Frame theory for binary vector spaces

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(Communicated by David Larson)

We develop the theory of frames and Parseval frames for finite-dimensional vector spaces over the binary numbers. This includes characterizations which are similar to frames and Parseval frames for real or complex Hilbert spaces, and the discussion of conceptual differences caused by the lack of a proper inner product on binary vector spaces. We also define switching equivalence for binary frames, and list all equivalence classes of binary Parseval frames in lowest dimensions, excluding cases of trivial redundancy.

1. Introduction

There are many conceptual similarities between frames and error-correcting linear codes. Frame theory is concerned with stable linear embeddings of Hilbert spaces obtained from mapping a vector to its frame coefficients [Duffin and Schaeffer 1952; Christensen 2003; Han et al. 2007]. The linear dependencies incorporated in the frame coefficients of a vector help recover from errors such as noise, quantization and data loss [Goyal et al. 1998; 2001; Rath and Guillemot 2003; 2004; Püschel and Kovačević 2006], just as linear codes help recover from symbol decoding errors and erasures [MacWilliams and Sloane 1977]. Frame design for specific purposes has been related to optimization problems of a geometric nature [Casazza and Kovačević 2003; Strohmer and Heath 2003; Holmes and Paulsen 2004] or even a discrete one [Bodmann and Paulsen 2005; Xia et al. 2005; Kalra 2006], including combinatorial considerations that are more commonly associated with error-correcting codes. On the other hand, one may ask whether concepts from frame theory yield insights in the binary setting. This is the motivation of the present paper.

MSC2000: 15A03, 15A33, 42C15.

Keywords: frames, binary numbers, Parseval frames, finite-dimensional vector spaces, binary numbers, binary vector spaces.

Parts of this research were supported by NSF Grant DMS-0807399, and by DMS-0604429 (REU supplement).

We translate many of the essential results on frames for finite-dimensional real or complex Hilbert spaces to analogous statements for vector spaces over the binary numbers. In the first part, we show that in the binary case, the spanning property of a family of vectors is equivalent to having a reconstruction identity with a dual family. This means, both properties can be used interchangeably as a definition of frames, as on finite dimensional real or complex Hilbert spaces. On the other hand, we demonstrate that an attempt to define binary frames similarly to the real or complex case via norm inequalities fails in binary vector spaces, because they lack an inner product and a polarization identity. In the main part of this paper, we focus on Parseval frames, which have a particularly simple reconstruction identity. We characterize binary Parseval frames in terms of their frame operator and develop a notion of switching equivalence for binary frames, similar to the concept for real or complex frames [Goyal et al. 2001; Holmes and Paulsen 2004; Bodmann and Paulsen 2005]. Moreover, we introduce the notion of trivial redundancy, caused by repeated vectors or the inclusion of the zero vector in the frame. Ignoring cases of trivial redundancy and choosing representatives from each switching equivalence class simplifies the enumeration of binary Parseval frames. By an exhaustive search, we have found that if $k \in \{4, 5, \dots, 11\}$, then all frames that are not trivially redundant in $\mathbb{Z}_2^4$ with k vectors belong to one switching equivalence class. Further simplifications for the search of all binary Parseval frames are obtained from a combinatorial consideration, which might be useful for a future effort to catalogue binary Parseval frames in larger dimensions.

The remainder of this paper is organized as follows. In Section 2, we define frames for finite-dimensional binary vector spaces. Section 3 specializes the discussion to Parseval frames. Finally, in Section 4, we define switching equivalence for binary frames and give a catalogue of representatives from each equivalence class of Parseval frames in lowest dimensions, excluding trivially redundant ones.

2. Preliminaries

In this section we first revisit the essentials of frames over the fields $\mathbb{R}$ or $\mathbb{C}$, the real or complex numbers. We then proceed to develop the concept of frames over the field $\mathbb{Z}_2$, that is, the field with two elements $\{0, 1\}$, where 0 is the neutral element with respect to addition, and 1 is the neutral element with respect to multiplication. The main insight of this section is that while there are equivalent characterizations of certain types of frames when the ground field is $\mathbb{R}$ or $\mathbb{C}$, this is not true over $\mathbb{Z}_2$, because the polarization identity is no longer available due to the lack of an inner product.

