary to form a smooth vector field on the double whose index can be computed in terms of the data given by the original vector field. We use this to prove Theorem 1.1. We also prove Corollary 5.2, which gives a similar formula where the left side is the orbifold index of the induced vector field on the inertia orbifold and on the right side, the Euler–Satake characteristics are replaced with the Euler characteristics of the underlying topological spaces.

Another generalization of the Poincaré–Hopf Theorem to orbifolds with boundary is explored in [Seaton 2008] and follows as a corollary to Satake's Gauss–Bonnet Theorem for orbifolds with boundary [Satake 1957]. In each of these cases, the boundary term is expressed by evaluation of an auxiliary differential form representing a global topological invariant of the boundary pulled back via the vector field. The generalization given in our paper expresses the boundary term in terms of Euler–Satake characteristics of suborbifolds determined by the vector field.

2. Preliminaries and definitions

Satake proved a Poincaré–Hopf Theorem for closed orbifolds; however, he worked with a slightly different definition of orbifold, the so-called V-manifold [Satake 1956; 1957]. A V-manifold corresponds to an *effective* or *reduced* (codimension-2) orbifold, an orbifold such that the group in each chart acts effectively [Chen and Ruan 2002]. That is, the only group element that acts trivially is the identity element. We adopt the language of his result and use it here.

Theorem 2.1 (Satake's Poincaré–Hopf Theorem for Closed Orbifolds). *Let Q be an effective, closed orbifold, and let X be a vector field on Q that has isolated zeros. Then*

$$\mathfrak{Ind}^{\mathrm{orb}}(X;\, Q) = \chi_{\mathrm{orb}}(Q).$$

Note that the requirement that Q is effective is unnecessary; as mentioned in [Chen and Ruan 2002], an ineffective orbifold can be replaced with an effective orbifold Q_{red}, and the differential geometry of the tangent bundle (or any other *good* orbifold vector bundle) is unchanged.

The *orbifold index* $\mathfrak{Ind}^{\mathrm{orb}}(X;\, p)$ at a zero p of the vector field X is defined in terms of the topological index of a vector field on a manifold. Let a neighborhood of p be uniformized by the chart $\{V, G, \pi\}$ and choose $x \in V$ with $\pi(x) = p$. Let $G_x \leq G$ denote the isotropy group of x. Then π^*X is a G-invariant vector field on V with a zero at x. The orbifold index at p is defined by

$$\mathfrak{Ind}^{\mathrm{orb}}(X;\, p) = \frac{1}{|G_x|}\mathfrak{Ind}\big(\pi^*X;\, x\big),$$

where $\mathfrak{Ind}\big(\pi^*X;\, x\big)$ is the usual topological index of the vector field π^*X on the manifold V at x [Guillemin and Pollack 1974; Milnor 1965]. Note that this definition does not depend on the chart, nor on the choice of x. We use the notation

$$\mathfrak{Ind}^{\mathrm{orb}}(X;\, Q) = \sum_{p \in Q,\, X(p)=0} \mathfrak{Ind}^{\mathrm{orb}}(X;\, p).$$

The *Euler–Satake characteristic* $\chi_{\mathrm{orb}}(Q)$ is most easily defined in terms of an appropriate simplicial decomposition of Q. In particular, let $\mathcal{T}$ be a simplicial decomposition of Q such that the isomorphism class of the isotropy group is constant on the interior of each simplex (such a simplicial decomposition always exists; see [Moerdijk and Pronk 1999]). For each simplex $\sigma \in \mathcal{T}$, the (isomorphism class of the) isotropy group on the interior of σ is denoted G_σ. The Euler–Satake characteristic of Q is then defined by

$$\chi_{\mathrm{orb}}(Q) = \sum_{\sigma \in \mathcal{T}} (-1)^{\dim \sigma} \frac{1}{|G_\sigma|}.$$

This coincides with Satake's *Euler characteristic of Q as a V-manifold*. Note that it follows from this definition that if $Q = Q_1 \cup Q_2$ for orbifolds Q_1 and Q_2 with $Q_1 \cap Q_2$ a suborbifold, then

$$\chi_{\text{orb}}(Q) = \chi_{\text{orb}}(Q_1) + \chi_{\text{orb}}(Q_2) - \chi_{\text{orb}}(Q_1 \cap Q_2). \tag{2-1}$$

In the case that Q has boundary, $\chi_{\text{orb}}(Q)$ is defined in the same way. We let

$$\chi_{\text{orb}}(Q, \partial Q) = \chi_{\text{orb}}(Q) - \chi_{\text{orb}}(\partial Q).$$

This coincides with Satake's *inner Euler characteristic of Q as a V-manifold with boundaries*. The reader is warned that there are many different Euler characteristics defined for orbifolds; both the topological index of a vector field and the Euler–Satake characteristic used here are generally rational numbers.

