On distances and self-dual codes over $F_q[u]/(u^t)$

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New metrics and distances for linear codes over the ring $F_q[u]/(u^t)$ are defined, which generalize the Gray map, Lee weight, and Bachoc weight; and new bounds on distances are given. Two characterizations of self-dual codes over $F_q[u]/(u^t)$ are determined in terms of linear codes over $F_q$. An algorithm to produce such self-dual codes is also established.

1. Introduction

Many optimal codes have been obtained by studying codes over general rings rather than fields. Lately, codes over finite chain rings (of which $F_q[u]/(u^t)$ is an example) have been a source of many interesting properties [Norton and Salagean 2000a; Ozbudak and Sole 2007; Dougherty et al. 2007]. Gulliver and Harada [2001] found good examples of ternary codes over $F_3$ using a particular type of Gray map. Siap and Ray-Chaudhuri [2000] established a relation between codes over $F_q[u]/(u^2 - a)$ and codes over $F_q$ which was used to obtain new codes over $F_3$ and $F_5$. In this paper we present a certain generalization of the method used in [Gulliver and Harada 2001] and [Siap and Ray-Chaudhuri 2000], defining a family of metrics for linear codes over $F_q[u]/(u^t)$ and obtaining as particular examples the Gray map, the Lee weight and the Bachoc weight. For the latter, we give a new bound on the distance of those codes. It also shows that the Gray images of codes over $F_2 + uF_2$ are more powerful than codes obtained by the so-called $u-(u+v)$ condition.

With these tools in hand, we study conditions for self-duality of codes over $F_q[u]/(u^t)$. Norton and Salagean [2000b] studied the case of self-dual cyclic codes in terms of the generator polynomials. In this paper we study self-dual codes in terms of linear codes over $F_q$ that are obtained as images under the maps defined on the first part of the paper. We provide a way to construct many self-dual codes over $F_q$ starting from a self-dual code over $F_q[u]/(u^t)$. We also study self-dual codes

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in terms of the torsion codes, and provide a way to construct many self-dual codes over $F_q[u]/(u^t)$ starting from a self-orthogonal code over $F_q$. Our results contain many of the properties studied by Bachoc [1997] for self-dual codes over $F_3 + uF_3$.

2. Metric for codes over $F_q[u]/(u^t)$

We will use $R(q, t)$ to denote the commutative ring $F_q[u]/(u^t)$. The $q^t$ elements of this ring can be represented in two different forms, and we will use the most appropriate in each case. First, we can use the polynomial representation with indeterminate $u$ of degree less than or equal to $(t-1)$ with coefficients in $F_q$, using the notation $R(q, t) = F_q + uF_q + u^2F_q + \cdots + u^{t-1}F_q$. We also use the $u$-ary coefficient representation as an $F_q$-vector space.

Let $B \in M_t(F_q)$ be an invertible $t \times t$ matrix, and let $B$ act as right multiplication on $R(q, t)$ (seen as $F_q$-vector space). We extend this action linearly to the $F_q$-module $(R(q, t))^n$ by concatenation of the images $\phi_B : (R(q, t))^n \rightarrow (F_q)^{tn}$ given by

$$\phi_B(x_1, x_2, \ldots, x_n) = (x_1B, x_2B, \ldots, x_nB)$$

An easy counting argument shows that $\phi_B$ is an $F_q$-module isomorphism and if $C$ is a linear code over $R(q, t)$ of length $n$, then $\phi_B(C)$ is a linear $q$-ary code of length $tn$.

Example 1. Consider the ring $R(3, 2) = F_3 + uF_3$ with $u^2 = 0$. Choosing

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

we obtain the Gray map $\phi_B : (F_3 + uF_3)^n \rightarrow F_3^{2n}$ with

$$(a + ub)B = (a \ b) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = (b \ a + b)$$

used by Gulliver and Harada [2001].

Each such matrix $B$ induces a new metric in the code $C$.

Definition 1. Let $C$ be a linear code over $R(q, t)$. Let $B$ be an invertible matrix in $M_t(F_q)$, and let $\phi_B$ be the corresponding map. The $B$-weight of an element $x \in R(q, t)$, $w_B(x)$, is defined as the Hamming weight of $xB$ in $(F_q)^t$. Also, the $B$-weight of a codeword $(x_1, \cdots, x_n) \in C$ is defined as:

$$w_B(x_1, \cdots, x_n) = \sum_{i=1}^n w_B(x_i).$$
Similarly, the B-distance between two codewords in C is defined as the B-weight of their difference, and the B-distance, \( d_B \), of the code C is defined as the minimal B-distance between any two distinct codewords.

**Example 2.** In the example above, the corresponding B-weight of an element of \( \mathbb{F}_3 + u\mathbb{F}_3 \) is given by

\[
    w_B(x) = w_B(a + ub) = w_H((a + ub)B) = w_H(b, a + b) = \begin{cases} 
    0 & \text{if } x = 0, \\
    1 & \text{if } x = 1, 2, 2 + u, 1 + 2u, \\
    2 & \text{otherwise},
  \end{cases}
\]

which coincides with the Gray weight given in [Gulliver and Harada 2001].

**Example 3.** Consider the matrix

\[
    B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\]

the corresponding B-weight of an element of \( \mathbb{F}_2 + u\mathbb{F}_2 \) is given by

\[
    w_B(x) = w_B(a + ub) = w_H((a + ub)B) = w_H(a + b, b) = \begin{cases} 
    0 & \text{if } x = 0, \\
    1 & \text{if } x = 1, 1 + u, \\
    2 & \text{if } x = u,
  \end{cases}
\]

which produces the Lee weight \( w_L \) for codes over \( \mathbb{F}_2 + u\mathbb{F}_2 \).

