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Karen Zangara and John Marafino

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Applications of full covers in real analysis

Karen Zangara and John Marafino

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In this paper we briefly introduce the reader to the concept of full covers and indicate how it can be used to prove theorems in an undergraduate analysis course. The technique exposes the student to the idea of covering an interval $[a, b]$ with a collection of sets and then extracting from this collection a subcollection that *partitions* $[a, b]$. As a consequence, the student is furnished with a unifying thread that ties together and simplifies the proofs of many theorems.

1. Introduction

We were first drawn to the concept of full covers after reading two papers by Botsko [1987; 1989]. We then pursued this idea in [Klaimon 1990] and will now provide full covering arguments for four more theorems: the Lebesgue Number Lemma, the Intermediate Value Theorem for Derivatives, Baire's Theorem, and Ascoli's Theorem.

The following definition and lemma are used in full covering arguments:

Definition. Let $[a, b]$ be a closed, bounded interval. A collection C of closed subintervals of $[a, b]$ is a *full cover* of $[a, b]$ if, for each x in $[a, b]$, there corresponds a number $\delta > 0$ such that every closed subinterval of $[a, b]$ that contains x and has length less than δ belongs to C.

Thomson's Lemma. *If C is a full cover of $[a, b]$, then C contains a partition of $[a, b]$. In other words, there is a partition of $[a, b]$ all of whose subintervals belong to C.*

The proof of the lemma is based upon a bisection argument and is easily accessible to undergraduates. It can be found in [Botsko 1987], [Botsko 1989] and [Thomson 1980]. Thomson's lemma is similar in concept and execution to the version of the Heine–Borel theorem, which states that every open cover of a closed

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interval has a finite subcover. So the proof of any theorem that uses the Heine–Borel theorem can be rewritten to use full covers. Our proof of the Lebesgue Number Lemma illustrates this.

The advantage of full covering arguments lies in that the resulting finite subcover of $[a, b]$ is a *partition* of the interval. This extra condition can often be used to streamline and simplify the proofs of certain theorems. For instance, in the proof of the Intermediate Value Theorem for derivatives, we obtain a finite subcover $\{J_k : k = 1, 2, \dots, m\}$ of $[a, b]$ having the property that a function f defined on $[a, b]$ has a constant sign on each J_k. Since $\{J_k : k = 1, 2, \dots, m\}$ is a partition of closed subintervals of $[a, b]$, we can order the J_k so that J_k abuts J_{k+1} to the left, J_1 contains a, and J_m contains b. It is then trivial to see that the sign of f on J_1 determines the sign of f on each of the J_k, $k = 2, 3, \dots, m$, and hence, determines the sign of f on $[a, b]$.

Another instance occurs in Baire's Theorem. In the proof we have a function f defined on $[a, b]$ and a finite subcollection $\{J_k : k = 1, 2, \dots, m\}$ such that f is bounded above on each J_k by a constant function Φ_k. Since $\{J_k : k = 1, 2, \dots, m\}$ forms a partition of $[a, b]$, the intervals can easily be ordered as before. Trivially, the interiors of J_k do not intersect, so each interior point of J_k can be associated with only one Φ_k value. By moving from left to right on $[a, b]$, we then indicate a well defined procedure that connects the graphs of each Φ_k and defines a continuous function h_ε on $[a, b]$. Finally, the full covering argument offers a very efficient iterative method for proving Ascoli's Theorem.

We close this section by listing those theorems that have been proved using this technique and that a student would normally encounter in an undergraduate analysis course. We categorize them under four main topics and give a reference for their proof.

Topology:

Heine–Borel Theorem [Botsko 1987, p. 452]. *Any open cover of* $[a, b]$ *has a finite subcover.*

Bolzano–Weierstrass Theorem [Botsko 1987, p. 452]. *If S is a bounded infinite set of real numbers, then S has an accumulation point.*

Continuity and differentiability:

Theorem [Botsko 1987, p. 451]. *If f is continuous on* $[a, b]$, *then f is bounded on* $[a, b]$.

Theorem [Klaimon 1990, p. 156]. *If f is a continuous function on* $[a, b]$, *there exists points M and m on* $[a, b]$ *such that $f(M) \geq f(x)$ and $f(m) \leq f(x)$ for all x on* $[a, b]$.

Theorem [Botsko 1987, p. 452]. *If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.*

Intermediate Value Theorem [Botsko 1987, p. 451]. *If f is continuous on $[a, b]$ with $f(a)f(b) < 0$, then there exists x_0 on (a, b) such that $f(x_0) = 0$.*

Remark. The same cover employed to prove this theorem will be used below to prove the Intermediate Value Theorem for derivatives.