Vector fields in *generic contact* with the boundary have orbifold exit regions, which we now describe. Let Q be a compact n-dimensional orbifold with boundary and X a smooth vector field on Q. In Lemma 3.1, we show that, as with the case of manifolds, there is a neighborhood of ∂Q in Q diffeomorphic to $\partial Q \times [0, \epsilon)$. Given a metric, the tangent bundle of Q on the boundary decomposes with respect to this product so that there is a well-defined normal direction to the boundary. Let R_-^1 be the closure of the subset of ∂Q where X points out of Q. Analogously, let R_+^1 be the closure of the subset of ∂Q where X points into Q. We require that R_-^1 and R_+^1 are $(n-1)$-dimensional orbifolds with boundary. The subset of ∂Q where the vector field is tangent to ∂Q is denoted Γ^1; we require that Γ^1 be a suborbifold of ∂Q of codimension 1. Note that, by the continuity of X, the component of the vector field pointing outward must approach zero near the boundary of R_-^1 and R_+^1. Hence $\Gamma^1 = \partial R_-^1 = \partial R_+^1$.

The vector field X is tangent to Γ^1, and so it may be considered a vector field on the orbifold Γ^1. We again require this vector field to have orbifold exit regions. Call R_-^2 the closure of the subset of Γ^1 where the vector field points out of R_-^1, and R_+^2 the closure of the subset where it points into R_-^1. The subset of Γ^1 where the vector field is tangent to Γ^1 is denoted Γ^2, and is required to be a codimension-1 suborbifold of Γ^1.

In the same way, we define Γ^i, R_-^i, R_+^i, requiring that these sets form a chain of closed suborbifolds $\{\Gamma^i\}_{i=1}^n$ and compact orbifolds with boundary $\{R_-^i\}_{i=1}^n$. We require that $\dim R_-^i = \dim R_+^i = n-i$ and $\dim \Gamma^i = n-i-1$. Since each successive Γ^i has strictly smaller dimension, we eventually run out of space, and so both of these sequences terminate. The last entry in the sequence of Γ^i is Γ^n, which is necessarily the empty set.

3. Formation of the double orbifold

In the proof of Theorem 1.1, we pass from an orbifold with boundary to a closed orbifold in order to employ Theorem 2.1. In this section, we construct the double of an orbifold with boundary. In the process, we develop charts near the boundary of a specific form which are required in the sequel. The construction of the double is similar to the case of a manifold [Munkres 1963].

Let $\mathbf{B}_x(r)$ denote the ball of radius r about x in $\mathbb{R}^n$ where $\mathbb{R}^n$ has basis $\{e_i\}_{i=1}^n$. For convenience, $\mathbf{B}_0$ denotes the ball of radius 1 centered at the origin in $\mathbb{R}^n$. We let $\mathbb{R}^n_+ = \{x_1, \dots, x_n : x_n \geq 0\}$ where the x_i are the coordinates with respect to the basis $\{e_i\}$, $\mathbf{B}^+_x(r) = \mathbf{B}_x(r) \cap \mathbb{R}^n_+$, and $\mathbf{B}^+_0 = \mathbf{B}_0 \cap \mathbb{R}^n_+$. Also, $\mathbf{B}^k_0$ denotes the ball of radius 1 about the origin in $\mathbb{R}^k$.

Let Q be a compact orbifold with boundary. For each point $p \in Q$, we choose an orbifold chart $\{V_p, G_p, \pi_p\}$ where V_p is $\mathbf{B}_0$ or $\mathbf{B}^+_0$ and $\pi_p(0) = p$. Let U_p denote $\pi_p(V_p) \subseteq Q$ for each p, and then the U_p form an open cover of Q. Choose a finite subcover of the U_p, and on each V_p corresponding to a U_p in the subcover, we put the standard Riemannian structure on V_p so that the $\{\partial/\partial x_i\}$ form an orthonormal basis. Endow Q with a Riemannian structure by patching these Riemannian metrics together using a partition of unity subordinate to the finite subcover of Q chosen above.

Now, let $p \in Q$, and then there is a geodesic neighborhood U_p about p uniformized by $\{V_p, G_p, \pi_p\}$ where $V_p = \mathbf{B}_0(r)$ or $\mathbf{B}^+_0(r)$ for some $r > 0$, and G_p acts as a subgroup of $O(n)$ [Chen and Ruan 2002]. Identifying V_p with a subset of $T_0 V_p$ via the exponential map, we can assume as above that $\{e_i\}$ forms an orthonormal basis with respect to which coordinates are denoted $\{x_i\}$. In the case with boundary, $\mathbf{B}^+_0(r)$ corresponds to points with $x_n \geq 0$. We call such a chart a *geodesic chart of radius r at p*. Note that in such charts, the action of $\gamma \in G_p$ on V_p and the action of $d\gamma = D(\gamma)_0$ on a neighborhood of 0 in $T_0 V_p$ (or in half-space in the case with boundary) are identified via the exponential map.

Lemma 3.1. *At every point p in ∂Q, there is a geodesic chart at p of the form $\{V_p, G_p, \pi_p\}$ where G_p fixes e_n. On the boundary, the tangent space $TQ|_{\partial Q}$ is decomposed orthogonally into $(T\partial Q) \oplus \nu$ where ν is a trivial 1-bundle on which each group acts trivially.*

Proof. Let $p \in \partial Q$, and let a neighborhood of p be uniformized by the geodesic chart $\{V_p, G_p, \pi_p\}$ so that $V_p = \mathbf{B}^+_0(r)$. Let $\langle \cdot, \cdot \rangle_0$ denote the inner product on $T_0 V_p$. Let T^+_0 correspond to the half-space in $T_0 V_p$ corresponding to vectors with nonnegative $(\partial/\partial x_n)$-component. The exponential map identifies an open ball about $0 \in T^+_0$ with V_p.

