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Construction and enumeration of Franklin circles

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Around 1752, Benjamin Franklin constructed a variant on the popular magic squares and what we call a magic (a, r)-circle. We provide a definition for magic (a, r)-circles, magic a-circles and, more specifically, Franklin magic a-circles. In this paper, we use techniques in computational algebraic combinatorics and enumerative geometry to construct and to count Franklin magic 8-circles. We also provide a description of its minimal Hilbert basis and determine the symmetry operations on Franklin magic 8-circles.

1. Introduction

Benjamin Franklin was a noted American scholar, politician, scientist, inventor, author of various books and scientific articles, publisher of *Poor Richard's Almanac*, and most notably an editor and signer of the Declaration of Independence, who eventually came to enjoy the recreational side of mathematics. Among his most cherished mathematical works are his famous 8×8 and 16×16 magic squares, which are many times more magical than ordinary magic squares. Here is one of Franklin's 8×8 squares, with sum 260:

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

MSC2000: primary 05A15; secondary 15A48.

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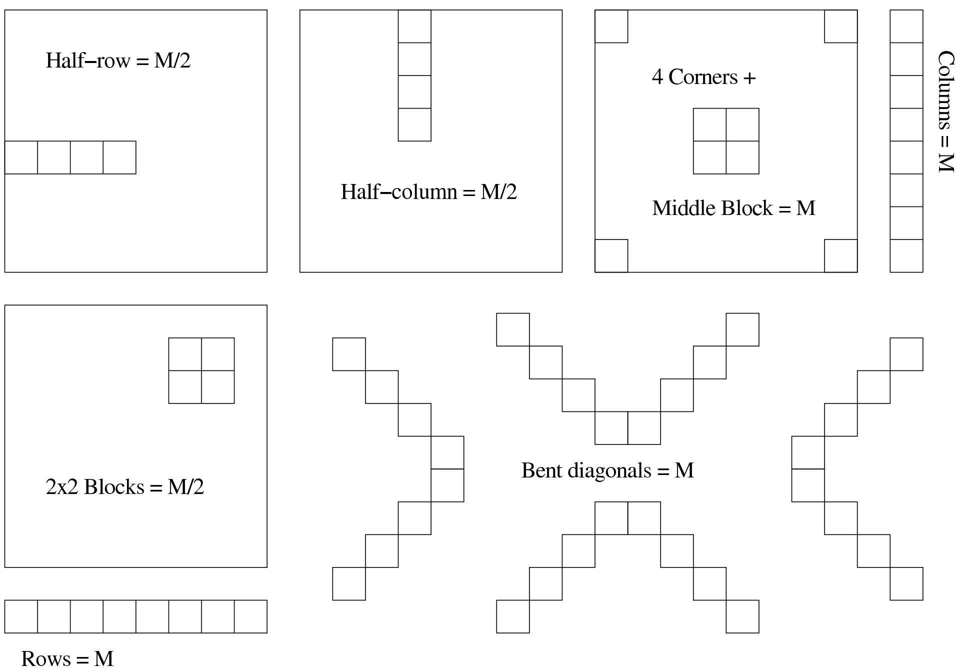
A magic square has three properties: that each row sum, each column sum, and each main diagonal sum is the same "magic" number. A Franklin square, however, has many more properties [Ahmed 2004]:

- Each row sum is the same magic number M.
- Each column sum is M.
- Each *bent* diagonal sum is M.
- Each half-row sum is $M/2$.
- Each half-column sum is $M/2$.
- For 8×8 Franklin squares, each 2×2 block sum is $M/2$.
- For 8×8 Franklin squares, the four corners with the middle four sums to M.
- For 16×16 Franklin squares, each 2×2 block sum is $M/4$, each 4×4 block sum is $M/4$.

These properties can be nicely visualized thus:



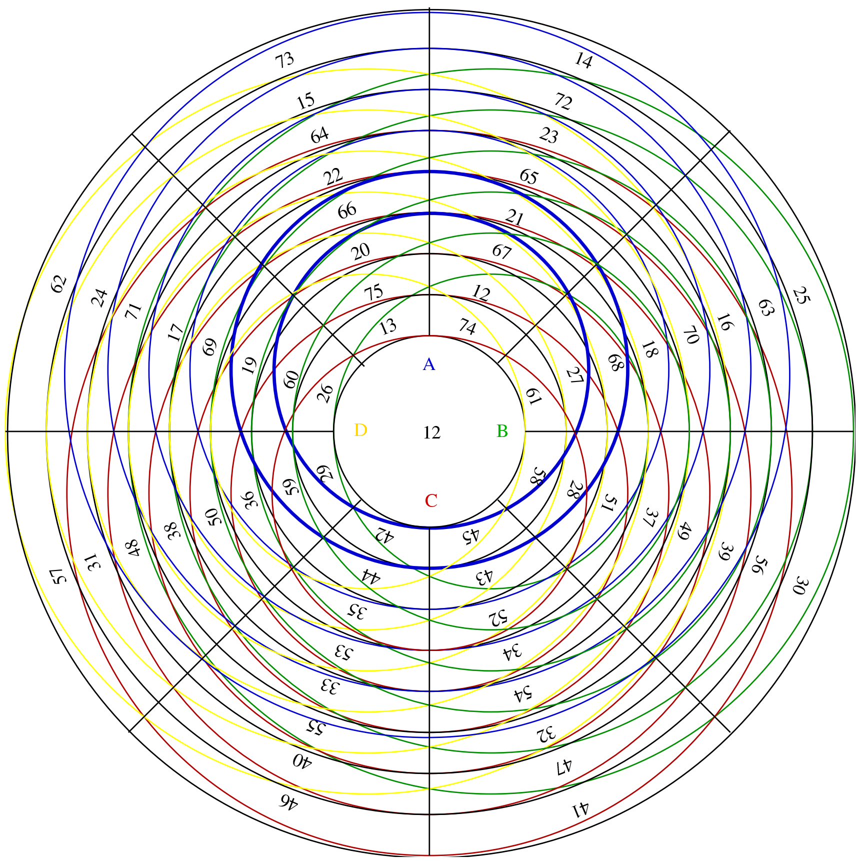
Several magic squares constructed by Franklin were described in a letter written around 1752 to fellow English botanist Peter Collinson [Pasles 2001]. Franklin noted in the same letter that these unusual squares were not his only construction. Franklin provided a similarly complex magic circle in a letter dated 1765 to fellow

English physicist John Canton, describing all of its properties with painstaking detail. In this article, we refer to this magic 8-circle as *the* Franklin magic 8-circle.

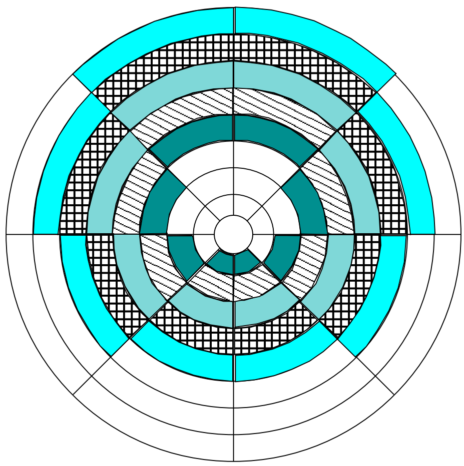
There is no standard definition for a magic circle. We will use the following definition, found also in [Nicholas 1955].

Definition 1. A *magic* (a, r)*-circle* is an arrangement of nonnegative integers in a circular grid consisting of a concentric annuli and r radii, with the property that each annular sum is M and each radial sum is M. In the case $a = r$, we shall call it a *magic* a*-circle*.

Franklin's magic circle, like his magic squares, has additional sophistication. Much like the bent diagonals of Franklin's magic squares, a Franklin magic circle has families of concentric annuli contained within the largest main circle, which however are *eccentric* relative to the basic circular grid. Here is Franklin's original magic 8-circle, with sum $M = 360$; one eccentric annulus is highlighted:



There are four different *excenters* labeled A, B, C, and D, located north, east, south and west of the center. Around each excenter are six concentric circles, forming five annuli. For example, the innermost annulus centered at A in the previous figure (bounded by the two thick circles) contains, clockwise from the bottom, the entries 42, 59, 19, 66, 21, 68, 28, 45. We call these values x_{17}, x_{28}, x_{31}, x_{42}, x_{43}, x_{34}, x_{25}, x_{16}, according to the cells in which they lie; that is, x_{ij} is the number in the i-th original annulus, counted outward, and in the j-th sector, counted clockwised starting with the lower of the two sectors in the first quadrant. Here is another way to visualize the eccentric circles centered at A:



Franklin's original magic circle satisfies the following four properties, which are also depicted graphically on the next page:

(i) Each radial sum plus the central number is the same magic number M.

(ii) Each upper- or lower-half annular sum plus half the central number is $M/2$, and consequently each annular sum plus the central number is M.

(iii) Each 2×2 block sum plus half the central number is $M/2$.

(iv) Each upper- or lower-half annular sum of vertically centered eccentric annuli plus half the central number is $M/2$, and similarly, each left- or right-half annular sum of horizontally centered eccentric annuli plus half the central number is $M/2$. Consequently, each eccentric annular sum plus the central number is M.

