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Given a graph G, a k-ranking is a labeling of the vertices using k labels so that every path between two vertices with the same label contains a vertex with a larger label. A k-ranking f is *minimal* if for all $v \in V(G)$ we have $f(v) \leq g(v)$ for all rankings g. We explore this problem for directed graphs. Here every directed path between two vertices with the same label contains a vertex with a larger label. The *rank* number of a digraph D is the smallest k such that D has a minimal k-ranking. The *arank* number of a digraph is the largest k such that D has a minimal k-ranking. We present new results involving rank numbers and arank numbers of directed graphs. In 1999, Kratochvíl and Tuza showed that the rank number of an oriented of a tree is bounded by one greater than the rank number of its longest directed path. We show that the arank analog does not hold. In fact we will show that the arank number of an oriented tree can be made arbitrarily large where the largest directed path has only three vertices.

1. Introduction

A labeling $f : V(G) \to \{1, 2, \ldots, k\}$ is a *k-ranking* of a graph G if, whenever $f(u) = f(v)$, every path joining u and v contains a vertex w such that $f(w) > f(u)$. A k-ranking f is *minimal* if $f(v) \leq g(v)$ for all $v \in V(G)$ and all rankings g. A ranking f has a *drop vertex* x if the labeling defined by $g(v) = f(v)$ when $v \neq x$ and $g(x) < f(x)$ is still a ranking. It was shown in [Jamison 2003; Isaak et al. 2009] that a ranking is minimal if and only if it contains no drop vertices. When the value of k is unimportant, we will refer to a k-ranking simply as a ranking.

Recall that an oriented graph is a directed graph D where for each pair of vertices x, y either (x, y) or (y, x) is an arc in D and D contains no self-loops. We will refer to a path where all of the arcs have the same orientation as a *directed path*. A path where the arcs are alternately oriented (so that each vertex is either a source or a sink) will be referred to as an *antidirected path*. An *undirected path* is simply a path where the edges are not oriented.

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We recall that a vertex coloring of a graph is a labeling of the vertices so that no two adjacent vertices receive the same label. Hence a k-ranking is a vertex coloring with an additional condition imposed. Following along the lines of the chromatic number, the *rank number of a graph* $\chi_r(G)$ is defined to be the smallest k such that G has a minimal k-ranking. At the other extreme, the *arank number of a digraph* $\psi_r(G)$ is defined to be the maximum k such that D has a minimal k-ranking. These rankings are known as *arankings* or *maximum minimal rankings*. We explore rankings and arankings of directed graphs. Here any directed path between two vertices with the same label contains a vertex with a higher label. The rank and arank number of oriented graphs are defined in the same way as they were for undirected graphs.

Early studies involving the rank number for undirected graphs were motivated by its numerous applications including its role in the design of very large scale integration (VLSI) layout and Cholesky factorizations associated with parallel processing [de la Torre et al. 1992; Ghoshal et al. 1999; Sen et al. 1992]. One of the first results involving minimal rankings was by Bodlaender et al. [1998], who determined the rank number of a path of length n to be

$$\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1.$$

A ranking of this form can be obtained by labeling the vertices $\{v_i \mid 1 \le i \le n\}$ with $\alpha + 1$ where 2^α is the highest power of 2 dividing i and this ranking is unique when n is a power of 2. We will refer to this ranking as the *standard ranking of a path*.

Many papers have since appeared on rankings of undirected graphs [Dereniowski 2006; 2004; Dereniowski and Nadolski 2006; Flórez and Narayan 2009; Ghoshal et al. 1999; 1996; Hsieh 2002; Jamison 2003; Kostyuk and Narayan $\geq$ 2009; Kostyuk et al. 2006; Leiserson 1980; Laskar and Pillone 2001; 2000; Novotny et al. 2009]. However only two papers are known to have investigated the ranking of oriented graphs, and to date there have not been any papers on the arank number of an oriented graph. Kratochvíl and Tuza [1999] gave a general bound on the ranking number of oriented trees. They also proved that deciding whether the rank number of an oriented graph is bounded by a constant is NP-complete. Flórez and Narayan [2009] established new results involving the rank number for all orientations of a cycle. In this paper, we build upon known results for oriented graphs and present the first results involving oriented graphs and arank numbers.