Suppose γ is an arbitrary element of G_p so that $d\gamma$ acts on $T_0 V_p$. Any $v \in T_0^+$ satisfies

$$\left\langle v, \frac{\partial}{\partial x_n} \right\rangle_0 \geq 0.$$

Furthermore, $(d\gamma)v \in T_0^+$, so

$$\left\langle (d\gamma)v, \frac{\partial}{\partial x_n} \right\rangle_0 \geq 0 \quad \text{or equivalently} \quad \left\langle v, d\gamma^{-1} \frac{\partial}{\partial x_n} \right\rangle_0 \geq 0,$$

for all $v \in T_0^+$.

We claim that G_p fixes $\partial/(\partial x_n)$. Pick $j \neq n$; since $\frac{\partial}{\partial x_j} \in T_0^+$,

$$\left\langle \frac{\partial}{\partial x_j}, d\gamma^{-1} \frac{\partial}{\partial x_n} \right\rangle_0 \geq 0.$$

However, $-\frac{\partial}{\partial x_j}$ is also a vector in T_0^+, and so

$$\left\langle -\frac{\partial}{\partial x_j}, d\gamma^{-1} \frac{\partial}{\partial x_n} \right\rangle_0 \geq 0.$$

By the linearity of the inner product, this is only possible if

$$\left\langle \frac{\partial}{\partial x_j}, d\gamma^{-1} \frac{\partial}{\partial x_n} \right\rangle_0 = 0.$$

Furthermore, since $j \neq n$ was arbitrary, this implies that $d\gamma^{-1}(\partial/\partial x_n)$ has no component in the direction of any $(\partial/\partial x_j)$, $j \neq n$. Since $d\gamma^{-1}$ is an isometry,

$$d\gamma^{-1} \frac{\partial}{\partial x_n} = \pm \frac{\partial}{\partial x_n},$$

but because $d\gamma^{-1} T_0^+ = T_0^+$, it must be the case that

$$d\gamma^{-1} \frac{\partial}{\partial x_n} = \frac{\partial}{\partial x_n}.$$

As $\gamma \in G_p$ was arbitrary, this implies G_p fixes $\partial/(\partial x_n)$.

Now, for each $p \in \partial Q$, pick a geodesic chart $\{V_p, G_p, \pi_p\}$ at p and let N_p denote the constant vector field $\partial/(\partial x_n)$ on V_p. Recall from [Satake 1957] that $\tilde{T}_0 V_p$ denotes the dG_p-invariant tangent space of $T_0 V_p$ on which the differential of π_p is invertible. If $q \in \pi_p(V_p) \subset Q$ with geodesic chart $\{V_q, G_q, \pi_q\}$ at q, then the fact that $D(\pi_q)_p^{-1} \circ D(\pi_p)_0 : \tilde{T}_0 V_p \to \tilde{T}_0 V_q$ maps $\tilde{T}_0 \partial V_p$ to $\tilde{T}_0 \partial V_q$ and preserves the metric ensures that the value of $N_q(0)$ coincides with that of $D(\pi_q)_p^{-1} \circ D(\pi_p)_0[N_p(0)]$ up to a sign. The sign is characterized by the property that for any curve $c : (-1, 1) \to V_p$ with derivative $c'(t) = N_p$, there is an $\epsilon > 0$ such that $c(t)$ is in the interior of V_p for $t \in (0, \epsilon)$; a curve in V_q with derivative

$D(\pi_q)_p^{-1} \circ D(\pi_p)_0[N_p(0)]$ has the same property. With this, we see that the N_p patch together to form a nonvanishing section of $TQ|_{\partial Q}$ that is orthogonal to $T\partial Q$ at every point; hence, it defines a trivial subbundle ν orthogonal to $T\partial Q$. Clearly, $TQ = (T\partial Q) \oplus \nu$. □

Let Q' be an identical copy of Q. In order to form a closed orbifold from the two, the boundaries of these two orbifolds are identified via

$$\partial Q \ni x \longleftrightarrow x' \in \partial Q'. \tag{3-1}$$

The resulting space inherits the structure of a smooth orbifold from Q as is demonstrated below.

Note by Lemma 3.1 that for each point $p \in \partial Q$, a geodesic chart $\{V_p, G_p, \pi_p\}$ can be restricted to a chart $\{C_p^+, G_p, \phi_p\}$ where $C_p^+ = \mathbf{B}_0^{n-1}(r/2) \times [0, \epsilon_p)$, ϕ_p is the restriction of π_p to C_p^+, and $\phi_p(\mathbf{B}_0^{n-1} \times \{0\}) = \partial\phi_p(C_p^+)$. We refer to such a chart as a *boundary product chart* for Q.