Because the central number, appropriately scaled, is added to each of the various types of sums, its role is merely to shift the magic sum. Thus, we will drop the inclusion of the central number in our computations throughout. In this paper, we define a *Franklin magic 8-circle* to be a magic 8-circle with nonnegative integer entries satisfying properties (i)–(iv) above (without a central number). In general, for $n \geq 2$, we define a *Franklin magic 2^n-circle* to be a magic 2^n-circle satisfying (i)–(iv), except that (iii) is modified to read that every 2×2 block sum is $2^{2-n}M$.

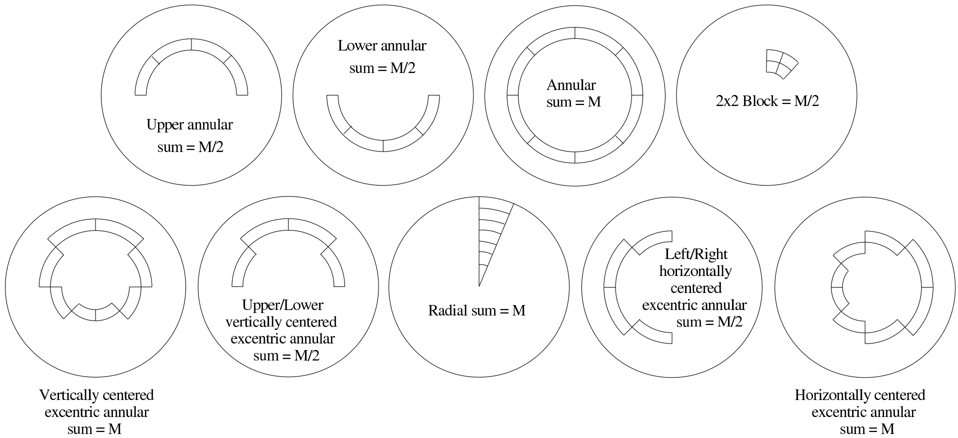


Figure 1. Properties of a Franklin magic circle.

In Section 2, we take a closer look at the tools in computational algebraic combinatorics that allow us to construct all Franklin magic 8-circles. We give a succinct description of the minimal Hilbert basis of the Franklin magic 8-circles, which is a special set of Franklin magic 8-circles that can be used to construct any Franklin magic 8-circle. Furthermore, we give a simple description for producing all possible Franklin magic 4-circles.

In Section 3, we discuss the symmetry operations on the Franklin magic 8-circles and we reveal a *new* Franklin magic 8-circle, that is, a Franklin magic 8-circle which cannot be obtained via symmetry operations on the original Franklin magic 8-circle. Finally, in Section 4, we provide the generating function for Franklin magic 8-circles $\mathrm{FC}_8(s)$, a function which determines the number of Franklin magic 8-circles with magic sum s.

2. Background and notation

We now describe the techniques used to derive the building blocks of all Franklin magic 2^n-circles, starting with $n = 3$. To this end, we view a generic Franklin magic 8-circle as a vector in $\mathbb{R}^{8^2}$, with variable entries $x_{11}, \ldots, x_{88}$, where, as before, the entry x_{ij} is in the i-th annulus and the j-th radius.

The four defining properties of the Franklin magic 8-circle can be viewed as linear relations in these variables. For example, the first property states that each radial sum is the (undetermined) magic number M; that is, the sums $\sum_{i=1}^{8} x_{ij}$ must be equal for all j. This gives seven independent linear equations such as

$$x_{11}+x_{21}+x_{31}+x_{41}+x_{51}+x_{61}+x_{71}+x_{81} = x_{12}+x_{22}+x_{32}+x_{42}+x_{52}+x_{62}+x_{72}+x_{82}.$$

The half-annular sum property yields sixteen linear equations, while there are fifty-six distinct 2×2 block linear equations coming from the eight radial and seven annular locations in the circular arrangement. There are altogether twenty eccentric annuli coming from the five eccentric annuli centered around each of the four ex-centers, with each eccentric annulus yielding two linear relations for the eccentric half annular sum. Rewriting these equations in matrix form, we have a 119×64 integral matrix M:

$$M = \begin{pmatrix} 1 & \cdots & 1 & -1 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & -1 & \cdots & -1 & 0 & \cdots & 0 & \cdots & 0 \\ & \vdots & & & \vdots & & & \vdots & & & & \vdots & & \end{pmatrix}.$$

We observe that for the Franklin magic 8-circle: (1) any nonnegative integer linear combination of Franklin magic 8-circle is a Franklin magic 8-circle, and (2) the set of all Franklin magic 8-circles is the integral solution set to the 119 integral linear equations mentioned above. This shows that the Franklin magic 8-circles are also the integral points inside the set