**Example 4.** Consider the matrix

\[
    B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

the corresponding B-weight of an element of \( \mathbb{F}_q + u\mathbb{F}_q \) is given by

\[
    w_B(x) = w_B(a + ub) = w_H((a + ub)B) = w_H(b, a) = \begin{cases} 
    0 & \text{if } x = 0, \\
    1 & \text{if exactly one of } a \text{ or } b \text{ is nonzero,} \\
    2 & \text{if both } a \text{ and } b \text{ are nonzero},
  \end{cases}
\]

which produces the Gray weight for codes in [Siap and Ray-Chaudhuri 2000].

The case \( B = I_t \) corresponds to the special weight studied in [Ozbudak and Sole 2007] with regards to Gilbert–Varshamov bounds. A theorem similar to [Ozbudak and Sole 2007, Theorem 3] can be obtained using special families of matrices \( B \). The definition leads immediately to the fact that \( \phi_B \) preserves weights and distances between codewords.

When the generator matrix of a code \( C \) is of the form \( G = (I \ M) \), \( C \) is called a free code over \( R(q, t) \). In this case, we can establish the correspondence between
the parameters of the codes; see [Siap and Ray-Chaudhuri 2000, Section 2.2]. The case of nonfree codes will be considered later in Proposition 4.

**Proposition 1.** Let \( B \) be an invertible matrix over \( M_t(\mathbb{F}_q) \), let \( C \) be a linear free code over \( R(q, t) \) of length \( n \) with \( B \)-distance \( d_B \), and let \( \phi_B \) be the corresponding map. Then \( \phi_B(C) \) is a linear \([tn, tk, d_B]_q\)-code over \( \mathbb{F}_q \). Furthermore, the Hamming weight enumerator polynomial of the linear code \( \phi_B(C) \) over \( \mathbb{F}_q \) is the same as the \( B \)-weight enumerator polynomial of the code \( C \) over \( R(q, t) \).

**Proof.** Since \( B \) is nonsingular, \( \phi_B(C) \) is a linear code over \( \mathbb{F}_q \), with the same number of codewords. A basis for \( \phi_B(C) \) can be obtained from a (minimal) set of generators for \( C \), say, \( y_1, y_2, \ldots, y_k \). The set \( \{ u^i y_j | i = 0..(t-1), j = 1..k \} \) forms a set of generators for \( C \) as an \( \mathbb{F}_q \)-submodule. Since \( C \) is free and \( B \) is invertible, it follows that \( \{ \phi_B(u^i y_j) | i = 0..(t-1), j = 1..k \} \) are linearly independent over \( \mathbb{F}_q \) and form a basis for the linear code \( \phi_B(C) \). Hence the dimension of the code \( \phi_B(C) \) is \( tk \). The equality of distance follows from the definition. \( \square \)

In matrix form, we can construct a generator matrix for the linear code \( \phi_B(C) \) as follows. Let \( G \) be a matrix of generators for \( C \). For each row \((x_1, x_2, \ldots, x_n)\) of \( G \) consider the matrix representation \((X_1, X_2, \ldots, X_n)\) of the elements of \( R(q, t) \) given by

\[
X_i = \begin{pmatrix}
    a_0 & a_1 & a_2 & \cdots & a_{t-1} \\
    0 & a_0 & a_1 & \cdots & a_{t-2} \\
    \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & 0 & 0 & \cdots & a_0
\end{pmatrix}.
\]

For a free code, the rows of the matrix \((X_1B, X_2B, \ldots, X_nB)\) produce \( t \) linearly independent generators for the linear code \( \phi_B(C) \). Repeating this process for each row of \( G \), we will obtain the \( tk \) generators for \( \phi_B(C) \). We denote this matrix by \( \phi_B(G) \). For the case of nonfree linear codes, several rows will become zero and need to be deleted from the matrix. A counting of these rows will be given in Section 3.

Some choices of \( B \) can produce some optimal ternary and quintic codes as we now illustrate.

**Example 5.** Consider a linear code \( C \) over \( \mathbb{F}_3 + u\mathbb{F}_3 \) of length 9 with generator matrix:

\[
G = \begin{pmatrix}
    1 & 0 & 0 & 0 & u & 2+u & 1+u & 1 & 0 \\
    0 & 1 & 0 & 0 & u & 2+u & 1+u & 1 & 0 \\
    0 & 0 & 1 & 0 & 1 & 0 & u & 2+u & 1+u \\
    0 & 0 & 0 & 1 & 1+u & 1 & 0 & u & 2+u
\end{pmatrix}
\]

Let \( B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \).
The $B$-weight enumerator polynomial is given by
\[
1 + 98x^7 + 206x^8 + 412x^9 + 780x^{10} + 1032x^{11} + 1308x^{12} + 1224x^{13} \\
+ 828x^{14} + 462x^{15} + 166x^{16} + 40x^{17} + 4x^{18}.
\]

The corresponding linear ternary code $\phi_B(C)$ is an optimal ternary $[18, 8, 7]$-code.

Notice that if we take $B = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$, we get a linear ternary code $\phi_B(C)$ of length 18, dimension 8, but now, with minimal distance 4. The challenge now is to look for matrices $B$ that produce optimal codes.