It follows, in particular, that there is a neighborhood of ∂Q in Q that is diffeomorphic to $\partial Q \times [0, \epsilon]$ for some $\epsilon > 0$ and that the metric respects the product structure. This can be shown by forming a cover of ∂Q of sets uniformized by charts of the form $\{C^+, G_p, \phi_p\}$, choosing a finite subcover, and setting $\epsilon = \min\{\epsilon_p/2\}$.

Lemma 3.2. *The glued set $\hat{Q}$, that is, the set of equivalence classes under the identification made by Equation* (3-1), *may be made into a smooth orbifold containing diffeomorphic copies of both Q and Q' such that $Q \cap Q' = \partial Q = \partial Q'$.*

Proof. For each point $p \in \partial Q$, form a boundary product chart $\{C_p^+, G_p, \phi_p\}$. Then glue each chart of the boundary of Q to its corresponding chart of Q' in the following way. Let $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ be the reflection that sends $e_n \mapsto -e_n$ and fixes all other coordinates. A point p in the boundary is uniformized by two corresponding boundary product charts on either side of ∂Q, $\{C_p^+, G_p, \phi_p\}$ and $\{C_p^{+\prime}, G_p', \phi_p'\}$. From these two charts, a new chart $\{C_p, G_p, \psi_p\}$ for a neighborhood of p in $\hat{Q}$ is constructed where $C_p = \mathbf{B}_0^{n-1}(r/2) \times (-\epsilon_p, \epsilon_p)$, and

$$\psi_p(x) = \begin{cases} \phi_p(x), & x_n \geq 0, \\ \phi_p' \circ \alpha(x), & x_n < 0. \end{cases}$$

These charts cover a neighborhood of $\partial Q = \partial Q'$ in $\hat{Q}$. By taking a geodesic chart at each point on the interiors of Q and Q' together with these new charts, the entire set $\hat{Q}$ is covered. Injections of charts at points in the interior of Q or Q' into charts of the form $\{C_p^+, G_p, \phi_p\}$ induce injections into $\{C_p, G_p, \psi_p\}$. Hence, $\hat{Q}$ is given the structure of a smooth orbifold with the desired properties. □

Again, it follows that a neighborhood of $\partial Q \subset \hat{Q}$ admits a tubular neighborhood diffeomorphic to $\partial Q \times [-\epsilon, \epsilon]$ such that the metric respects this product structure.

4. The Morse Index of a vector field on an orbifold

The definition of the Morse Index and its relation to the topological index of a vector field extend readily to orbifolds.

Let Q be a compact orbifold with or without boundary, and let X be a vector field on Q that does not vanish on the boundary. Suppose $X(p) = 0$ for $p \in Q$. We say that p is a *nondegenerate* zero of X if there is a chart $\{V, G, \pi\}$ for a neighborhood U_p of p and an $x \in V$ with $\pi(x) = p$ such that π^*X has a nondegenerate zero at x; that is, $D(\pi^*X)_x$ has trivial kernel. As in the manifold case, nondegenerate zeros are isolated in charts and hence isolated on Q. The Morse Index $\lambda(\pi^*X; x)$ of π^*X at x is defined to be the number of negative eigenvalues of $D(\pi^*X)_x$ [Milnor 1963]. Since the Morse Index is a diffeomorphism invariant, this index does not depend on the choice of chart nor on the choice of x. Since the isomorphism-class of the isotropy group does not depend on the choice of x, the expression $|G_p|$ is well-defined. Hence, for simplicity, we may restrict to charts of the form $\{V_p, G_p, \pi_p\}$ where $\pi_p(0) = p$ and G_p acts linearly. We define the *orbifold Morse Index* of X at p to be

$$\lambda^{\mathrm{orb}}(X; p) = \frac{1}{|G_p|}\lambda(\pi_p^*X; 0).$$

Note that this index differs from that recently defined in [Hepworth 2007]; however, it is sufficient for our purposes. We have

$$\begin{aligned}\mathfrak{Ind}^{\mathrm{orb}}(X; p) &= \tfrac{1}{|G_p|}\mathfrak{Ind}(\pi_p^*X; 0)\\ &= \tfrac{1}{|G_p|}(-1)^{\lambda(\pi_p^*X;0)}.\end{aligned}$$

Suppose X has only nondegenerate zeros on Q. For each $\lambda \in \{0, 1, \dots, n\}$, we let $\{p_i : i = 1, \dots, k_\lambda\}$ denote the points in Q at which the pullback of X in a chart has Morse Index λ. Then we let

$$C_\lambda = \sum_{i=1}^{k_\lambda} \frac{1}{|G_{p_i}|}$$

count these points, where the orbifold-contribution of each zero p_i is $1/|G_{p_i}|$. Note that as nondegenerate zeros are isolated, there is a finite number on Q.

As in the manifold case, we define

$$\Sigma^{\mathrm{orb}}(X; Q) = \sum_{\lambda=0}^{n}(-1)^\lambda C_\lambda,$$

and we have

$$\begin{aligned}\Sigma^{\mathrm{orb}}(X;\,Q) &= \textstyle\sum_{\lambda=0}^{n}(-1)^{\lambda}\sum_{i=1}^{k_\lambda}\frac{1}{|G_{p_i}|}\\ &= \textstyle\sum_{p\in Q,X(p)=0}\frac{1}{|G_p|}(-1)^{\lambda(\pi_p^*X;0)}\\ &= \textstyle\sum_{p\in Q,X(p)=0}\mathfrak{Ind}^{\mathrm{orb}}(X;\,p)\\ &= \mathfrak{Ind}^{\mathrm{orb}}(X;\,Q).\end{aligned}$$

In the case that Q is closed, this quantity is equal to $\chi_{\mathrm{orb}}(Q)$ by Theorem 2.1.