Theorem [Botsko 1989, p. 331; Klaimon 1990, p. 158]. *If $f'(x) = 0$ for all x on $[a, b]$, then f is constant on $[a, b]$.*

Theorem [Klaimon 1990, p. 160]. *If $f'(x) > 0$ (< 0) on (a, b), then f is increasing (decreasing) on (a, b).*

Remark. The proofs of these two theorems do not use the Mean Value Theorem as is typically done.

Rolle's Theorem [Klaimon 1990, p. 157]. *If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, there exists a point x_0 on (a, b) such that $f'(x_0) = 0$.*

Integration:

Theorem [Botsko 1989, p. 330]. *If f is continuous on $[a, b]$, then f is Riemann integrable.*

Theorem [Botsko 1989, p. 331]. *If f is bounded on $[a, b]$ and continuous almost everywhere, then f is Riemann integrable on $[a, b]$.*

Remark. By slightly modifying the cover in the first theorem, Botsko is able to prove the stronger second theorem! This also works with other results, as pointed out in [Botsko 1989].

Sequences:

Dini's Theorem [Klaimon 1990, p. 159]. *If $f_n(x)$ is a sequence of continuous functions on $[a, b]$ and $f_n(x) < f_{n+1}(x)$ for all n and for all x in $[a, b]$, and if $f_n(x)$ converges to $f(x)$ where $f(x)$ is continuous on $[a, b]$, then $f_n(x)$ converges uniformly to $f(x)$.*

2. Four theorems proven using full covers

Lebesgue Number Lemma. *Let $\Im$ be an open cover of $[a, b]$. There exists a number $\eta > 0$ such that if B is any subset of $[a, b]$ with diameter $B < \eta$, then there exists a set $A \in \Im$ such that $A \supseteq B$.*

Proof. For each x in $[a,b]$ and $\varepsilon > 0$, set $I_\varepsilon[x] = [x-\varepsilon, x+\varepsilon]$. Let

$$C = \{I_\varepsilon[x] : I_\varepsilon[x] \subset [a,b] \text{ and } I_{2\varepsilon}[x] \text{ is a subset of some } A \in \Im\}.$$

We first show that C is a full cover of $[a,b]$. Let x be an element of $[a,b]$. Since $\Im$ is an open cover of $[a,b]$, there exists an A in $\Im$ that contains x. Because A is open, one can find a $\delta = \delta(x)$ such that $I_{2\delta}[x]$ is a subset of A. Let J be a closed subinterval of $[a,b]$ containing x such that $|J| < \delta$. We can write J as $I_\varepsilon[x']$, where x' is the midpoint of J and $\varepsilon = \delta/2$. Since $I_{2\varepsilon}[x']$ is a subset of A, J is in C and so C is a full cover of $[a,b]$.

By Thomson's Lemma, C contains a partition of $[a,b]$; that is, there exists $a = p_0 < p_1 < \cdots < p_m = b$ such that $[p_{k-1}, p_k] = I_{\varepsilon(k)}[x_k]$ is in C for $k = 1, 2, \dots, m$. Note that $x_k = (p_k + p_{k-1})/2$ and $\varepsilon(k) = |p_{k-1} - p_k|/2$.

Let $\eta = \min\{\varepsilon(k)\}$ and let B be a subset of $[a,b]$ with diameter $B < \eta$. B intersects $I_{\varepsilon(k)}[x_k]$ for some $k = 1, \dots, m$ and hence B is a subset of $I_{2\varepsilon(k)}[x_k]$, which is contained in some $A \in \Im$. □

Intermediate Value Theorem for derivatives. *If $f(x)$ is the derivative for some function $g(x)$ on an open interval containing $[a,b]$, and if $f(a)f(b) < 0$, then there exists an x_0 in (a,b) such that $f(x_0) = 0$.*

Proof. Suppose to the contrary that for all x on (a,b), $f(x) \neq 0$. Then $f(x) \neq 0$ on $[a,b]$. Let

$$C = \{I : I \text{ is a closed subinterval of } [a,b] \text{ and } f(x) \text{ has one sign on } I\}.$$