Example 6. Consider a linear code $C$ over $\mathbb{F}_5 + u\mathbb{F}_5$ of length 5 with a generator matrix:
\[
G = \begin{pmatrix} 1 & 0 & 2u & 3+3u & 4 \\ 0 & 1 & 4 & 2u & 3+3u \end{pmatrix}.
\]

Let
\[
B = \begin{pmatrix} 3 & 0 \\ 2 & 3 \end{pmatrix}.
\]

The linear $\mathbb{F}_5$-code $\phi_B(C)$ is an optimal $[10, 4, 6]$-code, with generator matrix given by
\[
\phi_B(G) = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 3 & 2 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 & 3 & 3 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 3 & 2 & 3 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 & 3 & 3 & 1 \end{pmatrix}.
\]

Example 7. Consider a linear code $C$ over $R(5, 3) = \mathbb{F}_5 + u\mathbb{F}_5 + u^2\mathbb{F}_5$ of length 14 with generator matrix obtained by cyclic shifts of the first 5 components and cyclic shift of the last 9 components of the vector:
\[
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & u & 3+3u & 2+4u & 4u & 0 & 4 & 3+u^2 & 2+u+u^2 & u+u^2 \end{pmatrix}.
\]

Let
\[
B = \begin{pmatrix} 0 & 3 & 3 \\ 0 & 0 & 4 \\ 3 & 3 & 2 \end{pmatrix}.
\]

The $B$-weight enumerator polynomial is given by
\[
1+24x^{16}+32x^{17}+80x^{18}+150x^{19}+158x^{20}+140x^{21}+82x^{22}+44x^{23}+14x^{24}+4x^{25}
\]
and the linear $\mathbb{F}_5$-code $\phi_B(C)$ is an optimal $[42, 15, 16]$-code over $\mathbb{F}_5$. 
3. Metrics using the torsion codes

A generalization of the residue and torsion codes for \( \mathbb{F}_2 + u\mathbb{F}_2 \) has been studied in [Norton and Salagean 2000b] where a generator matrix for a code \( C \) over \( R(q, t) \) is defined as a matrix \( G \) over \( R(q, t) \) whose rows span \( C \) and none of them can be written as a linear combination of the other rows of \( G \). Recalling that two codes over \( R(q, t) \) are equivalent if one can be obtained from the other by permuting the coordinates or by multiplying all entries in a specified coordinate by an invertible element of \( R(q, t) \), and performing Gauss elimination (remembering not to multiply by nonunits) we can always obtain a generator matrix for a code (or equivalent code) which is in standard form, that is, in the form

\[
G = \begin{pmatrix}
I_k & B_{1,2} & B_{1,3} & \cdots & B_{1,t} & B_{1,t+1} \\
0 & uI_k & uB_{2,3} & \cdots & uB_{2,t} & uB_{2,t+1} \\
0 & 0 & u^2I_k & \cdots & u^2B_{3,t} & u^2B_{3,t+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & u^{t-1}I_k & u^{t-1}B_{t,t+1}
\end{pmatrix},
\]

where \( B_{i,j} \) is a matrix of polynomials in \( \mathbb{F}_q[u]/(u^i) \) of degrees at most \( j-i-1 \). In fact, we can think of \( B_{i,j} \) as a matrix of the form

\[
B_{i,j} = A_{i,j,0} + A_{i,j,1}u + \cdots + A_{i,j,j-i-1}u^{j-i-1},
\]

where the matrices \( A_{i,j,r} \) are matrices over the field \( \mathbb{F}_q \).

We define the following torsion codes over \( \mathbb{F}_q \):

\[
C_i = \{ X \in (\mathbb{F}_q)^n \mid \exists Y \in ((u^i))^n \text{ with } Xu^{i-1} + Y \in C \},
\]

for \( i = 1 \ldots t \). It is then easy to see that these are linear \( q \)-ary codes, and we have:

**Proposition 2.** Let \( C \) be a linear \( R(q, t) \) code of length \( n \), and let \( C_i, \ i = 1 \ldots t \) be the torsion codes defined above. Then

1. \( C_1 \subseteq C_2 \subseteq \cdots \subseteq C_t \);
2. a generator matrix for the code \( C_1 \) is given by

\[
G_1 = (I_k \ A_{1,2,0} \ A_{1,3,0} \ \cdots \ A_{1,t+1,0});
\]
3. if \( G_i \) is a generator matrix for the code \( C_i \), then a generator matrix \( G_{i+1} \) for the code \( C_{i+1} \) is given by

\[
G_{i+1} = \begin{pmatrix}
0 & \cdots & 0 & G_i \\
I_{i+1} & \cdots & 0 & A_{i+1,t+2,0} & \cdots & A_{i+1,t+1,0}
\end{pmatrix}.
\]

**Proof.** Let \( X \in C_i \), then there exists \( Y \in ((u^i))^n \mid z := Xu^{i-1} + Y \in C \). Then \( uz \in C \). But \( uz = Xu^i + uY \in C \). Hence \( X \in C_{i+1} \). Now, let \( X \in C_1 \). Then there
exist vectors $Y_i, i = 1..t - 1$ over $(\mathbb{F}_q)^n$ such that $X + Y_1u + \cdots + Y_{t-1}u^{t-1} \in C$. Thus, the coefficients of $X$ must come from independent coefficients of elements on the first row-group of the generator matrix $G$. A similar reasoning indicates that at each stage, the remaining generators come from the independent coefficients of elements in the next row-group of the matrix $G$.

Note that the code $C_i$ has dimension $k_1 + \cdots + k_i$. The code $C$ then contains all products $[v_1, v_2, \ldots, v_t]G$ where the components of the vectors $v_i \in (R(q, t))^{k_i}$ have degree at most $t - i$. The number of codewords in $C$ is then $q^{(t)k_1 + (t-1)k_2 + \ldots + k_t}$, which can also be seen as $q^{k_1q^{k_1+k_2} \cdots q^{k_1+k_2+\ldots+k_t}}$. For the case $\mathbb{F}_2 + u\mathbb{F}_2$, the code $C_1$ is called the residue code, and the code $C_t = C_2$ is called the torsion code.