We summarize these results as follows.

Proposition 4.1. *Let X be a smooth vector field on the compact orbifold Q that has nondegenerate zeros only, none of which occurring on ∂Q. Then*

$$\Sigma^{\mathrm{orb}}(X;\,Q) = \mathfrak{Ind}^{\mathrm{orb}}(X;\,Q).$$

If $\partial Q = \varnothing$, then

$$\Sigma^{\mathrm{orb}}(X;\,Q) = \chi_{\mathrm{orb}}(Q).$$

Remark 4.2. If Q is a compact orbifold (with or without boundary) and X a smooth vector field on Q that is nonzero on some compact subset Γ of the interior of Q, then X may be perturbed smoothly outside of a neighborhood of Γ so that it has only isolated, nondegenerate zeros. This is shown in [Waner and Wu 1986] for the case of a smooth global quotient M/G using local arguments, and so it extends readily to the case of a general orbifold by working in charts.

5. Proof of Theorem 1.1

Proof. Let Y be a vector field in generic contact with ∂Q that has isolated zeros on the interior of Q. Define $\hat{Y}$ on $\hat{Q}$ by letting $\hat{Y}$ coincide with Y on each copy of Q. Unfortunately, $\hat{Y}$ has conflicting definitions on ∂Q. However, as in the manifold case treated in [Pugh 1968], the vector field may be perturbed near the boundary to form a well-defined vector field using the product structure. We give an adaptation of Pugh's result to orbifolds.

Proposition 5.1. *Given a smooth vector field Y in generic contact with ∂Q with isolated zeros, none of which on ∂Q, there is a smooth vector field X on the double $\hat{Q}$ such that*

(i) *outside of a tubular neighborhood P_ϵ of ∂Q containing none of the zeros of Y, X coincides with Y on Q and Q';*

(ii) *$X|_{\partial Q}$ is tangent to ∂Q;*

(iii) *on Γ^1, X coincides with Y and in particular defines the same Γ^i, R_-^i, and R_+^i for $i > 1$;*

(iv) *the zeros of X are those of Y on the interior of Q and Q' and a collection of isolated zeros on ∂Q which are nondegenerate as zeros of $X|_{\partial Q}$.*

Proof. As above, $\hat{Y}$ is defined everywhere on $\hat{Q}$ except on ∂Q. Let P_ϵ be a normal tubular ϵ-neighborhood of ∂Q in $\hat{Q}$ of the form $\partial Q \times [-\epsilon, \epsilon]$ which we parameterize as $\{(x, v) : x \in \partial Q, v \in [-\epsilon, \epsilon]\}$. We assume that P_ϵ is small enough so that it does not contain any of the zeros of $\hat{Y}$. On P_ϵ, decompose $\hat{Y}$ respecting the product structure of P_ϵ into

$$\hat{Y} = \hat{Y}_h + \hat{Y}_v.$$

These are the horizontal and vertical components of $\hat{Y}$, respectively. The horizontal component $\hat{Y}_h$ is well-defined, continuous, and smooth when restricted to the boundary. However, $\hat{Y}_v$ has conflicting definitions on the boundary, although they only differ by a sign. Note that the restriction of $\hat{Y}_h$ to ∂Q may not have isolated zeros. However, as Y does not have zeros on ∂Q and $\hat{Y}_h \equiv Y$ on Γ^1, none of the zeros of $\hat{Y}_h|_{\partial Q}$ occur on Γ^1.

Define Z_h to be a smooth vector field on ∂Q that coincides with $\hat{Y}_h$ on an open subset of ∂Q containing Γ^1 and has only nondegenerate zeros (see Remark 4.2). Let $f(x, v)$ be the parallel transport of $Z_h(x, 0)$ along the geodesic from $(x, 0)$ to (x, v), and then Z_h is a horizontal vector field on P_ϵ. For $s \in (0, \epsilon)$, let $\phi_s : \mathbb{R} \to [0, 1]$ be a smooth bump function which is one on $[-s/2, s/2]$ and zero outside of $[s, s]$.

Define the vector field X_s to be $\hat{Y}$ outside of P_ϵ and

$$X_s(x, v) = \phi_s(v)\big(f(x, v) + |v|\hat{Y}_v(x, v)\big) + (1 - \phi_s(v))\hat{Y}(x, v)$$

on P_ϵ. Note that X_s is smooth. By picking s sufficiently small, it may be ensured that the zeros of X are the zeros of $\hat{Y}$ and the zeros of $Z_h|_{\partial Q}$ only. We prove this as follows.