Let x be in $[a,b]$. Assume for definiteness that $f(x) > 0$. We claim that there exists a δ neighborhood about x such that $f(y) > 0$ for all y in this neighborhood. Suppose the claim is false. Then one can find a sequence $\{y_n\}$, where $y_n \to x$ as $n \to \infty$ and $f(y_n) < 0$. Again, for definiteness, suppose $\{y_n\}$ approaches x from the left. Since $f(x) > 0$, there exists a δ_1 such that when $|h| < \delta_1$, $g(x+h) < g(x)$ if $h < 0$. By choosing n large enough, one can find y_n such that $|y_n - x| < \delta_1$ and so $g(y_n) < g(x)$. Since $f(y_n) < 0$, there exists a δ_2 neighborhood about y_n that is contained in the δ_1 neighborhood about x such that if $|h| < \delta_2$ and $h < 0$, then $g(y_n) < g(y_n + h)$. Choose $|h'| < \delta_2$ and $h' < 0$. Then $y_n + h' < y_n < x$ and $g(y_n) < g(y_n + h') < g(x)$. By the Intermediate Value Theorem for Continuous Functions, there exists an x_n in $[y_n, x]$ such that $g(x_n) = g(y_n + h')$. Using Rolle's Theorem on $[y_n + h', x_n]$ one can find an x_0 in $[y_n + h', x_n]$ such that $f(x_0) = 0$. This contradicts our assumption that $f(x) \neq 0$ for all x in $[a,b]$. So there must be a $\delta > 0$ such that $f(y) > 0$ for all y satisfying $|y - x| < \delta$. Let J be a closed interval containing x with $|J| < \delta$. Then J is in C. A similar δ can be found if $f(x) < 0$. Thus C is a full cover of $[a,b]$.

Using Thomson's Lemma, C contains a partition of $[a, b]$; that is, there exist $a = p_0 < p_1 < \cdots < p_m = b$ such that $[p_{k-1}, p_k] = I_k$ is in C for $k = 1, 2, \ldots, m$. Suppose $f(x) > 0$ on I_1. Since the intervals overlap at the endpoints, we have $f(x) > 0$ on I_k for $k = 2, \ldots, m$. But this contradicts our assumption $f(a)f(b) < 0$. The same contradiction results if $f(x) < 0$ on I_1. So our original assumption that $f(x) \neq 0$ on (a, b) is false. Consequently, there exists a point x_0 on (a, b) such that $f(x_0) = 0$. □

Next, we present the proofs of two sequence theorems. For these theorems we will need the following definitions. The function f is upper (lower) semi-continuous at x if $\limsup_{y \to x} f(y) \leq f(x)$ ($\liminf_{y \to x} f(y) \geq f(x)$). If f is upper (lower) semi-continuous for all x on $[a, b]$, then f is upper (lower) semi-continuous on $[a, b]$. The family of functions Ω is equicontinuous on $[a, b]$ if for each x in $[a, b]$ and $\varepsilon > 0$, there exists a $\delta = \delta(x, \varepsilon)$ such that if $|y - x| < \delta$, then $|f(y) - f(x)| < \varepsilon$ for all f in Ω. The family of functions Ω is uniformly bounded on $[a, b]$ provided there exists a constant $M > 0$ such that $|f(x)| < M$ for all $x \in [a, b]$ and for all $f \in \Omega$.

Baire's Theorem. *Let f be upper* (*lower*) *semi-continuous on* $[a, b]$ *and bounded above* (*below*) *by M on* $[a, b]$. *Then there exists a sequence of continuous functions* $\{h_n\}$ *such that, for all x in* $[a, b]$,

(i) $M \geq h_1(x) \geq \cdots \geq h_n(x) \geq \cdots \qquad (M \leq h_1(x) \leq \cdots \leq h_n, (x) \leq \cdots)$,

(ii) $\lim_{n \to \infty} h_n(x) = f(x)$.

Proof. Let M be the upper bound of $f(x)$ on $[a, b]$ and let ε be an arbitrarily small positive number.

Define

$$C = \big\{J : J \text{ is a closed subinterval of } [a, b],\ |J| < \varepsilon, \text{ and there exists } x \text{ in } J \text{ such that } f(y) \leq f(x) + \varepsilon \text{ for all } y \text{ in } J\big\}.$$

Let x be an element in $[a, b]$. Since f is upper semi-continuous at x, there is a $\delta(x) > 0$ such that $|y - x| < \delta(x)$ implies $f(y) \leq f(x) + \varepsilon$. We can further assume that $\delta(x) < \varepsilon$. Now let J be any closed interval of $[a, b]$ containing x with $|J| < \delta(x)$. If we set $\Phi(y) = f(x) + \varepsilon$ for all $y \in J$, then $f(y) < \Phi(y)$ on J. Thus J is in C and C is a full cover of $[a, b]$.

By Thomson's Lemma there exists a partition $\{J_k, k = 1, 2, \ldots m\}$ of $[a, b]$ contained in C. Hence, $|J_k| < \varepsilon$ and on each J_k, $k = 1, 2, \ldots, m$, there is a point x_k such that the constant functions $\Phi_k(x) = f(x_k) + \varepsilon$ defined on J_k satisfy $f(x) \leq \Phi_k(x)$ on J_k.