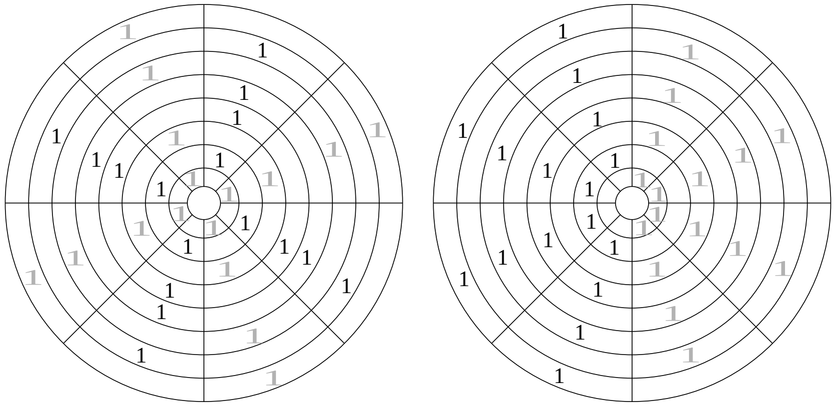
$$C = \{\boldsymbol{x} = (x_{11}, \ldots, x_{88}) \in \mathbb{R}^{64}_{\geq 0} \,:\, M\boldsymbol{x} = 0\},$$

which is itself a pointed rational polyhedral cone.

By [Schrijver 1986, Theorem 16.4], there is a unique finite set H of integral points in C, such that every integral point in C is a linear combination of elements in H. This set is known as *the minimal Hilbert basis* of C. Thus with this minimal Hilbert basis, every Franklin magic 8-circle is some linear combination of the elements in H. Using the software 4ti2, we computed the seventy-four elements of the minimal Hilbert basis for the Franklin magic 8-circles.

We observed two distinct subsets of elements in this minimal Hilbert basis. Among the minimal Hilbert basis elements of the first type, we observed that the first radial arrangement determines all others for the following reasons: the second radial arrangement must be the complement of the first, and this alternating pattern repeats for the third through the eighth radial arrangements. We also noted that the first radial arrangement is completely determined by the placement of the four 1s. Since there are eight possible places, that yields precisely $\binom{8}{4} = 70$ Hilbert basis elements of the first type.

To illustrate the construction of a minimal Hilbert basis element of the first type, see Figure 2. Here, the first radial arrangement is in the second quadrant, just above the horizontal diameter. We chose to place the four 1s in annuli 2, 4, 5, and 7. The second radius, clockwise, is then determined by placing a 1 in the complementary annuli: 1, 3, 6, and 8, shown in bold gray type in Figure 2. This pattern alternates in the subsequent radii.



First type Second type

Figure 2. Examples of Hilbert basis elements.

Finally, there are four Hilbert basis elements of the second type. Their description is based on two observations: all radial arrangements consists of alternating 0 and 1, and the two radial arrangements in each of the four quadrants are duplicates, with the upper half consisting of complementary quadrants and likewise for the lower half. This yields precisely four elements. See Figure 2 for an illustration.

With this minimal Hilbert basis, constructing new Franklin magic 8-circles boils down to simple arithmetic. We present another Franklin magic 8-circle in Figure 3.

We computed the integer linear combination that produces the original Franklin magic 8-circle. This combination is shown in Figure 4 and uses eleven minimal

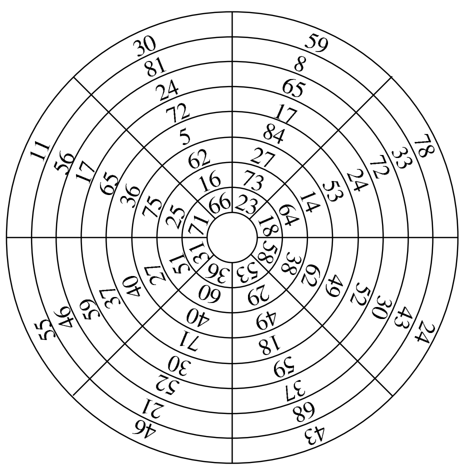


Figure 3. A new Franklin magic 8-circle.