On points (x, v) where $x \in \Gamma^1$ and $|v| \le s$, the horizontal component of X is $\phi_s(v) f(x, v) + (1 - \phi_s(v))\hat{Y}_h(x, v)$. Note that $f(x, 0) = \tilde{Y}_h(x, 0)$ for $x \in \Gamma^1$ and $f(x, 0) \ne 0$ on Γ^1. Let $m > 0$ be the minimum value of $\|f(x, 0)\|$ on the compact set Γ^1, and then as $\Gamma^1 \times [-\epsilon, \epsilon]$ is compact and $\tilde{Y}_h(x, v)$ continuous, there is an s_0 such that

$$\|\hat{Y}_h(x, 0) - \hat{Y}_h(x, v)\| = \|f(x, 0) - \hat{Y}_h(x, v)\| < m/2$$

whenever $|v| < s_0$. Hence, for such v and for any $t \in [0, 1]$,

$$\begin{aligned} \|tf(x, v) + (1 - t)\hat{Y}_h(x, v)\| &= \|\hat{Y}_h(x, v) + t[f(x, v) - \hat{Y}_h(x, v)]\| \\ &\ge \|\hat{Y}_h(x, v)\| - t\|f(x, v) - \hat{Y}_h(x, v)\| \\ &> m - \tfrac{tm}{2} \\ &\ge \tfrac{m}{2} > 0. \end{aligned}$$

Therefore, the horizontal component is nonvanishing, implying that $X_s(v, h)$ does not vanish here.

Now let $\{x_i : i = 1, \dots, k\}$ be the zeros of Z_h on ∂Q. Each x_i is contained in a ball $B_{\epsilon_i} \subset \partial Q$ whose closure does not intersect Γ^1. Hence, $\hat{Y}_v(x, 0) \neq 0$ on each B_{ϵ_i}. Therefore, for each i, there is an s_i such that $\hat{Y}_v(x, v) \neq 0$ on $B_{\epsilon_i} \times (-s_i, s_i)$. This implies that the vertical component of $X_s(x, v)$, and hence $X_s(x, v)$ itself, does not vanish on $B_{\epsilon_i} \times (-s_i, s_i)$ except where $v = 0$; i.e. on ∂Q.

Letting s be less than the minimum of $\{s_0, s_1, \dots, s_k\}$, we see that X_s does not vanish on P_ϵ except on ∂Q, where it coincides with Z_h. Therefore, $X = X_s$ is the vector field which was to be constructed. $\square$

It follows that the index of the vector field X constructed in the proof of Proposition 5.1 is

$$\mathfrak{Ind}^{\text{orb}}(X; \hat{Q}) = 2\mathfrak{Ind}^{\text{orb}}(Y; Q) + \sum_{p \in \partial Q} \mathfrak{Ind}^{\text{orb}}(X; p). \tag{5-1}$$

Let p be a zero of X on ∂Q, i.e. it is a zero of Z_h. We express the index of X at p in terms of the index of Z_h.

Because of Lemma 3.1, the isotropy group of p as an element of Q is the same as the isotropy group of p as an element of ∂Q, and so we may refer to G_p without ambiguity. For a neighborhood of p in Q small enough to contain no other zeros of X, choose a boundary product chart $\{C_p^+, G_p, \phi_p\}$. Then, as in Lemma 3.2, $\{C_p, G_p, \psi_p\}$ forms a chart about p in $\hat{Q}$. The product structure (y, w) of

$$C_p = \mathbf{B}_0^{n-1}(r/2) \times (-\epsilon_p, \epsilon_p)$$

coincides with that of P_ϵ near the boundary, so within the preimage of

$$\partial Q \times [-s/2, s/2]$$

by ψ_p, we have that

$$\psi_p^* X = \psi_p^* f + |w| \psi_p^* \hat{Y}_v.$$

Note that $\psi_p(0, 0) = p$, and then

$$\begin{aligned} D(\psi_p^* X)_{(0,0)} &= \begin{pmatrix} D(\psi_p^* Z_h)_0 & \left(\frac{\partial \psi_p^* f}{\partial w}\right)_0 \\ D\big((|w| \psi_p^* \hat{Y}_v)|_{\partial C_p}\big)_0 & \left(\frac{\partial}{\partial w} |w| \psi_p^* \hat{Y}_v\right)_0 \end{pmatrix} \\ &= \begin{pmatrix} D(\psi_p^* Z_h)_0 & 0 \\ 0 & \psi_p^* \hat{Y}_v(0, 0) \end{pmatrix}. \end{aligned}$$

As $\psi_p^* \hat{Y}_v(0,0)$ is positive if $p \in R^1_+$ and negative if $p \in R^1_-$, we see that

$$\lambda\bigl(\psi_p^* X; (0,0)\bigr) = \begin{cases} \lambda\bigl(\psi_p^* X|_{\partial C_p}; 0\bigr), & p \in R^1_+, \\ \lambda\bigl(\psi_p^* X|_{\partial C_p}; 0\bigr) + 1, & p \in R^1_-. \end{cases}$$

Hence

$$\mathfrak{Ind}\bigl(\psi_p^* X; (0,0)\bigr) = \begin{cases} \mathfrak{Ind}\bigl(\psi_p^* Z_h|_{\partial C_p}; 0\bigr), & p \in R^1_+, \\ -\mathfrak{Ind}\bigl(\psi_p^* Z_h|_{\partial C_p}; 0\bigr), & p \in R^1_-. \end{cases}$$