For $X \in C_i$, we know there exists $Y \in ((u^i))^n$ such that $Xu^{i-1} + Y \in C$. $Y$ can be written as

$$Y = u^i \overline{Y} + \text{hot}, \quad \text{with } \overline{Y} \in \mathbb{F}_q^n,$$

where ‘hot’ designates higher order terms. With this notation, define the map

$$F_i : C_i \rightarrow \mathbb{F}_q^n/C_{i+1}$$

by $F_i(X) = \overline{Y} + C_{i+1}$. If two such vectors $Y_1, Y_2 \in ((u^i))^n$ exist, we have

$$Y_1 = u^i \overline{Y}_1 + \text{hot} \quad \text{and} \quad Y_2 = u^i \overline{Y}_2 + \text{hot}.$$

Then,

$$Y_2 - Y_1 = u^i(\overline{Y}_2 - \overline{Y}_1) + \text{hot} \in C.$$

Therefore $\overline{Y}_2 - \overline{Y}_1 \in C_{i+1}$ and $F_i$ is well defined. It is easy to see that the maps $F_i$ are $\mathbb{F}_q$-morphisms. By its very definition, it can be seen that the image of these maps consist of direct sums of the matrices $A_{i,j,r}$ in a generator matrix $G$ for $C$ in standard form. We then have:

**Theorem 1.** Let $C$ be a code over $R(q, t)$ with a generator matrix $G$ in standard form. $C$ is determined uniquely by a chain of linear codes $C_i$ over $\mathbb{F}_q$ and $\mathbb{F}_q$-module homomorphisms $F_i : C_i \rightarrow \mathbb{F}_q^n/C_{i+1}$.

**Example 8.** If $G = (I_k, A)$ then $C_1 = C_2 = \cdots = C_t$. Also $k_i = 0$ for all $i \geq 2$ and hence the code $C$ has $(q^t)^k$ elements. These are called free codes since they are free $R(q, t)$-modules. Furthermore, if $A = A_0 + uB_1 + u^2B_2 + \cdots + u^{t-1}B_{t-1}$, where $B_i$ is a matrix over $\mathbb{F}_q$, then $C_1$ determines $A_0$ and $F_1(C_i)$ determines $B_i$.

**Example 9.** Let

$$G = \begin{pmatrix}
1 & 0 & 2 & 2+u & 1+u+u^2 \\
0 & 1 & 1 & 1+2u & u+u^2 \\
0 & 0 & u & 2u & u+u^2 \\
0 & 0 & 0 & u^2 & 2u^2
\end{pmatrix}$$
be a generator matrix for a code $C$ over $R(3, 3)$. The corresponding generator matrices for the linear codes are:

$$C_1 = \begin{pmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad \text{a } [5, 2, 3]\text{-code over } \mathbb{F}_3,$$

$$C_2 = \begin{pmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}, \quad \text{a } [5, 3, 2]\text{-code over } \mathbb{F}_3,$$

$$C_3 = \begin{pmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}, \quad \text{a } [5, 4, 1]\text{-code over } \mathbb{F}_3,$$

and the code $C$ has $(3^3)^2(3^2)^1(3)^1 = 27^3$ codewords.

Utilizing the torsion codes of $C$ we can define a new weight on $C$ and obtain a bound for their minimum distance.

**Definition 2.** Let $x \in R(q, t)$ and let $p$ be the characteristic of the field $\mathbb{F}_q$. Let $i_0 = \max\{i \mid x \in \langle u^i \rangle\}$. Define the $p$-weight of $x$ as $w_{tp}(x) = p^i$, if $x \neq 0$ and $w_{tp}(0) = 0$. For an element of $(R(q, t))^n$ define the $p$-weight as the sum of the $p$-weights of its coordinates.

**Note.** For the case $R(2, 2) = \mathbb{F}_2 + u\mathbb{F}_2$, the $p$-weight coincides with the Lee weight, and for $R(p, 2) = \mathbb{F}_p + u\mathbb{F}_p$, the $p$-weight coincides with the Bachoc weight defined in [Bachoc 1997].

**Theorem 2.** Let $C$ be a linear code over $R(q, t)$, and let $C_1, C_2, \ldots, C_t$ be the associated torsion codes over $\mathbb{F}_q$. Let $d_i$ be the Hamming distance of the codes $C_i$, then the minimum weight $d$ of the code $C$ with respect to the $p$-weight satisfies

$$\min\{p^{i-1}d_i \mid i = 1, \ldots, t\} \leq d \leq p^{t-1}d_t.$$

**Proof.** Let $W = (y_1, y_2, \ldots, y_n) \in C$ with minimum weight. Then for some $i$, $W = u^iX + Y$ with $Y \in \langle u^{i+1} \rangle$. Thus $X \in C_{i+1}$ and $w_{tp}(W) \geq p^i \cdot w_{tp}(X) \geq p^i d_{i+1}$. Now take $X_1 \in C_i$ to be a word of minimum weight $d_i$, then $u^{i-1}X_1 \in C$, and, by the minimality of $W$, we have $w_{tp}(W) \leq w_{tp}(u^{i-1}X_1) = p^{i-1}d_i$. \hfill \Box

It is well known [Bonnecaze and Udaya 1999; Ling and Sole 2001], that the Lee weight for a cyclic code $C$ over $\mathbb{F}_2 + u\mathbb{F}_2$ is the lower bound above. Here we show an example over $\mathbb{F}_2 + u\mathbb{F}_2$ that attains the upper bound.

**Example 10.** Let $C$ be the linear code over $\mathbb{F}_2 + u\mathbb{F}_2$, with generator matrix

$$G = \begin{pmatrix} 1 & 0 & u & 1 \\ 0 & 1 & 1+u & u \end{pmatrix}.$$
The codeword \((u, u, u, u)\) has Lee (or 2-) weight 8, while all the other nonzero codewords have weight 4. On the other hand \(C_1\) and \(C_2\) are equal with generator matrix

\[
G = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}.
\]

Hence \(d_1 = d_2 = 2\), and \(\min\{d_1, 2d_2\} = 2 \neq d\).