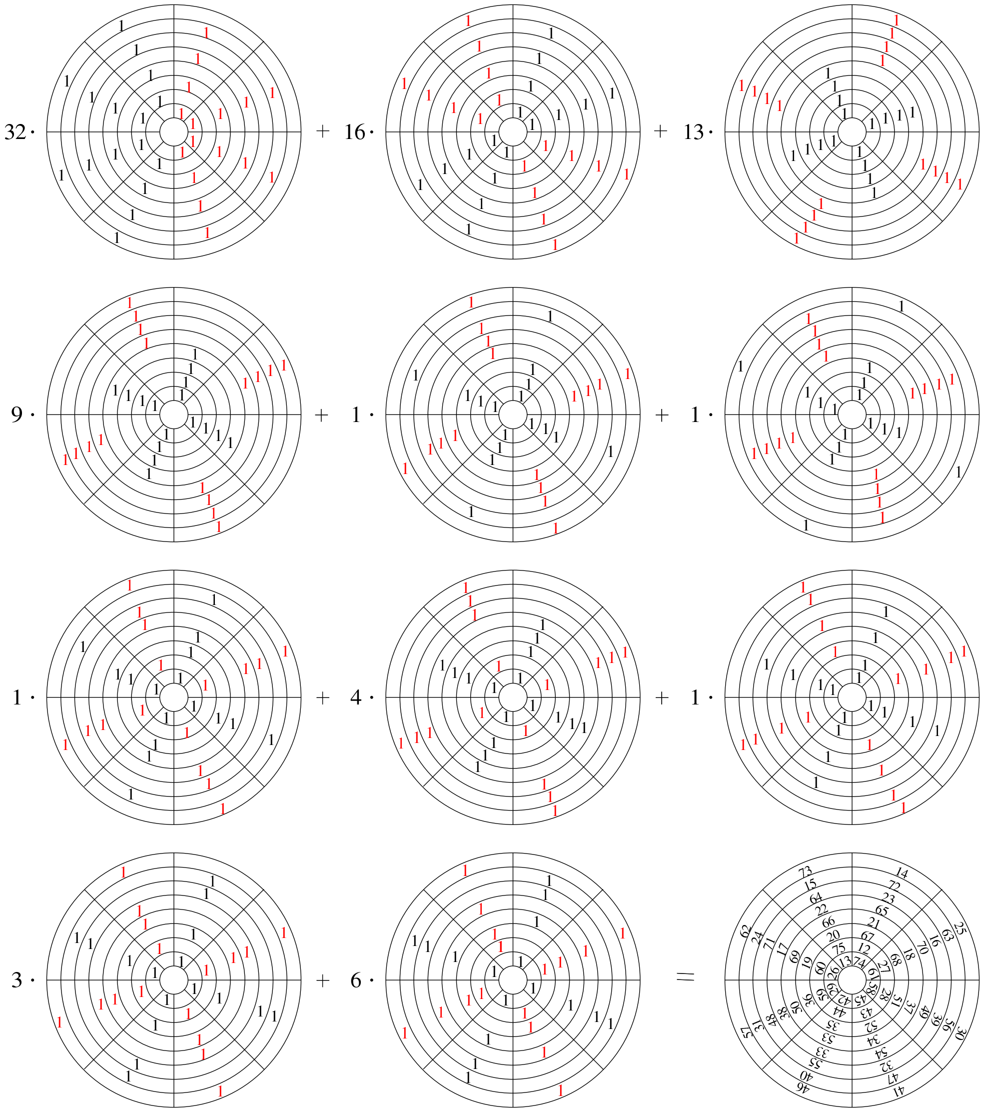


Figure 4. Linear combination of the original Franklin magic 8-circle.

Hilbert basis elements. In terms of the two types of minimal Hilbert basis elements, this linear combination uses two of the second type and nine of the first type.

As for Franklin magic 4-circles, there are precisely six elements in its minimal Hilbert basis. They, too, were computed using 4ti2. Figure 5 shows three elements, with the remaining three obtained by flipping these along the horizontal diameter. The simplicity of its minimal Hilbert basis forces all Franklin magic 4-circles to have repeated entries, whose arrangement is described in Theorem 3.

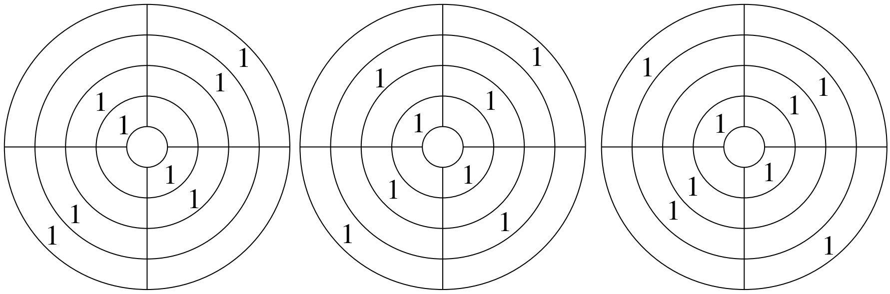
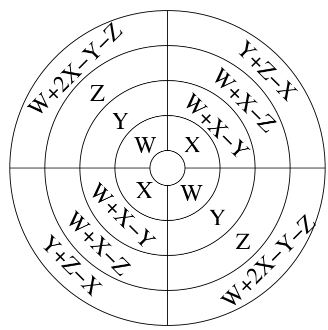


Figure 5. Three elements of the minimal Hilbert basis for Franklin magic 4-circles.