If $\mathcal{H}$ is a finite-dimensional Hilbert space over $\mathbb{R}$ or $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$, then a family of vectors $\mathcal{F} := \{f_1, f_2, \dots, f_k\}$ in $\mathcal{H}$ is called a *frame* if there exist real numbers A and B such that $0 < A \leq B < \infty$ and

$$A\|x\|^2 \leq \sum_{j=1}^{k} |\langle x, f_j\rangle|^2 \leq B\|x\|^2 \qquad \text{for all } x \in \mathcal{H}. \tag{2-1}$$

The inequalities displayed in (2-1) are known as the *frame condition*, and it can be shown that when $\mathcal{H}$ is finite dimensional, then the set $\mathcal{F}$ satisfies the frame condition if and only if $\operatorname{span}\mathcal{F} = \mathcal{H}$ [Han et al. 2007, Proposition 3.18]. In this case, there exist vectors $\{g_1, g_2, \dots, g_k\}$ which provide the *reconstruction identity*

$$x = \sum_{j=1}^{k} \langle x, f_j\rangle g_j \qquad \text{for all } x \in \mathcal{H}.$$

While the family $\{g_j\}_{j=1}^k$ may not be unique, there is a canonical choice. If we define the so-called *frame operator* S on $\mathcal{H}$ by $Sx = \sum_{j=1}^{k} \langle x, f_j\rangle f_j$, then setting $g_j = S^{-1} f_j$ for $j \in \{1, 2, \dots, k\}$ yields the reconstruction identity [Christensen 2003]. The family $\{g_j\}_{j=1}^k$ is also called the *canonical dual frame*.

A frame $\mathcal{F} = \{f_1, \dots, f_k\}$ is called a *Parseval frame* (or sometimes a *normalized tight frame*) if we can choose $A = B = 1$ in the frame condition, so that

$$\sum_{j=1}^{k} |\langle x, f_j\rangle|^2 = \|x\|^2 \qquad \text{for all } x \in \mathcal{H}. \tag{2-2}$$

Using the polarization identity, it can be shown (see [Han et al. 2007, Proposition 3.11]) that $\mathcal{F}$ is a Parseval frame if and only if

$$x = \sum_{j=1}^{k} \langle x, f_j\rangle f_j \qquad \text{for all } x \in \mathcal{H}. \tag{2-3}$$

The simple form of the reconstruction formula for Parseval frames has many practical uses in engineering and computer science [Goyal et al. 1998; 2001; Kovačević and Chebira 2008].

We now turn to frames over the binary numbers.

The first two goals in this paper are to develop the notion of frames and of Parseval frames for finite-dimensional vector spaces over the field $\mathbb{Z}_2$. Any such vector space has the form $\mathbb{Z}_2^n = \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$ for some $n \in \mathbb{N}$.

Definition 2.1. A family of vectors $\mathcal{F} = \{f_1, f_2, \dots, f_k\}$ in $\mathbb{Z}_2^n$ is a *frame* if it spans $\mathbb{Z}_2^n$.

We have chosen this form of the definition because the field $\mathbb{Z}_2$ has no notion of positive elements, so that it is impossible to find a properly defined inner product, let alone a norm on $\mathbb{Z}_2^n$, which would be needed to formulate a direct analogue of the frame condition (2-1).

Nevertheless, we want to show that an analogue of the reconstruction identity can be deduced with the help of a $\mathbb{Z}_2$-valued "dot product" in place of an inner product.

Definition 2.2. We define a bilinear map $(\cdot\,,\cdot) : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{Z}_2$, called the *dot product* on $\mathbb{Z}_2^n$, by

$$\left(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right) := \sum_{i=1}^{n} a_i b_i .$$

We see that the dot product $(\cdot\,,\cdot)$ is symmetric and $\mathbb{Z}_2$-linear in each component, but it is degenerate: It is possible to have $x \in \mathbb{Z}_2^n$ with $(x, x) = 0$ but $x \neq 0$. Furthermore, because the dot product is degenerate, it does not provide a norm on $\mathbb{Z}_2^n$. Nonetheless, we will use the dot product as an analogue of the inner products on $\mathbb{R}^n$ and $\mathbb{C}^n$, and for expressions in $\mathbb{R}^n$ or $\mathbb{C}^n$ involving $\langle x, y\rangle$ or $\|x\|^2$, we shall consider analogous expressions in $\mathbb{Z}_2^n$ involving (x, y) or (x, x), respectively.

To establish the equivalence between the spanning property and the reconstruction identity for frames, we unfortunately cannot simply use the same strategy as in the real or complex case. If we take the dot product instead of an inner product to define the frame operator, then the spanning property of the frame does not guarantee that the frame operator is invertible. To see this, we note that the family $\{1, 1\}$ is spanning for $\mathbb{Z}_2$, but the analogue of the frame operator maps every $x \in \mathbb{Z}_2$ to $x + x = 0$. A similar family can be obtained for any $\mathbb{Z}_2^n$, $n \geq 1$, by repeating vectors of an arbitrary spanning set.

To build an alternative strategy that relates the spanning property with the existence of a reconstruction identity, we first recall that the dot product mediates a canonical mapping between vectors and linear functionals.