Since the \(p\)-weight coincides with the Lee weight for codes over \(\mathbb{F}_2 + u\mathbb{F}_2\), we obtain the general version for the Lee weight of those codes as a corollary of Theorem 2.

**Corollary 1.** The minimum Lee weight of a code \(C\) over \(\mathbb{F}_2 + u\mathbb{F}_2\), satisfies

\[
\min \{d_1, 2d_2\} \leq d \leq 2d_2
\]

where \(d_1, d_2\) are respectively the Hamming distance of the residue code \(C_1\) and the torsion code \(C_2\).

**Example 11.** Return to Example 9 over \(R(3, 3)\) with \(d_1 = 3, d_2 = 2, d_3 = 1\). Hence \(3 \leq d \leq 9\). The first and second generators combine to form a codeword of \(p\)-weight 3. Hence \(d = 3\), and in this example the minimum weight attains the lower bound.

**Example 12.** Let \(C\) be the linear code over \(\mathbb{F}_3 + u\mathbb{F}_3\), with generator matrix

\[
G = \begin{pmatrix}
1 & 0 & u & 2 \\
0 & 1 & 1+u & u
\end{pmatrix}.
\]

There are only 4 codewords with 2 zero entries, and they have Bachoc weight (and hence \(p\)-weight) 6. There are no codewords with Bachoc weight 3, and the Bachoc distance \(d\) of the code is 4. On the other hand the associated ternary codes are

\[
C_1 = C_2 = \begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 1 & 0
\end{pmatrix}.
\]

Thus \(d_1 = d_2 = 2\) and the Bachoc weight \(d\) lies strictly between the bounds given above.

**Corollary 2.** For free codes the \(p\)-weight \(d\) satisfies: \(d_1 \leq d \leq p^{t-1} d_1\).

We can also use the torsion codes to study the Hamming weight of the code \(C\). The results given here use a straightforward proof in comparison with the proof given in [Norton and Salagean 2000a].

For a code \(C\) over \(R(q, t)\) and \(w \in C\), we denote \(w_H(w)\) the usual Hamming weight of \(w\). Accordingly, the minimum Hamming distance of the code will be denoted by \(d_H(C)\).
Proposition 3. Let \( C \) be a linear code over \( R(q, t) \), and let \( C_1, C_2, \ldots, C_i \) be the associated torsion codes over \( \mathbb{F}_q \). Let \( d_i \) be the Hamming distance of the codes \( C_i \), then the minimal Hamming weight \( d_H \) of the code \( C \) satisfies

\[
    d_H = d_i \leq d_{i-1} \leq \cdots \leq d_1.
\]

Proof. Since \( C_i \subseteq C_{i+1} \), it follows that \( d_{i+1} \leq d_i \), for \( i = 1 \ldots t \). Now let \( X \in C_i \). Then \( X u^i \in C \) and hence \( d_H \leq d_i \). Conversely, let \( w^* \) be a codeword in \( C \) with minimum Hamming weight \( d_H \). Let \( j \) be the maximum integer such that \( u^j \) divides \( w^* \). Then \( w^* = u^j v \) and \( z = u^{i-j-1} w^* = u^{i-1} v \in C \). Thus \( \hat{v} \in C_i \), where \( \hat{v} \) denotes the canonical projection from \( R(q, t)^n \) into \( \mathbb{F}_q^n \). We then have \( w_H(w^*) \geq w_H(\hat{v}) \geq d_i \), and therefore \( d_H \geq d_i \). \( \square \)

From the above proof, the Singleton bound for \( C_i \), and the comment after Proposition 2, we have:

Corollary 3. Let \( C \) be a linear code over \( R(q, t) \), and let \( C_1, C_2, \ldots, C_i \) be the associated torsion codes. Then:

\[
    d_H \leq n - (k_1 + k_2 + \cdots + k_i) + 1.
\]

Proposition 4. \( \phi_B(C) \) is a \( [nt, \sum_{i=1}^{t} k_i (t - i + 1), d^*] \) linear code over \( \mathbb{F}_q \), with \( d^* \leq td_i \).

Proof. Since \( u^{i-1} \) divides \( y_j \) for each \( y_j \) in the \( i \)-th row-block of \( G \), \( u^s y_j = 0 \) for \( s \geq t - i + 1 \). Furthermore, the generators \( u^s y_j \neq 0 \) for \( s < t - i + 1 \) are linearly independent. Since there are \( k_i \) such \( y_j \), we have

\[
    \dim(\phi_B(C)) = \sum_{i=1}^{t} k_i (t - (i - 1)).
\]

\( \square \)

4. Self-dual codes over \( \mathbb{F}_q[u]/(u^t) \) using torsion codes

Duality for codes over \( \mathbb{F}_q[u]/(u^t) \) is understood with respect to the inner product \( x \cdot y = \sum x_i y_i \), where \( x_i, y_i \in R(q, t) \). As usual, a code is called self-dual if \( C = C^\perp \), and is called self-orthogonal is \( C \subseteq C^\perp \).

First, we give an examples of self-dual codes over \( R(q, t) \) of length \( n \) when \( t \) is even and \( n \) is a multiple of \( p \) (the characteristic of the field \( \mathbb{F}_q \)). The construction mimics the \( C_n \) codes studied by Bachoc [1997] for the case \( t = 2 \).