We observe that in this case, the minimal Hilbert basis is only of the first type; that is, the first radial arrangement determines all other radial arrangements. We also noted that the first radial arrangement is completely determined by the placement of the two 1s. Since there are four possible places, then this yields precisely $\binom{4}{2} = 6$ Hilbert basis elements of the first kind. With this observation, we make the following conjecture for the minimal Hilbert basis for Franklin magic 16-circles.

Conjecture 2. *There are* $\binom{16}{8}+16$ *elements in the minimal Hilbert basis for Franklin magic* 16*-circles.*

Theorem 3. *A Franklin magic* 4*-circle is of the form*



Proof. Consider the matrix whose rows correspond to integer multiples of an element in the minimal Hilbert basis:

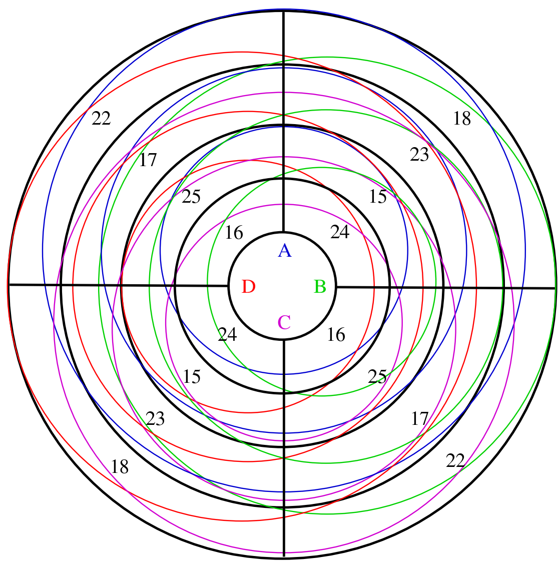
$$\begin{pmatrix}
0 & a & 0 & a & a & 0 & a & 0 & 0 & a & 0 & a & a & 0 & a & 0\\
b & 0 & b & 0 & b & 0 & b & 0 & 0 & b & 0 & b & 0 & b & 0 & b\\
0 & c & 0 & c & 0 & c & 0 & c & c & 0 & c & 0 & c & 0 & c & 0\\
d & 0 & d & 0 & 0 & d & 0 & d & d & 0 & d & 0 & 0 & d & 0 & d\\
e & 0 & e & 0 & 0 & e & 0 & e & 0 & e & 0 & e & e & 0 & e & 0\\
0 & f & 0 & f & f & 0 & f & 0 & f & 0 & f & 0 & 0 & f & 0 & f
\end{pmatrix}$$

Any Franklin magic 4-circle is an integral linear combination of the elements in the minimal Hilbert basis. Thus, any Franklin magic 4-circle corresponds to the sum of the rows of this matrix:

$$(b+d+e,\ a+c+f,\ b+d+e,\ a+c+f,\ a+b+f,\ c+d+e,\ a+b+f,\ c+d+e,\\ c+d+f,\ a+b+e,\ c+d+f,\ a+b+e,\ a+c+e,\ b+d+f,\ a+c+e,\ b+d+f),$$

where the first four entries represent the first annulus in the Franklin magic 4-circle, the next four entries represent the second annulus, and so forth. Setting $W = b+d+e$, $X = a+c+f$, $Y = a+b+f$ and $Z = c+d+f$, demonstrates that any Franklin magic 4-circle is of the desired form. $\square$

Example 4. Choosing fixed nonnegative integer values for $W = 16$, $X = 24$, $Y = 25$, and $Z = 17$ yields the following Franklin magic 4-circle:



3. Symmetry operations on Franklin magic 8-circles

A *symmetry operation* on the set of Franklin magic 8-circles is defined to be a map σ from the set of all Franklin magic 8-circles to itself, that permutes the entries in a Franklin magic 8-circle. From this definition, one can easily see that there are three obvious such symmetry operations: 180° rotation and reflection along the horizontal and vertical diameters.

There are operations on Franklin magic 8-circles which yield magic 8-circles that do not preserve all defining properties properties (i)–(iv). For example, rotation by 90° is *not* a symmetry operation. This can be readily seen by considering this operation on the first minimal Hilbert basis element in Figure 4 (page 364). Here, the upper half annular sum is 0, while the lower half annular sum is 4. In addition, we also note that the *transpose*, that is, exchanging annuli for radii, is not a symmetry operation. This can be seen easily from trying to *transpose* an element of the minimal Hilbert basis, such as the multiple of 13 in Figure 4.

We observed that the elements of the minimal Hilbert basis hold the key to finding all symmetry operations on Franklin magic circles, as we see in Theorem 5.