Lemma 2.3. *If $\phi : \mathbb{Z}_2^n \to \mathbb{Z}_2$ is a linear functional then there exists a unique $z \in \mathbb{Z}_2^n$ such that $\phi(x) = (x, z)$ for all $x \in \mathbb{Z}_2^n$.*

Proof. Let ϕ be a linear functional. Let $\{e_1,\dots,e_n\}$ be the canonical basis for $\mathbb{Z}_2^n$, and let $z = \phi(e_1)e_1 + \cdots + \phi(e_n)e_n$. We now observe that if $x \in \mathbb{Z}_2^n$, with $x = \sum_{i=1}^n a_i e_i$ for $a_i \in \mathbb{Z}_2$, then $\phi(x) = \sum_{i=1}^n a_i \phi(e_i) = (x, z)$.

To verify the uniqueness, assume there is z' such that $\phi(x) = (x, z') = (x, z)$. Choosing x among the canonical basis vectors gives $\phi(e_i) = (e_i, z') = (e_i, z)$ and thus z and z' are identical. □

Theorem 2.4. *Given a family $\mathcal{F} = \{f_j\}_{j=1}^k$ in $\mathbb{Z}_2^n$, then $\mathcal{F}$ is a frame if and only if there exist vectors $\{g_j\}_{j=1}^k$ such that for all $y \in \mathbb{Z}_2^n$*

$$y = \sum_{j=1}^{k} (y, g_j) f_j . \tag{2-4}$$

Proof. We note that if (2-4) is true, then necessarily $\{f_j\}_{j=1}^k$ is spanning.

Conversely, assume that $\{f_j\}_{j=1}^k$ is a frame for $\mathbb{Z}_2^n$. In a first step, we prove that there are linear functionals $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ such that $y = \sum_{j=1}^k \gamma_j(y) f_j$ for all $y \in \mathbb{Z}_2^n$. For any family of linear functionals $\gamma_1, \gamma_2, \ldots, \gamma_k$, we note that the expression $\sum_{j=1}^k \gamma_j(y) f_j$ is linear in y, so it is enough to show that there exist linear functionals giving

$$w_i = \sum_{j=1}^k \gamma_j(w_i) f_j \qquad \text{for all vectors in some basis } w_1, \ldots, w_n \text{ of } \mathbb{Z}_2^n.$$

To establish this, we choose a subset of $\{f_1, \ldots, f_k\}$ which is spanning and linearly independent, that is, a basis. Without loss of generality, assume that this set is $\{f_1, \ldots, f_n\}$. Choosing the dual basis $\{\gamma_1, \ldots, \gamma_n\}$ to $\{f_1, \ldots, f_n\}$, characterized by

$$\gamma_j(f_i) = \delta_{ij}, \quad \text{for all } i, j \in \{1, 2, \ldots n\},$$

we obtain

$$\sum_{j=1}^n \gamma_j(f_i) f_j = f_i.$$

Thus if we enlarge the set $\{\gamma_j\}_{j=1}^n$ by setting $\gamma_j = 0$ if $j > n$, then

$$f_i = \sum_{j=1}^k \gamma_j(f_i) f_j$$

and by linearity

$$y = \sum_{j=1}^k \gamma_j(y) f_j \qquad \text{for any } y \in \mathbb{Z}_2^n.$$

In the final step of the proof, we apply the preceding lemma which yields for each γ_j a corresponding vector g_j satisfying $\gamma_j(y) = (y, g_j)$ for all $y \in \mathbb{Z}_2^n$. □

3. Parseval frames for $\mathbb{Z}_2^n$

In this section we present the definition of Parseval frames for $\mathbb{Z}_2^n$ and illustrate the conceptual differences between such frames in the real or complex case and in the binary case.

Definition 3.1. A family of vectors $\mathcal{F} = \{f_1, \ldots, f_k\}$ in $\mathbb{Z}_2^n$ is a *Parseval frame* if

$$x = \sum_{j=1}^k (x, f_j) f_j \qquad \text{for all } x \in \mathbb{Z}_2^n. \tag{3-1}$$

Observe that a binary Parseval frame necessarily spans $\mathbb{Z}_2^n$, and moreover if $\mathcal{F}$ is a Parseval frame, we must have $k \geq n$.

It is natural to ask if, in analogy with the real and complex cases, being a Parseval frame in $\mathbb{Z}_2^n$ is equivalent to having a Parseval identity as in (2-2). It turns out that this is not the case.

Proposition 3.2. *If $\mathcal{F} = \{f_1, \dots, f_k\}$ is a Parseval frame for $\mathbb{Z}_2^n$, then*

$$\sum_{j=1}^{k} (x, f_j)^2 = (x, x) \qquad \text{for all } x \in \mathbb{Z}_2^n. \tag{3-2}$$

However, in general, the converse does not hold.