We first construct a function h_ε that will approximate f in a sense to be clarified below. For each k, $k = 1, 2, \ldots, m$, let Φ_k also denote the constant value of $\Phi_k(x)$

on J_k. If $\Phi_1 > \Phi_2$, then connect the horizontal graph of $\Phi_1(x)$ to that of $\Phi_2(x)$ by the line segment P_1P_2 where P_1 coincides with the end point of the graph of $\Phi_1(x)$ and P_2 lies $1/3$ of the distance on the graph of $\Phi_2(x)$. If $\Phi_1 < \Phi_2$, then connect the horizontal graphs by the line segment P_1P_2 where P_1 is $2/3$ of the distance on the graph of $\Phi_1(x)$ and P_2 coincides with the initial point of the graph of $\Phi_2(x)$. Doing this for $k = 1, 2, \ldots, m$, we can construct a continuous function $h_\varepsilon(x)$ on $[a, b]$ with the property $f(x) \le h_\varepsilon(x)$ for all x on $[a, b]$. We will say that h_ε approximates f in the following sense. Given any $x \in [a, b]$, $x \in J_k$ for some k and either $(x, h_\varepsilon(x))$ will be on a horizontal step of h_ε or it will be on a line segment connecting two consecutive steps. In either case, it is evident that

$$h_\varepsilon(x) \le \sup_{|y-x|<2\varepsilon} f(y) + \varepsilon,$$

since all J_k are of width at most ε.

We now construct the desired sequence $\{h_n\} \downarrow f$. For each n, let $\varepsilon = 1/n$ and let g_n be a function constructed as h_ε was above. Define $h_1 = \inf(g_1, M)$, $h_2 = \inf(g_2, h_1)$, $h_3 = \inf(g_3, h_2)$, and so on. For each x in $[a, b]$, the sequence $\{h_n(x)\}$ converges, since it is decreasing and bounded below by $f(x)$. Moreover, $\lim_{n\to\infty} h_n(x) \ge f(x)$. Since

$$h_n(x) \le \sup_{|y-x|<2/n} f(y) + 1/n,$$

it follows that $\lim_{n\to\infty} h_n(x) \le \limsup_{y\to x} f(y) \le f(x)$ for all x. Hence $\{h_n\} \downarrow f$, as claimed. $\square$

Ascoli's Theorem. *Let Ω be a family of functions uniformly bounded and equicontinuous at every point of a closed interval $[a, b]$. Then every sequence of functions $\{f_n\}$ in Ω contains a subsequence that converges uniformly on $[a, b]$.*

Proof. Consider a family of functions Ω, equicontinuous and uniformly bounded on $[a, b]$. We first establish this claim:

For any sequence $\{g_n\}_{n=1}^{\infty} \subseteq \Omega$ and positive number ε, there exists a subsequence $\{g_{n(\varepsilon,k)}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ such that $n(\varepsilon, 1) > 1$ and

$$|g_{n(\varepsilon,k)}(y) - g_{n(\varepsilon,i)}(y)| < \varepsilon \text{ for all } y \in [a, b] \text{ and all } k, i. \tag{1}$$

To see this, fix $\varepsilon > 0$ and let

$$C = \Big\{ J : J \text{ is a closed subinterval on } [a, b] \text{ and for any } \{g_n\} \subseteq \Omega, \text{ there exists a subsequence } \{g_{n(k)}\} \text{ of } \{g_n\} \text{ such that } n(1) > 1 \text{ and } |g_{n(k)}(y) - g_{n(i)}(y)| < \varepsilon \text{ on } J \text{ for all } k, i \Big\}.$$

Let x be an element of $[a, b]$. Since Ω is equicontinuous, there exists a $\delta = \delta(x, \varepsilon)$ such that $|f(x) - f(y)| < \varepsilon/3$ for all f in Ω and y such that $|y - x| < \delta$. Let J be a closed interval containing x such that $|J| < \delta$. We must show J is in C; so we begin with any sequence $\{g_n\}$ from Ω. Since $\{g_n(x)\}$ is bounded, it has a subsequence $\{g_{n(s)}(x)\}$, $s = 1, 2, \ldots$, that converges. Convergent sequences are Cauchy sequences, so we can choose $S > 1$ such that $|g_{n(i)}(x) - g_{n(j)}(x)| < \varepsilon/3$ if $i, j > S$. Consider the collection of functions $g_{n(s)}$, $s > S$, on $[a, b]$. Using the triangle inequality, we have $|g_{n(i)}(y) - g_{n(j)}(y)| < \varepsilon$ for all y in J and $i, j > S$. In addition, $n(S + 1) > 1$. We relabel the sequence by writing $s > S$ as $S + k$, $k = 1, 2, \ldots$ and setting $n(k) = n(S + k)$. The sequence $\{g_{n(k)}\}$ now satisfies the conditions defining C. Thus J is in C and C is a full cover of $[a, b]$. Using Thomson's Lemma, we see that C contains a partition $\{J_h, h = 1, 2, \ldots, m\}$ of $[a, b]$.