Theorem 5. *Let $\mathscr{F}_8$ denote the set of all Franklin magic 8-circles, and $\mathrm{H}\mathscr{F}_8$ denote the minimal Hilbert basis of the Franklin magic 8-circles. Then $\sigma : \mathscr{F}_8 \longrightarrow \mathscr{F}_8$ is a symmetry operation on $\mathscr{F}_8$ if and only if its restriction $\overline{\sigma} : \mathrm{H}\mathscr{F}_8 \longrightarrow \mathrm{H}\mathscr{F}_8$ is a symmetry operation on $\mathrm{H}\mathscr{F}_8$.*

Proof. ($\Longrightarrow$) By definition. ($\Longleftarrow$) Let $\overline{\sigma} : \mathrm{H}\mathscr{F}_8 \longrightarrow \mathrm{H}\mathscr{F}_8$ be a symmetry operation on $\mathrm{H}\mathscr{F}_8$. Let (x_{ij}) denote a Franklin magic 8-circle. Then, by definition, (x_{ij}) is some integral linear combination of the elements in $\mathrm{H}\mathscr{F}_8$,

$$(x_{ij}) = \sum_{k=1}^{74} n_k (\mathrm{H}[k]_{ij}),$$

where $\mathrm{H}[k]_{ij}$ is the entry in the i-th annulus and j-th radius of the k-th element in $\mathrm{H}\mathscr{F}_8$. By definition, $\overline{\sigma}$ is a permutation on the entries of $\big(\mathrm{H}[k]_{ij}\big)$. Under $\overline{\sigma}$, the entry in position ij is permuted to a new position $\overline{ij}$, and thus $\overline{\sigma}(\mathrm{H}[k]_{ij}) = (\mathrm{H}[k]_{\overline{ij}})$, which is, by definition of $\overline{\sigma}$, another minimal Hilbert basis element, which we will denote, for the sake of convenience, as $\mathrm{H}[\overline{\sigma}(k)]$. Observe that if $\overline{\sigma}(H[k]) = \overline{\sigma}(H[j])$, then $k = j$, since all the entries move in precisely the same manner.

Given $\overline{\sigma}$, define

$$\sigma : \mathscr{F}_8 \longrightarrow \mathscr{F}_8, \quad (x_{ij}) \mapsto \sum_{k=1}^{74} n_k (\mathrm{H}[\overline{\sigma}(k)]_{ij}). \tag{1}$$

The image of (x_{ij}) under σ in (1) is an integral linear combination of elements in $\mathrm{H}\mathscr{F}_8$ and is therefore a Franklin magic 8-circle. □

From Theorem 5, we observe that all symmetry operations can be obtained by finding symmetry operations on the minimal Hilbert basis. We used the description of the minimal Hilbert basis given in Section 2 to observe that the operations described in Theorem 6 are in fact symmetry operations on the set of all Franklin 8-circles.

Theorem 6. *Let $\mathscr{F}_8$ denote the set of all Franklin magic 8-circles. The following are symmetry operations on $\mathscr{F}_8$:*

(i) *Rotation by 180°, and reflections along the horizontal and vertical diameters.*

(ii) *Exchanging two consecutive annuli*

$$x_i, x_{i+1} \quad \textit{with} \quad x_{i+2k}, x_{i+2k+1},$$

with $1 \le i \le 5$ and $0 \le k \le 3$ and the restriction that $i + 2k + 1 \le 8$.

Proof. The operations in (i) clearly preserve the four Franklin magic 8-circle properties. For the operations in (ii), recall that there are two types of elements in the minimal Hilbert basis $\mathrm{H}\mathcal{F}_8$ of $\mathcal{F}_8$. The first type of elements come from placing 1s in four of the eight possible locations in the first radial arrangement. This pattern is duplicated in the third, fifth and seventh radial arrangements, while the complementary arrangement is duplicated in the second, fourth, sixth and eighth radii. Thus, permuting the annuli among these elements produces another element in $\mathrm{H}\mathcal{F}_8$. The second type of elements in $\mathrm{H}\mathcal{F}_8$ impose the conditions given in (ii). □

4. Enumeration of Franklin magic 8-circles

In this section, we answer the question *For any positive integer s, how many Franklin magic 8-circles have a magic sum of s?* This is an example of the general question of enumerating integer lattice points contained in polyhedra. For an excellent resource on this general topic, see [Beck and Robins 2007].

As we noted earlier, we view a generic Franklin magic 8-circle as an integer vector $(x_{11}, \dots, x_{88})$ in $\mathbb{R}^{8^2}$, where the entry x_{ij} is in the i-th annulus and the j-th radius. Also any Franklin magic 8-circle is an integer linear combination of the minimal Hilbert basis described in Section 2. Since each element of the minimal Hilbert basis has magic sum 4, that implies that every Franklin magic 8-circle must have a magic sum divisible by 4.