Proof. If $\mathcal{F}$ is a Parseval frame, then using the $\mathbb{Z}_2$-linearity of the first component of the dot product, for any $x \in \mathbb{Z}_2^n$ we have

$$(x, x) = \left(\sum_{j=1}^{k} (x, f_j) f_j, x \right) = \sum_{j=1}^{k} (x, f_j)(f_j, x) = \sum_{j=1}^{k} (x, f_j)^2.$$

To see that the converse does not hold in general, consider $\binom{1}{1} \in \mathbb{Z}_2^2$, then for any $x = \binom{a_1}{a_2} \in \mathbb{Z}_2^2$ we have

$$\left(x, \binom{1}{1}\right) = a_1 + a_2 = a_1^2 + a_2^2 = (x, x).$$

Hence $\mathcal{F} = \left\{\binom{1}{1}\right\}$ satisfies (3-2). However, $\mathcal{F}$ contains one element, so $\mathcal{F}$ does not span $\mathbb{Z}_2^2$, and $\mathcal{F}$ is not a Parseval frame. □

Remark 3.3. More generally, we can produce counterexamples for any $n \geq 2$, meaning sets which give the Parseval property without spanning $\mathbb{Z}_2^n$. First we consider even n. Let $\{f_1, \dots, f_k\}$ be the family of all vectors which contain exactly two 1's. Thus, there are $k = \binom{n}{2}$ such vectors. If the first vector is chosen as $f_1 = (1, 1, 0, \dots, 0)^t$ and $x = (a_1, a_2, \dots, a_n)^t$, then over $\mathbb{Z}_2$,

$$(x, f_1)^2 = (a_1 + a_2)^2 = a_1^2 + a_2^2 .$$

Evaluating other dot products similarly gives

$$\sum_{j=1}^{k} (x, f_j)^2 = \sum_{i=1}^{n} a_i^2$$

because each a_i^2 appears in $n - 1$ terms in the sum, and $n - 1 \bmod 2 = 1$ by the assumption that n is even.

However, the vectors $\{f_j\}_{j=1}^k$ are not spanning for $\mathbb{Z}_2^n$, because they contain an even number of 1's and so does any linear combination of them.

If n is odd, then we split $\mathbb{Z}_2^n = \mathbb{Z}_2 \oplus \mathbb{Z}_2^{n-1}$ and construct the above family $\{f_1, f_2, \dots, f_k\}$ for the second summand. Now this family can be enlarged by the first canonical basis vector e_1 to $\{e_1, f_1, f_2, \dots, f_k\}$ which has the Parseval property but is not spanning, because $\{f_1, f_2, \dots, f_k\}$ does not span $\mathbb{Z}_2^{n-1}$.

4. Towards a catalogue of binary Parseval frames

In principle, all Parseval frames for $\mathbb{Z}_2^n$ could be catalogued individually, but even for relatively small n this is already an extensive list. In order to obtain a more efficient way of enumerating Parseval frames, we use an equivalence relation which has been called switching equivalence for real or complex frames [Goyal et al. 2001; Holmes and Paulsen 2004; Bodmann and Paulsen 2005]. It is most easily formulated in terms of the Grammian of a Parseval frame, as defined below. The catalogue of frames can then be reduced to representatives of each equivalence class. To prepare the definition of the equivalence relation, we discuss certain matrices related to frames.

We write $A \in M_{m,n}(\mathbb{Z}_2)$ when A an $m \times n$ matrix with entries in $\mathbb{Z}_2$. We often view A as a linear map from $\mathbb{Z}_2^m$ to $\mathbb{Z}_2^n$ by left multiplication. In particular, $A \in M_n$ denotes an $n \times n$ matrix which is associated with a map from $\mathbb{Z}_2^n$ to itself. We write $A_{i,j}$ for the (i, j)th entry of A, and we let A^* denote the transpose of A; that is, $A^* \in M_{n,m}(\mathbb{Z}_2)$ with $A^*_{i,j} := A_{j,i}$. By the rules of matrix multiplication, we have $(Ax, y) = (x, A^*y)$ for all $A \in M_n(\mathbb{Z}_2)$.

Definition 4.1. If $U \in M_n(\mathbb{Z}_2)$, then we say U is a *unitary* if U is invertible and $U^{-1} = U^*$.

Lemma 4.2. *If $x \in \mathbb{Z}_2^n$ and $(x, y) = 0$ for all $y \in \mathbb{Z}_2^n$, then $x = 0$.*

Proof. Write

$$x = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

If $\{e_1, \dots, e_n\}$ is the standard basis for $\mathbb{Z}_2^n$, then for all $1 \le i \le n$ we have $a_i = (x, e_i) = 0$. Thus $x = 0$. □

Proposition 4.3. *Let $U \in M_n(\mathbb{Z}_2)$, then U is a unitary if and only if for all $x, y \in \mathbb{Z}_2^n$ we have $(Ux, Uy) = (x, y)$.*