Example 13. For \( t \) even, let \( I = (u^{t/2}) \subseteq R(q, t) \). Define the set:

\[
    D_n := \{(x_1, x_2, \ldots, x_n) \in R(q, t)^n \mid \sum_{i=1}^{n} x_i = 0 \text{ and } x_i - x_j \in I \text{ for all } i \neq j \}.
\]
Let \( X, Y \in D_n \).
\[
X \cdot Y = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} (x_i - y_i)(y_i - x_i) + \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} x_1 y_i - n x_1 y_1.
\]
The first term is in \( I^2 = 0 \), the next two terms are zero by definition and the third term is zero since \( p \mid n \). Thus \( D_n \subseteq D_n^\perp \). Now, for each \( i = 1 \ldots n \), we can write \( x_i = a + b_i \) where \( a \) is a common polynomial of degree less than \( t/2 \), and \( b_i \in I \) with \( \sum b_i = 0 \). There are \( q^{t/2} \) choices for \( a \), and \( (q^{n-1})^{t/2} \) choices for the \( b_i \)'s, thus
\[
|D_n| = q^{t/2}(q^{n-1})^{t/2} = q^{nt/2},
\]
and hence \( D_n \) is self-dual.

The torsion \( q \)-ary codes are as follows: for \( i = 1, \ldots t/2 \), \( C_i \) is the code generated by the \( 1 \) word, with \( d_i = n \); and for \( i = t/2 + 1 \ldots t \), \( C_i \) is the parity check code of length \( n \) and dimension \( n - 1 \), thus \( d_i = 2 \). Applying Theorem 2, we obtain
\[
\min \{ n, 2p^{t/2} \} \leq d \leq 2p^{t-1}.
\]
But \( 1 \) and \( (0, 0, \ldots, 0, u^{t/2}, -u^{t/2}, 0, \ldots, 0) \in D_n \), hence \( d = \min \{ n, 2p^{t/2} \} \).

We study self-orthogonal and self-dual codes over \( R(q, t) \) taking two different approaches. We look at the linear codes \( \phi_B(C) \), and also look at the torsion codes corresponding to \( C \).

To study the latter we need some results on the parity check matrix of these codes, which can be defined in terms of block matrices using the recurrence relation
\[
D_{i,j} = \sum_{k=i+1}^{t+2-j} -B_{i,k} D_{k,j}
\]
for blocks, such that \( i + j \leq t + 1 \). For blocks such that \( i + j = t + 2 \), \( D_{i,j} = u^{-j} + I_{k,j} \) for \( i = 1, \ldots, t \) and \( D_{t+1,1} = I_{n - (k_1 + k_2 + \ldots + k_t)} \). All remaining blocks are 0. From here a generator matrix for the dual code can be obtained and we easily observe the following relations: \( k_1(C^\perp) = n - (k_1 + \ldots + k_t) \) and \( k_h(C^\perp) = k_{t-h+2}(C) \) for \( h = 2, \ldots, t \).

A different recurrence relation for the definition of the parity check matrix is given in [Norton and Salagean 2000a].

**Proposition 5.** Let \( C \) be an \( R(q, t) \) code, and let \( C_i \)'s be its corresponding torsion codes. Then
\[
(C^\perp)_i = (C_{t-i+1})^\perp, \quad i = 1, t.
\]

**Proof.** Let \( w \in (C^\perp)_i \) and \( v \in C_{t-i+1} \). Then there exists \( z \in (u^t)^n \) with \( a := wu^{i-1} + z \in C^\perp \), and \( y \in (u^{t-i+1})^n \) with \( b := vu^{-i} + y \in C \). Since \( a \cdot b = 0 \), we
have
\[0 = (wu^{i-1} + z) \cdot (vu^{t-i} + y) = (w \cdot v)u^{t-1},\]
which implies \(w \cdot v = 0\), and \(w \in (C_{t-i+1})^\perp\). So \((C^\perp)_i \subseteq (C_{t-i+1})^\perp\). Looking at dimensions
\[\dim((C^\perp)_i) = \sum_{j=1}^{i} k_j(C^\perp) = n - (k_1 + \ldots + k_i) + \sum_{j=2}^{i} k_{t-j+2}(C),\]
\[= n - \sum_{j=1}^{t-i+1} k_j(C) = n - \dim(C_{t-i+1}) = \dim((C_{t-i+1})^\perp). \quad \square\]

Using the generator in standard form of a code \(C\) and forming the inner products of its row-blocks we obtain:

**Proposition 6.** Let \(C\) be an \(R(q, t)\) code with a generator matrix in standard form. \(C\) is self-orthogonal if and only if
\[\sum_{h=0}^{k} \sum_{j=\max\{i,k\}}^{t+1} A_{i,j,h} A_{i,j,h-k}^T = 0, \quad \text{for each} \quad k = 0, \ldots, t - (i + l - 2) - 1.\]

This gives us the first characterization of self-dual codes:

**Theorem 3.** Let \(C\) be an \(R(q, t)\) code; and let \(C_i\)’s be its corresponding torsion codes. The code \(C\) is self-orthogonal and \(C_i = C_{t-i+1}^\perp\) if and only if \(C\) is self-dual.

**Proof.** By Proposition 5 we have \((C^\perp)_i = C_{t-i+1}^\perp = C_i\) for all \(i = 1 \ldots t\). Furthermore, \(\text{rk}(C) = \dim(C_i) = \dim((C^\perp)_i) = \text{rk}(C^\perp)\); but \(C\) is self-orthogonal, hence \(C = C^\perp\). Similarly, the converse follows immediately from Proposition 5. \(\quare\)

As an immediate consequence we have:

**Corollary 4.** If \(C\) is self-dual, then \(C_i\) is self-orthogonal for all \(i \leq (t + 1)/2\).

Note that when \(t\) is odd, \(C_{(t+1)/2}\) is self-dual and hence \(n\) must be even. For the case \(t\) even, we can construct self-dual codes of even or odd length.

Proposition 6 and Theorem 3 provide us with an algorithm to produce self-dual codes over \(R(q, t)\) starting from self-orthogonal codes over \(F_q\).