Therefore, for $p \in R^1_+$,

$$\begin{aligned} \mathfrak{Ind}^{\mathrm{orb}}(X, p) &= \tfrac{1}{|G_p|}\mathfrak{Ind}\bigl(\psi_p^* X; 0\bigr) \\ &= \tfrac{1}{|G_p|}\mathfrak{Ind}\bigl(\psi_p^* Z_h|_{\partial C^+}; 0\bigr) \\ &= \mathfrak{Ind}^{\mathrm{orb}}\bigl(Z_h; p\bigr), \end{aligned}$$

and similarly

$$\mathfrak{Ind}^{\mathrm{orb}}\bigl(X; p\bigr) = -\mathfrak{Ind}^{\mathrm{orb}}\bigl(Z_h; p\bigr),$$

for $p \in R^1_-$.

With this, Equation (5-1) becomes

$$\mathfrak{Ind}^{\mathrm{orb}}(X; \hat{Q}) = 2\mathfrak{Ind}^{\mathrm{orb}}(Y; Q) + \mathfrak{Ind}^{\mathrm{orb}}(Z_h; R^1_+) - \mathfrak{Ind}^{\mathrm{orb}}(Z_h; R^1_-).$$

By Theorem 2.1 and Equation (2-1), $\mathfrak{Ind}^{\mathrm{orb}}(X; \hat{Q}) = 2\chi_{\mathrm{orb}}(Q) - \chi_{\mathrm{orb}}(\partial Q)$, with the result that

$$2\chi_{\mathrm{orb}}(Q) - \chi_{\mathrm{orb}}(\partial Q) = 2\mathfrak{Ind}^{\mathrm{orb}}(Y; Q) + \mathfrak{Ind}^{\mathrm{orb}}(Z_h; R^1_+) - \mathfrak{Ind}^{\mathrm{orb}}(Z_h; R^1_-).$$

Note that ∂Q is also a closed orbifold, so

$$\chi_{\mathrm{orb}}(\partial Q) = \mathfrak{Ind}^{\mathrm{orb}}(X; \partial Q) = \mathfrak{Ind}^{\mathrm{orb}}(X; R^1_+) - \mathfrak{Ind}^{\mathrm{orb}}(X; R^1_-).$$

Hence, restricting X to ∂Q,

$$\begin{aligned} \mathfrak{Ind}^{\mathrm{orb}}(Y; Q) &= \chi_{\mathrm{orb}}(Q) + 1/2\bigl(-\chi_{\mathrm{orb}}(\partial Q) + \mathfrak{Ind}^{\mathrm{orb}}(X; R^1_-) - \mathfrak{Ind}^{\mathrm{orb}}(X; R^1_+)\bigr) \\ &= \chi_{\mathrm{orb}}(Q) + 1/2\bigl[-\chi_{\mathrm{orb}}(\partial Q) + 2\mathfrak{Ind}^{\mathrm{orb}}(X; R^1_-) \\ &\qquad - \bigl(\mathfrak{Ind}^{\mathrm{orb}}(X; R^1_+) + \mathfrak{Ind}^{\mathrm{orb}}(X; R^1_-)\bigr)\bigr] \\ &= \chi_{\mathrm{orb}}(Q) + 1/2\bigl(-2\chi_{\mathrm{orb}}(\partial Q) + 2\mathfrak{Ind}^{\mathrm{orb}}(X; R^1_-)\bigr) \\ &= \chi_{\mathrm{orb}}(Q) - \chi_{\mathrm{orb}}(\partial Q) + \mathfrak{Ind}^{\mathrm{orb}}(X; R^1_-) \\ &= \chi_{\mathrm{orb}}(Q, \partial Q) + \mathfrak{Ind}^{\mathrm{orb}}(X; R^1_-). \end{aligned} \tag{5-2}$$

Because X coincides with Y on Γ^1, it defines the same Γ^i that Y does. Since X is a smooth vector field defined on R^1_- that does not vanish on $\partial R^1_- = \Gamma^1$, we may recursively apply this formula to higher and higher orders of R_-^i until R_-^i is empty, and there is no longer an index sum term. Hence,

$$\mathfrak{Ind}^{\mathrm{orb}}(X;\, R_-^1) = \sum_{i=1}^{n} \chi_{\mathrm{orb}}(R_-^i, \Gamma^i).$$

Along with Equation (5-2), this completes the proof of Theorem 1.1. □

Let $\tilde{Q}$ denote the inertia orbifold of Q and $\pi : \tilde{Q} \to Q$ the projection (see [Chen and Ruan 2004]). It is shown in [Seaton 2008] that a vector field Y on Q induces a vector field $\tilde{Y}$ on $\tilde{Q}$, and that $\tilde{Y}\big(p, (g)\big) = 0$ if and only if $Y(p) = 0$.