Proof. If U is a unitary, then $U^* = U^{-1}$ and for all $x, y \in \mathbb{Z}_2^n$ we have

$$(Ux, Uy) = (x, U^*Uy) = (x, Iy) = (x, y).$$

Conversely, if $(Ux, Uy) = (x, y)$ for all $x, y \in \mathbb{Z}_2^n$, then for a given $x \in \mathbb{Z}_2^n$ we see that $(U^*Ux, y) = (Ux, Uy) = (x, y)$ for all $y \in \mathbb{Z}_2^n$, and Lemma 4.2 implies that $U^*Ux = x$. Since x was arbitrary, this shows that $U^*U = I$, and because U is square, we have that U is invertible and $U^{-1} = U^*$. □

In contrast the case of Hilbert spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the condition $\langle Ux, Ux\rangle = \langle x, x\rangle$ for all $x \in \mathbb{F}^n$ is not equivalent to unitarity when the field $\mathbb{F}$ is $\mathbb{Z}_2$.

We have the following counterexamples for $n \geq 2$.

Proposition 4.4. *For any $n \geq 2$, there exist $A \in M_n(\mathbb{Z}_2)$ such that $(Ax, Ax) = (x, x)$ for all $x \in \mathbb{Z}_2^n$ but A is not invertible, and thus not unitary.*

Proof. We define the matrix A by

$$A_{i,j} = \begin{cases} 1 & \text{if } i = j = 1 \text{ or } j - i = 1\,, \\ 0 & \text{else.} \end{cases}$$

This means, the last row of A contains only zeros and thus A does not have full rank and is not invertible.

However, given $x = (a_1, a_2, \ldots, a_n)^t$ we have

$$Ax = \begin{pmatrix} a_1 + a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_n \\ 0 \end{pmatrix},$$

and thus

$$(Ax, Ax) = (a_1 + a_2)^2 + a_3^2 + \cdots + a_n^2 = \sum_{i=1}^{n} a_i^2 = (x, x). \qquad \square$$

Definition 4.5. Let $\mathcal{F} = \{f_1, \ldots, f_k\} \subseteq \mathbb{Z}_2^n$. The *analysis operator* for $\mathcal{F}$ is the $k \times n$ matrix containing the frame vectors as rows,

$$\Theta_{\mathcal{F}} = \begin{pmatrix} \leftarrow & f_1 & \rightarrow \\ & \vdots & \\ \leftarrow & f_k & \rightarrow \end{pmatrix}.$$

The *synthesis operator* for $\mathcal{F}$ is the $n \times k$ matrix

$$\Theta_{\mathcal{F}}^* = \begin{pmatrix} \uparrow & & \uparrow \\ f_1 & \cdots & f_k \\ \downarrow & & \downarrow \end{pmatrix},$$

with the elements of $\mathcal{F}$ as columns. The *frame operator* for $\mathcal{F}$ is the $n \times n$ matrix

$$S_{\mathcal{F}} := \Theta_{\mathcal{F}}^* \Theta_{\mathcal{F}},$$

and the *Grammian operator* for $\mathcal{F}$ is the $k \times k$ matrix

$$G_{\mathcal{F}} := \Theta_{\mathcal{F}} \Theta_{\mathcal{F}}^*.$$

Note that $(G_{\mathcal{F}})_{i,j} = (f_j, f_i)$ for all $1 \le i, j \le k$. When there is no ambiguity in the choice of $\mathcal{F}$, we shall omit the $\mathcal{F}$ subscript on these matrices and simply write Θ, Θ^*, S, and G.

Theorem 4.6. *Let $\mathcal{F} = \{f_1, \dots, f_k\} \subseteq \mathbb{Z}_2^n$, then $\mathcal{F}$ is a Parseval frame if and only if $S_{\mathcal{F}}$ is equal to the identity matrix.*

Proof. Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{Z}_2^k$. Observe that for any $x \in \mathbb{Z}_2^n$ we have $\Theta_{\mathcal{F}} x = \sum_{i=1}^k (x, f_i) e_i$. Also, for any $1 \le i \le n$ we have $\Theta_{\mathcal{F}}^* e_i = f_i$. Thus we have

$$S_{\mathcal{F}} x = \Theta_{\mathcal{F}}^* \Theta_{\mathcal{F}} x = \Theta_{\mathcal{F}}^* \left(\sum_{i=1}^k (x, f_i) e_i \right) = \sum_{i=1}^k (x, f_i) \Theta_{\mathcal{F}}^* e_i = \sum_{i=1}^k (x, f_i) f_i.$$

It follows that $\sum_{i=1}^k (x, f_i) f_i = x$ for all $x \in \mathbb{Z}_2^n$ if and only if $S_{\mathcal{F}} x = x$ for all $x \in \mathbb{Z}_2^n$. Thus $\mathcal{F}$ is a Parseval frame if and only if $S_{\mathcal{F}}$ is the identity matrix. □

Definition 4.7. Given two families $\mathcal{F} = \{f_1, \dots, f_k\}$ and $\mathcal{G} = \{g_1, \dots, g_k\}$ in $\mathbb{Z}_2^n$, then we say $\mathcal{F}$ *is unitarily equivalent to* $\mathcal{G}$ if there exists a unitary $U \in M_n(\mathbb{Z}_2)$ such that $U f_i = g_i$ for all $1 \le i \le k$.