1. Take a self-orthogonal code \(C_1\) over \(F_q\).
2. Define \(C_t := C_{t-1}^\perp\).
3. Choose a set of self-orthogonal words \(\{R_1, R_2, \ldots, R_l\}\) in \(C_t\) that are linearly independent from \(C_1\). Define
\[C_2 := \langle C_1 \cup \{R_1, R_2, \ldots, R_l\} \rangle \quad \text{and} \quad C_{t-1} = C_2^\perp.\]
(4) Repeat, if possible, the step above defining \( C_i \) and \( C_{t-i+1} = C_i^\perp \) until you produce \( C_{\lfloor (t+1)/2 \rfloor} \).

(5) For each \( i = 1..t \), multiply the generators of \( \{C_{i+1} - C_i\} \) by \( u^i \). This will produce a self-dual code.

Additional self-dual codes are obtained as follows:

(6) Form a generator matrix \( G \) in standard form, adding, where appropriate, variables to represent higher powers of \( u \).

(7) Now we find the system of equations on the defined variables arising from Proposition 6. Note that for fixed \( i, l = 1 \ldots t \) each \( k \) will produce a matrix equation, which in turn produces several nonlinear equations.

(8) Write this system of equations in terms of the independent variables. There will be

\[
\sum_{i=1}^{\lfloor t/2 \rfloor} \sum_{j=i}^{t-i} (t - i - j + 1)k_i k_j
\]

equations on

\[
\sum_{i=1}^{t-1} \sum_{j=i+2}^{t+1} (j - i - 1)k_i k_j
\]

total variables.

(9) By Theorem 3 every solution to this system of equations will produce a self-dual code (some may be equivalent).

We now provide an example of this construction.

**Example 14.** Self-dual codes in \( F(3, 4) \):

Consider the self-orthogonal code

\[
C_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 2 \\
0 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}
\]

Define

\[
C_4 := C_1^\perp = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 2 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

Since there are no more self-orthogonal words in \( C_4 \) to append to \( C_1 \), we let \( C_2 := C_1 \), and since \( C_2^\perp = C_4 \) we let \( C_3 := C_4 \). Multiplying the rows in \( C_3 - C_2 \) by \( u^2 \) we obtain a generator matrix for a self-dual code over \( R(3, 4) \):
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 2 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & u^2 & 0 & 0 & 0 \\
0 & 0 & 0 & u^2 & 0 & 0 \\
\end{pmatrix}
\]

Now we can form a generator matrix using variables to represent higher powers of \( u \) obtaining
\[
\begin{pmatrix}
1 & 0 & a & b & u & 1+cu+du^2+eu^3 & 2+fu+gu^2+hu^3 \\
0 & 1 & i & j & u & 1+ku+lu^2+mu^3 & 1+nu+pu^2+qu^3 \\
0 & 0 & u^2 & 0 & r & su^3 \\
0 & 0 & 0 & u^2 & t & vu^3 \\
\end{pmatrix}.
\]

The equation
\[
\sum_{h=0}^{k} \sum_{j=\max\{i,k\}}^{t+1} A_{i,j,h} A^T_{i,j,k-h} = 0
\]
produces a system of equations over \( \mathbb{F}_q \). For example, for \( i = 1, l = 2, k = 3 \) we obtain the equation
\[
\begin{align*}
a + r + 2s &= 0, \\
b + t + 2v &= 0, \\
i + r + s &= 0, \\
j + t + v &= 0.
\end{align*}
\]

Likewise, the remaining equations can be obtained, and we solve in terms of a set of independent variables \( \{a, b, h, i, j, n, p\} \):
\[
\begin{align*}
c &= n, \\
d &= ai + bj + i^2 + j^2 + p + 2a^2 + 2b^2, \\
e &= n(ai + bj + i^2 + j^2 + 2n^2 + p) + h, \\
f &= n, \\
g &= a^2 + b^2 + ai + bj + i^2 + j^2 + n^2 + p, \\
k &= 2n, \\
l &= i^2 + j^2 + 2p + 2n^2, \\
m &= n(i^2 + j^2 + 2a^2 + 2b^2 + ai + bj) + 2h, \\
q &= n(a^2 + b^2 + p + 2ai + 2bj + 2n^2) + h, \\
r &= a - 2i, \\
s &= i - a, \\
t &= b - 2j, \\
v &= j - b.
\end{align*}
\]
These equations allow us to generate up to $3^7$ self-dual codes over $R(3, 4)$. As an example, letting all the independent variables take the value 1 except for $b = 0$, we obtain the self-dual code

$$\begin{pmatrix}
1 & 0 & u & 0 & 1+u+u^3 & 2+u+u^3 \\
0 & 1 & u & 0 & 1+2u+u^3 & 1+u+u^2+u^3 \\
0 & 0 & u^2 & 0 & 2u^3 & 0 \\
0 & 0 & 0 & u^2 & u^3 & u^3
\end{pmatrix}.$$

5. Self-dual codes over $\mathbb{F}_q[u]/(u^t)$ using linear images

As discussed in Section 2, given a code $C$ over $R(q, t)$ of length $n$ and a nonsingular $t \times t$ matrix $B$ over $\mathbb{F}_q$, we can define a linear code $\phi_B(C)$ over $\mathbb{F}_q$ of length $nt$. In this section, we will consider an element $x \in R(q, t)$ in its polynomial representation, and will use $\bar{x}$ for its vector representation.

Let $w = (w_1, w_2, \ldots, w_n)$ be a codeword in $C$. Recall that

$$\phi_B(w) = (\bar{w}_1 B, \bar{w}_2 B, \ldots, \bar{w}_n B).$$

Let $E$ denote the square matrix

$$\begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 0
\end{pmatrix}$$

over $\mathbb{F}_q$.