Let $\{g_n\}$ be any sequence from Ω. Since J_1 is in C, there exists a subsequence $\{g_{n(1,k)}\}$ of $\{g_n\}$ such that for all y on J_1, $|g_{n(1,k)}(y) - g_{n(1,i)}(y)| < \varepsilon$ for all k, i and $n(1, 1) > 1$. Furthermore, $\{g_{n(1,k)}\}$ is a family of equicontinuous functions uniformly bounded on $[a, b]$. Thus, on J_2 there exists a subsequence $\{g_{n(2,k)}\}$ of $\{g_{n(1,k)}\}$ such that for all y in $J_1 \cup J_2$, $|g_{n(2,k)}(y) - g_{n(2,i)}(y)| < \varepsilon$ for all k, i and $n(2, 1) > n(1, 1)$. By continuing this process on each subinterval J_h, for $h = 3, 4, \ldots, m$, we obtain a subsequence $\{g_{n(m,k)}\}$ of $\{g_n\}$ with the properties that $n(m, 1) > 1$ and $|g_{n(m,k)}(y) - g_{n(m,i)}(y)| < \varepsilon$ for all k, i and all y on $[a, b]$. This sequence, with the change in labeling $n(\varepsilon, k) = n(m, k)$, satisfies (1), so the claim is proved.

We are now ready to conclude the proof of our theorem. Let $\{f_n\}$ be any sequence in Ω. Apply the claim with $\{f_n\}$ and 1 in place of $\{g_n\}$ and ε to obtain a subsequence $\{f_{n(1,k)}\}$ of $\{f_n\}$ such that $n(1, 1) > 1$ and $|f_{n(1,k)}(y) - f_{n(1,i)}(y)| < 1$ for all k, i and all y on $[a, b]$. Apply the claim again with $\{f_{n(1,k)}\}$ and $1/2$ in place of $\{g_n\}$ and ε to obtain a subsequence $\{f_{n(2,k)}\}$ of $\{f_{n(1,k)}\}$ with the properties that $n(2, 1) > n(1, 1)$ and $|f_{n(2,k)}(y) - f_{n(2,i)}(y)| < 1/2$ for all k, i and all y on $[a, b]$. Continuing in this manner, construct for each $p = 2, 3, \ldots$ a subsequence $\{f_{n(p,k)}\}$ of $\{f_{n(p-1,k)}\}$ such that $n(p, 1) > n(p - 1, 1)$ and $|f_{n(p,k)}(y) - f_{n(p,i)}(y)| < 1/p$ for all k, i and all y in $[a, b]$.

Now take the subsequence $\{f_{n(p,1)} : p = 1, 2, \ldots\}$ of $\{f_n\}$. If $q > p$ then $f_{n(q,1)}$ is equal to $f_{(p,k)}$ for some $k = 2, 3, \ldots$. Thus, for all y in $[a, b]$,

$$|f_{n(q,1)}(y) - f_{n(p,1)}(y)| = |f_{n(p,k)}(y) - f_{n(p,1)}(y)| < 1/p. \tag{2}$$

This shows that $\{f_{n(p,1)}(y)\}$ is a Cauchy sequence, hence convergent. Let $f(y)$ denote its limit. By letting $q \to \infty$ in (2) we obtain $|f(y) - f_{n(p,1)}(y)| < 1/p$, which implies $\{f_{n(p,1)} : p = 1, 2, \ldots\}$ converges uniformly to f on $[a, b]$. □

3. Conclusion

Because of their wide applicability, full covering arguments should seriously be considered as part of — or a supplement to — any elementary analysis course. In addition, these arguments prepare the student for the more intricate covering-based proofs of approximate and symmetric derivative theorems [Thomson 1980].

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zangark@gmail.com *Cheiron, One Greentree Centre, Suite 201, Marlton, NJ 08053, United States*

marafijt@jmu.edu *Department of Mathematics and Statistics, 305 Roop Hall, MSC 1911, James Madison University, Harrisonburg, VA 22807, United States*