It is easy to show that unitary equivalence is an equivalence relation.

Proposition 4.8. *Let $\mathcal{F} = \{f_1, \dots, f_k\} \subseteq \mathbb{Z}_2^n$ and $\mathcal{G} = \{g_1, \dots, g_k\} \subseteq \mathbb{Z}_2^n$ be Parseval frames, then $\mathcal{F}$ is unitarily equivalent to $\mathcal{G}$ if and only if $G_{\mathcal{F}} = G_{\mathcal{G}}$.*

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ are Parseval frames, it follows from Theorem 4.6 that $S_{\mathcal{F}}$ and $S_{\mathcal{G}}$ are the identity matrices. Suppose that $G_{\mathcal{F}} = G_{\mathcal{G}}$. Define U to be the $n \times n$ matrix $U := \Theta_{\mathcal{G}}^* \Theta_{\mathcal{F}}$, then

$$\begin{aligned} U^* U &= (\Theta_{\mathcal{G}}^* \Theta_{\mathcal{F}})^* \Theta_{\mathcal{G}}^* \Theta_{\mathcal{F}} = \Theta_{\mathcal{F}}^* \Theta_{\mathcal{G}} \Theta_{\mathcal{G}}^* \Theta_{\mathcal{F}} = \Theta_{\mathcal{F}}^* G_{\mathcal{G}} \Theta_{\mathcal{F}} \\ &= \Theta_{\mathcal{F}}^* G_{\mathcal{F}} \Theta_{\mathcal{F}} = \Theta_{\mathcal{F}}^* \Theta_{\mathcal{F}} \Theta_{\mathcal{F}}^* \Theta_{\mathcal{F}} = S_{\mathcal{F}} S_{\mathcal{F}} = I. \end{aligned}$$

Since U is square, it follows that U is invertible and $U^{-1} = U^*$, so that U is a unitary. Furthermore,

$$U \Theta_{\mathcal{F}}^* = \Theta_{\mathcal{G}}^* \Theta_{\mathcal{F}} \Theta_{\mathcal{F}}^* = \Theta_{\mathcal{G}}^* G_{\mathcal{F}} = \Theta_{\mathcal{G}} G_{\mathcal{G}} = \Theta_{\mathcal{G}}^* \Theta_{\mathcal{G}} \Theta_{\mathcal{G}}^* = S_{\mathcal{G}} \Theta_{\mathcal{G}}^* = \Theta_{\mathcal{G}}^*.$$

Thus U times the ith column of $\Theta_{\mathcal{F}}^*$ is equal to the ith column of $\Theta_{\mathcal{G}}^*$. Thus for all $1 \le i \le k$ we have $U f_i = g_i$, so that $\mathcal{F}$ and $\mathcal{G}$ are unitarily equivalent.

Conversely, if $\mathcal{F}$ and $\mathcal{G}$ are unitarily equivalent, then there exists a unitary $U \in M_n(\mathbb{Z}_2)$ such that $U f_i = g_i$ for all $1 \le i \le k$. Thus Proposition 4.3 implies that

$$(G_{\mathcal{F}})_{i,j} = (f_j, f_i) = (U f_j, U f_i) = (g_j, g_i) = (G_{\mathcal{G}})_{i,j}.$$

Hence $G_{\mathcal{F}} = G_{\mathcal{G}}$. □

Example 4.9. We present two examples of unitary equivalence. First set

$$\mathcal{F} = \left\{ \begin{pmatrix} 0\\1\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\1\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\1\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\1\\1\\1 \end{pmatrix}, \begin{pmatrix} 1\\1\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\1\\1\\1\\0 \end{pmatrix} \right\},$$

$$\mathcal{H} = \left\{ \begin{pmatrix} 0\\1\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\1\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\1\\1\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\1\\1\\1 \end{pmatrix}, \begin{pmatrix} 1\\1\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 1\\1\\1\\0\\1 \end{pmatrix} \right\}.$$

Computing the Grammian for both $\mathcal{F}$ and $\mathcal{H}$ we find

$$G_{\mathcal{F}} = G_{\mathcal{H}} = \begin{bmatrix} 0&0&0&1&0&1\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&1\\ 0&0&0&0&1&0\\ 1&0&0&1&0&0 \end{bmatrix}.$$