For each point $p \in Q$ and $g \in G_p$, a chart $\{V_p, G_p, \pi_p\}$ induces a chart

$$\{V_p^g, C(g), \pi_{p,g}\} \quad \text{at} \quad \big(p, (g)\big) \in \tilde{Q},$$

where V_p^g denotes the points in V_p fixed by g and $C(g)$ denotes the centralizer of g in G_p. Clearly, $\partial V_p^g = (\partial V_p) \cap V_p^g$. An atlas for $\tilde{Q}$ can be taken consisting of charts of this form, so it is clear that $\partial \tilde{Q} = \widetilde{\partial Q}$.

Let $p \in \partial Q$ and pick a boundary product chart $\{C_p^+, G_p, \phi_p\}$. Then for $g \in G_p$, there is a chart $\{(C_p^+)^g, C(g), \phi_{p,g}\}$ for $(p, (g)) \in \tilde{Q}$. As the normal component to the boundary of C_p^+ is G_p-invariant,

$$\begin{aligned} (C_p^+)^g &= \big(\mathbf{B}_0^{n-1}(r/2) \times [0, \epsilon_p)\big)^g \\ &= \big(\mathbf{B}_0^{n-1}(r/2)\big)^g \times [0, \epsilon_p), \end{aligned}$$

and so

$$T_0(C_p^+)^g = T_0\big(\mathbf{B}_0^{n-1}(r/2)\big)^g \times \mathbb{R}.$$

It follows that $\tilde{Y}$ points out of $\partial \tilde{Q}$ at $\big(p, (g)\big)$ if and only if Y points out of ∂Q at p. With this, applying Theorem 1.1 to $\tilde{Y}$ yields

$$\begin{aligned} \mathfrak{Ind}^{\mathrm{orb}}\Big(\tilde{Y};\, \tilde{Q}\Big) &= \chi_{\mathrm{orb}}\Big(\tilde{Q}, \partial \tilde{Q}\Big) + \textstyle\sum_{i=1}^{n} \chi_{\mathrm{orb}}\Big(\widetilde{R_-^i}, \widetilde{\Gamma^i}\Big) \\ &= \chi_{\mathrm{orb}}\Big(\tilde{Q}) - \chi_{\mathrm{orb}}(\partial \tilde{Q}\Big) + \textstyle\sum_{i=1}^{n} \chi_{\mathrm{orb}}\Big(\widetilde{R_-^i}\Big) - \chi_{\mathrm{orb}}\Big(\widetilde{\Gamma^i}\Big). \end{aligned} \tag{5-3}$$

Each of the Γ^i and ∂Q are closed orbifolds, so it follows from the proof of Theorem 3.2 in [Seaton 2008] (note that the assumption of orientability is not used to establish this result) that

$$\chi_{\mathrm{orb}}(\widetilde{\Gamma^i}) = \chi(\mathbb{X}_{\Gamma^i}),$$

and

$$\chi_{\mathrm{orb}}(\partial \tilde{Q}) = \chi_{\mathrm{orb}}(\widetilde{\partial Q}) = \chi(\mathbb{X}_{\partial Q}), \tag{5-4}$$

where $\mathbb{X}_{\Gamma^i}$ and $\mathbb{X}_{\partial Q}$ denote the underlying topological spaces of Γ^i and ∂Q, respectively, and χ the usual Euler characteristic.

Letting $\hat{Q}$ denote, as above, the double of Q, it is easy to see that $\hat{\tilde{Q}} = \tilde{\hat{Q}}$. Hence, applying the same result to $\hat{\tilde{Q}}$ yields

$$\begin{aligned}\chi(\mathbb{X}_{\hat{Q}}) &= \chi_{\text{orb}}(\tilde{\hat{Q}}) \\ &= \chi_{\text{orb}}(\hat{\tilde{Q}}) \\ &= 2\chi_{\text{orb}}(\tilde{Q}) - \chi_{\text{orb}}(\partial\tilde{Q}).\end{aligned} \tag{5-5}$$

However, as

$$\begin{aligned}\chi(\mathbb{X}_{\hat{Q}}) &= 2\chi(\mathbb{X}_Q) - \chi(\mathbb{X}_{\partial Q}) \\ &= 2\chi(\mathbb{X}_Q) - \chi_{\text{orb}}(\partial\tilde{Q}),\end{aligned}$$

it follows from Equation (5-5) that $\chi_{\text{orb}}(\tilde{Q}) = \chi(\mathbb{X}_Q)$. The same holds for each R^i_- so that Equation (5-3) becomes the following.

Corollary 5.2. *Let Q be an n-dimensional, smooth, compact orbifold with boundary, and let Y be a smooth vector field on Q. If $\tilde{Y}$ denotes the induced vector field on $\tilde{Q}$, then*

$$\mathfrak{Ind}^{\text{orb}}(\tilde{Y};\, \tilde{Q}) = \chi(\mathbb{X}_Q, \mathbb{X}_{\partial Q}) + \sum_{i=1}^{n} \chi(\mathbb{X}_{R^i_-}, \mathbb{X}_{\Gamma^i}).$$

Acknowledgments

We would like to thank the referee for helpful comments and suggestions regarding this paper. The first author would like to thank Michele Intermont for guiding him through much of the background material required for this work. The second author would like to thank Carla Farsi for helpful conversations and suggesting this problem.

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Received: 2008-06-11 Revised: Accepted: 2009-02-19

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