**Theorem 4.** If $C$ is self-orthogonal and $BB^T = cE$ where $c \neq 0 \in \mathbb{F}_q$, then $\phi_B(C)$ is self-orthogonal.

**Proof.** Let $R_j$ denote the $j$-th row of $B$. Then $R_j R_k^T = c$, for all $j + k < t + 2$ and $R_j R_k^T = 0$, for all $j + k \geq t + 2$. If $w, v \in C$, then

$$\phi_B(w)\phi_B(v)$$

$$= \sum_{i=1}^{n} \bar{w}_i B(\bar{v}_i B)^T = \sum_{i=1}^{n} \bar{w}_i B B^T \bar{v}_i$$

$$= \sum_{i=1}^{n} \sum_{j,k=0}^{t-1} w_{i,j} R_{j+1} R_{k+1}^T v_{i,k} = c \sum_{i=1}^{n} \sum_{j+k<t} w_{i,j} v_{i,k} + 0 \sum_{i=1}^{n} \sum_{j+k \geq t} w_{i,j} v_{i,k},$$

but since $C$ is self-orthogonal, the sum in the first term is 0. Therefore,

$$\phi_B(w)\phi_B(v) = 0,$$

and thus $\phi_B(C)$ is self-orthogonal. \qed
Corollary 5. If $C$ is self-dual, $BB^T = cE$, and
$$\sum_{i=2}^{t} k_i(t - 2i + 2) = 0,$$
then $\phi_B(C)$ is self-dual.

Proof. Splitting the equation from the hypothesis we have
$$\sum_{i=2}^{t} k_i(t - i + 1) = \sum_{i=2}^{t} k_i(i - 1),$$
$$2\sum_{i=2}^{t} k_i(t - i + 1) = \sum_{i=2}^{t} k_i(i - 1) + \sum_{i=2}^{t} k_i(t - i + 1) = \sum_{i=2}^{t} tk_i,$$
$$2\sum_{i=1}^{t} k_i(t - i + 1) = 2k_1t + \sum_{i=2}^{t} tk_i.$$
Since $C$ is self-dual, we know
$$C_1^\perp = C_t$$
and
$$\dim(C_t) = \text{rk}(C).$$
Thus,
$$\dim(C_1^\perp) = \text{rk}(C)$$
and
$$n - k_1 = \sum_{i=1}^{t} k_i.$$
Therefore,
$$2\sum_{i=1}^{t} k_i(t - i + 1) = nt,$$
making the length of $\phi_B(C)$ twice its dimension. By Theorem 4, $\phi_B(C)$ is self-orthogonal and hence $\phi_B(C)$ is self-dual. □

Let $M, N$ be two matrices over $\mathbb{F}_q$. We say they are root-equivalent ($M \sim N$) if $M$ can be obtained from $N$ by a column permutation, or a column multiplication by an element $\alpha \in \mathbb{F}_q$ such that $\alpha^2 = 1$. This implies $MM^T = NN^T$, and by the definition of $\phi_B$, we obtain the following

Corollary 6. If $B \sim D$ in the hypothesis of Corollary 5 then $\phi_B(C)$ and $\phi_D(C)$ are equivalent self-dual codes.

Example 15. For $R(3,3)$, all matrices $B$ that satisfy $BB^T = cE$ are root-equivalent, and therefore produce equivalent codes. Hence we can restrict ourselves to just one such matrix, for example,
$$B = \begin{pmatrix}
1 & 1 & 0 \\
0 & 2 & 1 \\
1 & 1 & 1
\end{pmatrix}.$$
ON DISTANCES AND SELF-DUAL CODES OVER $F_q[u]/(u')$

The cases of $R(2,2)$ and $R(3,3)$ are singular. For $R(3,4)$ we have 6 different classes of root-equivalent matrices.

In general, note that there exist self-dual codes $A$ and matrices $B$ with $BB^T \neq cE$ whose image $\phi_B(A)$ is self-dual. For example, consider the self-dual code $A$ over $R(3,4)$ with a generator matrix

$$G = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1+2u+u^2 & 1+2u & 1+2u & 1+2u & 1+2u & 1+2u \\
1+u^2 & 1+u^2 & 1+u^2 & 1+u^2 & 1+u^2 & 1+u^2 \\
u+u^2 & u & u & u+u^2 & u+u^2 & u \\
0 & 0 & 0 & 0 & 0 & u^2 \\
0 & 0 & 0 & 0 & 0 & 2u^2
\end{pmatrix}.$$

Passing to standard form,

$$G_1 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & u^2 & 0 & 0 & 0 & 2u^2 \\
0 & 0 & u^2 & 0 & 0 & 2u^2 \\
0 & 0 & 0 & u^2 & 0 & 2u^2 \\
0 & 0 & 0 & 0 & u^2 & 2u^2
\end{pmatrix}.$$

Consider the matrix

$$B = \begin{pmatrix}
1 & 0 & 0 \\
1 & 2 & 1 \\
0 & 1 & 2 \\
2 & 1 & 1
\end{pmatrix},$$

for which $BB^T \neq cE$ for any $c$. The image code $\phi_B(A)$ is a self-dual code:

$$\phi_B(A) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 0 & 2 & 2 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 2
\end{pmatrix}.$$

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ralfaro@umflint.edu Mathematics Department, University of Michigan–Flint, Flint, MI 48502, United States

stbennet@umflint.edu Mathematics Department, University of Michigan–Flint, Flint, MI 48502, United States

joshuaha@umflint.edu Mathematics Department, University of Michigan–Flint, Flint, MI 48502, United States

cthornbu@umflint.edu Mathematics Department, University of Michigan–Flint, Flint, MI 48502, United States