First notice the structure of Θ^* created by $\mathcal{F}$ and $\mathcal{H}$:

$$\Theta^*_{\mathcal{F}} = \begin{bmatrix} 0&0&0&1&1&1\\ 1&1&1&0&1&1\\ 0&0&1&1&0&1\\ 0&1&0&1&0&1\\ 1&1&1&1&1&0 \end{bmatrix}, \quad \Theta^*_{\mathcal{H}} = \begin{bmatrix} 0&0&0&1&1&1\\ 1&1&1&0&1&1\\ 0&0&1&1&0&1\\ 1&1&1&1&1&0\\ 0&1&0&1&0&1 \end{bmatrix}.$$

The fourth and fifth rows have swapped places, so naturally one would expect the unitary operator to reflect that. In fact, the proof gives a direct way to compute U, namely if $f_i = Uh_i$ then $U = \Theta^*_{\mathcal{F}}\Theta_{\mathcal{H}}$. Computing U confirms this:

$$\Theta^*_{\mathcal{F}}\Theta_{\mathcal{H}} = \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{bmatrix}.$$

Next, take for $\mathcal{F}$ and $\mathcal{H}$ two Parseval frames found in $\mathbb{Z}_2^5$ with six elements:

$$\mathcal{F} = \left\{ \begin{pmatrix} 0\\0\\0\\1\\1 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\1\\1\\1\\0 \end{pmatrix}, \begin{pmatrix} 1\\1\\1\\1\\1 \end{pmatrix} \right\},$$

$$\mathcal{H} = \left\{ \begin{pmatrix} 0\\1\\1\\1\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\1\\1\\1\\1 \end{pmatrix} \right\}.$$

Here, while not quite as obvious, differences in the two Parseval frames can be expressed in terms of row manipulations of the synthesis operator, which amount to left multiplication with a unitary U, $\Theta_{\mathcal{F}}^* = U\Theta_{\mathcal{H}}^*$.

We introduce an additional way to identify frames which coarsens the equivalence relation.

Definition 4.10. Two families $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ and $\mathcal{G} = \{g_1, g_2, \ldots, g_k\}$ in $\mathbb{Z}^n$ are called *switching equivalent* if there is a unitary U and a permutation π of the set $\{1, 2, \ldots, k\}$ such that

$$f_j = U g_{\pi(j)} \qquad \text{for all } j \in \{1, 2, \ldots\}.$$

Theorem 4.11. *Two Parseval frames $\mathcal{F}$ and $\mathcal{H}$ are switching equivalent if and only if there exists a permutation π of the index set such that $(G_{\mathcal{F}})_{i,j} = (G_{\mathcal{H}})_{\pi(i),\pi(j)}$.*

Proof. The condition on the Grammians amounts to the identity

$$G_{\mathcal{F}} = M G_{\mathcal{H}} M^*$$

for a permutation matrix with entries

$$M_{i,j} = \begin{cases} 1 & \text{if } \pi(i) = j, \\ 0 & \text{else}. \end{cases}$$

Being identical up to conjugation by permutation matrices defines an equivalence relation for Grammians, and thus for frames, which is coarser than unitary equivalence.

Moreover, with a similar proof as in the preceding proposition, we see that the two Grammians are related by conjugation with a permutation matrix M if and only if there exists a unitary U such that

$$\Theta_{\mathcal{F}}^* = U\Theta_{\mathcal{H}}^* M^*. \qquad \square$$

Apart from switching equivalence, there are other simple ways in which two Parseval frames can be related to each other. For example, adding zero vectors to a Parseval frame gives another Parseval frame. Moreover, adding pairs of arbitrary vectors to a Parseval frame preserves the Parseval property. In both cases, we have artificially increased the redundancy by enlarging the frame. In our catalogue, we discard Parseval frames with such a trivial source of redundancy.

Definition 4.12. A Parseval frame $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ for $\mathbb{Z}_2^n$ is called *trivially redundant* if there is $j \in \{1, 2, \ldots k\}$ with $f_j = 0$, or if there are two indices $i \neq j$ with $f_i = f_j$.

After repeated vectors are removed, Parseval frames can be interpreted as sets of vectors. We consider the set-theoretic complement of such a Parseval frame.

Theorem 4.13. *Let $n \geq 3$. Let $\mathcal{F} = \{f_i\}_{i=1}^{k}$ be a family without repeated vectors in $\mathbb{Z}_2^n$ and $\mathcal{G} = \mathbb{Z}_2^n \setminus \mathcal{F}$. If $\mathcal{F}$ is a Parseval frame for $\mathbb{Z}_2^n$, then $\mathcal{G}$ is also a Parseval frame.*

Proof. Let $\mathcal{X} = \{x \in \mathbb{Z}_2^n\}$, then we count $2^n - 1$ nonzero elements in $\mathcal{X}$. Thinking of $\mathcal{X}$ as a sequence $\{f_i\}_{i=0}^{m}$ where $m = 2^n - 1$ and the entries of f_i are given by the binary expansion of i, let $\Theta_{\mathcal{X}}^*$ be the matrix with f_i as the ith column.