Kratochvíl and Tuza [1999] showed that the rank number of an oriented tree is bounded by one plus the rank number of its longest directed path. In Theorem 10 we show that this property does not hold for the arank number. In fact we will show that the arank number of an oriented tree can be made arbitrarily large where the longest directed path has three vertices.

2. Rankings for oriented graphs

In this section we begin by determining the rank and arank numbers for orientations of stars. Later we investigate orientations of a tree.

2.1. *Oriented stars.* The next two theorems give the rank and arank number of an oriented star. We show that for oriented stars on two or more vertices, the rank number is 2 and the arank number is either 2 or 3.

Theorem 1. *Let $DS(n)$ be a digraph that is any orientation of a star. Then*

$$\chi_r(DS(n)) = 2.$$

Proof. A minimal 2-ranking can be formed by labeling the center vertex with a 2 and all other vertices with a 1. □

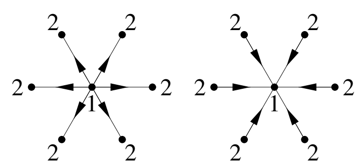
Theorem 2. *Let $DS^+(n)$ denote a directed out-star, $DS^-(n)$ a directed in-star, and $DS^H(n)$ the directed hybrid star that contains a directed P_3. Then*

(i) $\psi_r(DS^+(n)) = 2$,

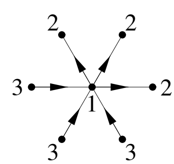
(ii) $\psi_r(DS^-(n)) = 2$,

(iii) $\psi_r(DS^H(n)) = 3$.

Proof. We consider the digraphs $DS^+(n)$ and $DS^-(n)$. There are only two possible rankings. First, if the center of the star is labeled 1, each vertex of degree 1 must be labeled 2, leading to the following minimal 2-rankings:



Next we consider $DS^H(n)$. Suppose we have a minimal ranking with the center vertex labeled 1. Then any directed path with labels a-1-b could be relabeled with 2-1-3, leading to a minimal 3-ranking such as the one on the right.



Hence any minimal k-ranking of $DS^H(n)$ must have $k \leq 3$. □

2.2. *Oriented paths.* We begin by recalling a theorem of [Bodlaender et al. 1998] that gives the rank number of an undirected path.

Theorem 3. $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$.

The directed case the follows immediately.

Corollary 4. *Let $\vec{P}_n$ denote the path on n vertices where all of the arcs have the same direction. Then $\chi_r(\vec{P}_n) = \lfloor \log_2 n \rfloor + 1$.*

Recall that the antidirected path AP_n is a path on vertices $v_1, v_2, \ldots, v_n$ where the arcs alternate in direction.

Theorem 5. *Let AP_n be the antidirected path on n vertices. Then $\chi_r(AP_n) = 2$.*

Proof. Each vertex of an antidirected path is either a source or a sink. Labeling each source with a 1 and each sink with a 2 creates a minimal ranking. □

We restate a known result involving the rank number of an oriented path.

Theorem 6 [Kratochvíl and Tuza 1999]. *Let P_l be the longest directed path contained in the orientation of a path OP_n. Then $\chi_r(OP_n) = \chi_r(P_l)$ or $\chi_r(P_l) + 1$.*

We next consider the arank number of an oriented path. For some oriented paths we may simply join together two ψ_r-rankings on the directed subpaths. Consider the example $\overset{3}{\bullet}\rightarrow\overset{1}{\bullet}\rightarrow\overset{2}{\bullet}\leftarrow\overset{1}{\bullet}$. The first three vertices are labeled according to a ψ_r-ranking of a directed P_3 and the last two vertices are labeled according to a ψ_r-ranking of P_2. Note that there is an overlap on the vertex of the third label. Since none of these labels can be reduced the arank number of this oriented path is at least 3. A 4-ranking would imply that each vertex receives a different label. In either case the label of the end vertex not adjacent to a vertex labeled 1 can be reduced to a 1. Hence the rank number is 3 which equals the arank number of its longest directed path.

Next consider $\overset{1}{\bullet}\rightarrow\overset{2}{\bullet}\leftarrow\overset{3}{\bullet}\rightarrow\overset{1}{\bullet}$. The arank number is at least 3 since no label can be reduced. However the arank number of its longest directed path is 2. It would seem that the difference between the arank number of an oriented path and the arank number of its longest directed path can differ by at most 1. This is in fact the case. Before proving this result we state the following lemma.