Alternatively, we can apply commutative algebra, as in [Ahmed et al. 2003; Cox et al. 1998]. Consider the map of polynomial rings

$$\phi : \mathbb{R}[\mathrm{H}[1], \dots, \mathrm{H}[74]] \longrightarrow \mathbb{R}[x_{11}, \dots, x_{88}]$$

defined by mapping the indeterminate $\mathrm{H}[k]$ to $\prod_{i,j=1}^{8} x_{ij}^{\mathrm{H}[k]_{ij}}$. Integral linear combinations of the elements in $\mathrm{H}\mathcal{F}_8$ correspond to products of integral powers of monomials in $\mathbb{R}[\mathrm{H}[1], \dots, \mathrm{H}[74]]$. Thus if $\sum_{k=1}^{74} n_k(\mathrm{H}[k]_{ij}) = \sum_{k=1}^{74} m_k(\mathrm{H}[k]_{ij})$, we have

$$\phi\Big(\prod_{k=1}^{74} \mathrm{H}[k]^{n_k} - \prod_{k=1}^{74} \mathrm{H}[k]^{m_k}\Big) = 0.$$

Define the homogeneous ideal $I_{\mathcal{F}_8} = \big\langle \prod_{k=1}^{74} \mathrm{H}[k]^{n_k} - \prod_{k=1}^{74} \mathrm{H}[k]^{m_k} \big\rangle$, and consider the weighted, graded ring

$$R = \mathbb{R}[\mathrm{H}[1], \dots, \mathrm{H}[74]]/I_{\mathcal{F}_8},$$

where the degree of each variable $\mathrm{H}[k]$ is 4, for all k. The Hilbert function $\mathrm{H}_R(s)$ is defined by

$$\mathrm{H}_R(s) = \dim_{\mathbb{R}} \mathbb{R}[\mathrm{H}[1], \dots, \mathrm{H}[74]]_s - \dim_{\mathbb{R}} I_{\mathcal{F}_8, s},$$

where $\mathbb{R}[\mathrm{H}[1], \dots, \mathrm{H}[74]]_s$ is the finite-dimensional vector space over $\mathbb{R}$ of homogeneous polynomials in $\mathbb{R}[\mathrm{H}[1], \dots, \mathrm{H}[74]]$ of degree s and $I_{\mathcal{F}_8,s}$ is the finite-dimensional vector space over $\mathbb{R}$ of homogeneous polynomials of degree s in $I_{\mathcal{F}_8}$. Applying [Ahmed 2004, Lemma 2.2], since the weight of each variable $\mathrm{H}[i]$ in the polynomial ring $\mathbb{R}[\mathrm{H}[1], \dots, \mathrm{H}[74]]$ is the magic sum of the corresponding magic circle, then the value $\mathrm{H}_R(s)$ is the number of distinct Franklin magic 8-circles with magic sum s. Thus, the Hilbert–Poincaré series

$$\mathrm{HP}_R(t) = \sum_{s=1}^{\infty} \mathrm{H}_R(s) t^s$$

is also the Ehrhart series of the Franklin magic 8-circles.

To count the number of Franklin magic 8-circles reduces to computing the generating function of the Hilbert–Poincaré series

$$\mathrm{HP}_R(t) = \sum_{s=1}^{\infty} H_R(s) t^s.$$

We used the computer algebra software `LattE` [De Loera et al. 2003], to find the $\mathrm{HP}_R(t)$ as a rational function, and then used `LattE` to compute the first sixty-four values of the enumerating function for the Franklin magic 8-circles. Using Mathematica, we found the interpolating polynomial for these values and computationally verified that the function given in Theorem 7 enumerates the Franklin magic 8-circles.

Theorem 7. *Let* $\mathrm{FC}_8(s)$ *denote the number of Franklin magic* 8*-circles with sum* s. *The Ehrhart series of the Franklin magic* 8*-circles has the rational form*

$$\sum_{s=0}^{\infty} \mathrm{FC}_8(s) t^s = \frac{t^8+64t^7+700t^6+2352t^5+3430t^4+2352t^3+700t^2+64t+1}{(t^9-9t^8+36t^7-84t^6+126t^5-126t^4+84t^3-36t^2+9t-1)(t-1)}.$$

A partial expansion of this series is

$$\sum_{s=0}^{\infty} \mathrm{FC}_8(s) t^s = 1+74t+1395t^2+13092t^3+80245t^4+367774t^5+\cdots.$$

The generating function for the Ehrhart series of the Franklin magic 8*-circles is*

$$\mathrm{FC}_8(s) = \frac{1}{1486356480}(1486356480+1980628992s+1233911808s^2 + 448643072s^3+103670784s^4+16004352s^5 + 1677312s^6+117888s^7+5436s^8+151s^9)$$

when 4 *divides* s; *otherwise*, $\mathrm{FC}_8(s) = 0$.

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