By simple counting, there are 2^{n-1} elements with 1 in the ith position. This means, in each row of $\Theta_{\mathcal{X}}^*$ the number 1 appears exactly 2^{n-1} times. Furthermore there are 2^{n-2} elements with 1 in the ith and jth position, similarly for any fixed choice of 1 or 0 in the ith and jth position. If $n \geq 3$, then 2^{n-2} is even and consequently, the dot product of any row of $\Theta_{\mathcal{X}}^*$ with itself or any other row is equal to 0, i.e. $\Theta_{\mathcal{X}}^* \Theta_{\mathcal{X}} = 0$.

If $\mathcal{F}$ is a Parseval frame, then $\Theta_{\mathcal{F}}^* \Theta_{\mathcal{F}} = I$ which implies via the matrix product that there is an odd number of elements in $\mathcal{F}$ with 1 in the ith position, and that among the elements with a 1 in the ith position there is an even number of elements with a 1 in the jth position.

As remarked above, in the entire space there is an even number of elements with 1 in the ith position and an even number of elements with 1 in the ith and jth position. Thus there is an odd number of elements in the complement $\mathcal{G} = \mathcal{X} \setminus \mathcal{F}$ with 1 in the ith position and an even number of such elements with 1 in the jth position, that is $\Theta_{\mathcal{G}}^* \Theta_{\mathcal{G}} = I$. Hence $\mathcal{G}$ is a Parseval Frame. □

Corollary 4.14. *If $\mathcal{F}$ is a Parseval frame for $\mathbb{Z}_2^n$ which is not trivially redundant, and $\mathcal{G}$ is its set-theoretic complement, then both $\mathcal{F}$ and $\mathcal{G} \setminus \{0\}$ are Parseval frames and one of them contains at most $2^{n-1} - 1$ vectors.*

Proof. After removing the zero vector from $\mathcal{G}$, the union of both Parseval frames $\mathcal{F}$ and $\mathcal{G} \setminus \{0\}$ contains $2^n - 1$ vectors. This implies that one of the two frames contains at most half this number, meaning at most $2^{n-1} - 1$ vectors. □

To complete the catalogue of binary Parseval frames for $\mathbb{Z}_2^n$, it is only necessary to find one representative from each switching equivalence class of Parseval frames with at most $2^{n-1} - 1$ vectors. Once these Parseval frames have been found, their complements complete the list, because the switching equivalence of a pair of frames is equivalent to that of their complements.

Proposition 4.15. *Two frames that are not trivially redundant are switching equivalent if and only if their set-theoretic complements are.*

Proof. This is a consequence of the fact that unitaries are one-to-one maps on the set $\mathbb{Z}_2^n$. Thus, if a unitary U maps a frame $\mathcal{F}$ to a frame $\mathcal{G}$, then it also maps the complement of $\mathcal{F}$ to the complement of $\mathcal{G}$, and vice versa. □

n	k	vectors										
3	3	1	2	4								
	4	3	5	6	7							
4	4	1	2	4	8							
	5	1	6	10	12	14						
	6	1	3	5	9	14	15					
	7	1	2	3	7	11	12	15				
	8	4	5	6	8	9	10	13	14			
	9	2	4	6	7	8	10	11	12	13		
	10	2	3	4	5	7	8	9	11	13	15	
	11	3	5	6	7	9	10	11	12	13	14	15

Table 1. Representatives of all switching-equivalence classes, excluding trivially redundant Parseval frames, for $\mathbb{Z}_2^3$ and $\mathbb{Z}_2^4$.

We conclude with Table 1, a complete list of representatives of switching-equivalence classes of Parseval frames for $n = 3$ and $n = 4$, excluding ones that are trivially redundant. Each frame vector in our list is recorded by the integer obtained from the binary expansion with the entries of the vector. For example, if a frame vector in $\mathbb{Z}_2^4$ is $f_1 = (1, 0, 1, 1)$, then it is represented by the integer $2^0+2^2+2^3 = 13$. Accordingly, in $\mathbb{Z}_2^4$, the standard basis is recorded as the sequence of numbers $1, 2, 4, 8$.

As explained above, the part of the table with $k > 2^{n-1} - 1$ vectors has been obtained by taking complements of Parseval frames with $k \leq 2^{n-1} - 1$ vectors, according to Corollary 4.14 and Proposition 4.15. An exhaustive search shows that there is only one switching equivalence class for $n = 3$ and $k \in \{3, 4\}$ and for $n = 4$ and each $k \in \{4, 5, 6, 7\}$, consequently also for $k \in \{8, 9, 10, 11\}$.

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Received: 2009-08-06 Accepted: 2009-08-12

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