Lemma 7. *Let P_n be a path on vertices $v_1, v_2, \ldots, v_n$. Then there exists a ψ_r-ranking f of P_n where $f(v_1) = \psi_r(P_n) + 1$.*

Proof. We first find the largest value less than or equal to m that is either one less than a power of 2, or one less than the average of two consecutive powers of 2. A construction was given in [Kostyuk et al. 2006] showing that ψ_r-rankings can be constructed for paths with these lengths where the endpoints receive the largest two labels. We can construct a ranking for P_n by starting with the endpoint with the largest label and extending the other end of the path, labeling additional vertices so that the new vertex i is labeled $\alpha + 1$ where 2^α is the largest power of 2 that divides i.

By the monotonicity property $\psi_r(P_s) \le \psi_r(P_t)$ whenever $s \le t$ mentioned in [Ghoshal et al. 1996] it follows that this ranking is a ψ_r-ranking. □

We next prove the main result of this section.

Theorem 8. *Let OP_n be an orientation of a path with longest directed path P_l. Then $\psi_r(OP_n) = \psi_r(P_l)$ or $\psi_r(P_l) + 1$.*

Proof. Let OP_n be the union of oppositely directed paths $P_{i_1}, P_{i_2}, \ldots, P_{i_j}$. We will proceed by induction on j. When $j = 1$ the result is immediate. We assume the result is true for $j-1$. There are two cases to consider depending on whether or not the P_{i_j} has the largest arank number of all paths in OP_n.

Case (i): $\psi_r(P_{i_j}) \geq \psi_r(P_{i_k})$ for all k, $1 \leq k \leq j$. Then by Lemma 7 the vertices of P_{i_j} can be labeled using a ψ_r-ranking where the largest label is given either to the first or last vertex of this path. We may need to use a larger label for this vertex; however, any larger label for this vertex can be reduced to at most $\psi_r(P_{i_j})+1$. We may have to reduce other labels to obtain a minimal ranking, but since these reductions will not increase the largest label we have $\psi_r(OP_n) \leq \psi_r(P_l)+1$.

Case (ii): $\psi_r(P_{i_j}) < \max_k \psi_r(P_{i_k})$ for all k, $1 \leq k \leq j$. We append a ψ_r-ranking of P_{i_j} as in Case (i). Since the largest label did not increase we again have $\psi_r(OP_n) \leq \psi_r(P_l)+1$. □

3. Arankings for oriented trees

We recall the following theorem involving the rank number of an oriented tree.

Theorem 9 [Kratochvíl and Tuza 1999]. *Let P_l be the longest directed path contained in the orientation of a tree T_n. Then $\chi_r(T_n) = \chi_r(P_l)$ or $\chi_r(P_l)+1$.*

Our next theorem shows that the analog does not hold for the arank number of a directed path. In fact we will show that the arank number of an oriented tree can be made arbitrarily large where the longest directed path has only three vertices.

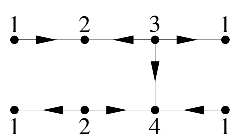
Theorem 10. *For any positive integer t, there exists a directed tree T without a directed P_3 such that $\psi_r(D) = t$.*

Proof. When $t=1$, D consists of a single vertex with no edges. When $t=2$, D is a directed K_2.

For the case where $t=3$, the minimal 3-ranking 

shows that $\psi_r(D) \geq 3$.

This same digraph can be extended to one with a minimal 4-ranking as follows:



We now give a general extension from the case where $t=j$ to the case where $t=j+1$.

Suppose $t=j$. We start with the digraph D which contains a vertex v such that $f(v)=j$. Let D' be a copy of the digraph D, with the orientation of all of the arcs in D' reversed. We note that the digraph D' contains a vertex v' such that $f(v')=j$. We then construct the graph D^* from D, D' along with an arc between v and v', where the direction is from the source to the sink. In D^* we let $f(v')=j+1$, and label all other vertices as they were in D or D'. The construction gives a minimal $(j+1)$-ranking of D^*. Hence $\psi_r(D^*) \geq j+1$. □

4. Conclusion

At the current time, the rank and arank number are only known for a few families of graphs. Even less is known about these numbers for oriented graphs. We note that Theorem 8 suggests the problem of partitioning oriented paths into two classes. Class 1 contains all oriented paths OP_n where $\psi_r(OP_n) = \psi_r(P_l)$ and Class 2 contains of all paths where $\psi_r(OP_n) = \psi_r(P_l) + 1$.

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