Some results on the size of sum and product sets of finite sets of real numbers

Derrick Hart and Alexander Niziolek

(Communicated by Andrew Granville)

Let A and B be finite subsets of positive real numbers. Solymosi gave the sum-product estimate $\max(|A+A|, |A\cdot A|) \geq (4\lceil\log|A|\rceil)^{-1/3}|A|^{4/3}$, where $\lceil\ \rceil$ is the ceiling function. We use a variant of his argument to give the bound

$$\max(|A+B|, |A\cdot B|) \geq (4\lceil\log|A|\rceil\lceil\log|B|\rceil)^{-1/3}\,|A|^{2/3}\,|B|^{2/3}.$$

(This isn't quite a generalization since the logarithmic losses are worse here than in Solymosi's bound.)

Suppose that A is a finite subset of real numbers. We show that there exists an $a\in A$ such that $|aA+A|\geq c|A|^{4/3}$ for some absolute constant c.

1. Introduction

Given finite subsets A and B of an additive group, the *sum set* of A and B is

$$A+B=\{a+b : a\in A,\ b\in B\}.$$

Similarly, define the *product set* by

$$A\cdot B=\{ab : a\in A,\ b\in B\}.$$

If M and N are numbers (depending on A and B) we write $M \gtrsim N$ to mean that $M\geq cN$ for some constant $c>0$ (independent of A and B). We write $M\approx N$ to mean that $cN\leq M\leq c'N$ for $c, c'>0$.

Suppose that $A=B$ is an arithmetic progression. Then

$$|A+A|\lesssim|A| \quad\text{and}\quad |A\cdot A|\gtrsim|A|^{2-\delta},$$

where here and throughout $\delta\to 0$ as $|A|\to\infty$ and $|\cdot|$ denotes the size of the set. In contrast, if $A=B$ is a geometric progression then

$$|A+A|\gtrsim|A|^2 \quad\text{and}\quad |A\cdot A|\lesssim|A|.$$

MSC2000: 11B13, 11B75.
Keywords: sum-product estimate, multiplicative energy, Solymosi bound.

These examples led Erdős and Szemerédi [1983] to ask whether both the product set and sum set can be small at the same time. They conjectured that it is not possible in the following sense.

Conjecture 1. *Let A be a finite subset of $\mathbb{Z}$. Then*

$$\max(|A+A|, |A\cdot A|) \geq |A|^{2-\delta}.$$

They showed that

$$\max(|A+A|, |A\cdot A|) \geq |A|^{1+\varepsilon},$$

for a positive ε.

The explicit bound $\varepsilon \geq \frac{1}{31}$ was given by Nathanson [1997], and $\varepsilon \geq \frac{1}{15}$ by Ford [1998]. A breakthrough was achieved by Elekes [1997], who connected the problem to incidence geometry and applied the Szemerédi–Trotter incidence theorem to obtain $\varepsilon \geq \frac{1}{4}$. This was improved by Solymosi [2005] to $\varepsilon \geq \frac{3}{14} - \delta$. These bounds hold in the more general context of finite subsets of $\mathbb{R}$.

Recently, by a short and ingenious argument it was shown by Solymosi [2009] that $\varepsilon \geq \frac{1}{3} - \delta$. In Section 3 we mimic Solymosi's argument with a few changes to give an analogous estimate for sums and products of different sets.

Given the strong relationship between sums and products one may ask a related question: how large is the set $A\cdot B + C$ guaranteed to be? Elekes (see [Alon and Spencer 2000]) showed that $|A\cdot B + C| \gtrsim \sqrt{|A|\,|B|\,|C|}$ with certain size restrictions on the three sets. His argument relied on the aforementioned Szeremedi–Trotter incidence theorem and is short enough to present in the next few lines.

Let P be a set of points in $\mathbb{R}^2$ and L a set of lines. We say a point $p \in P$ is incident to a line $l \in L$ if p lies on l. In this case, we denote this incidence by $(p, l) \in P \times L$.

Theorem 2 [Szemerédi and Trotter 1983]. *Let $I_{P,L}$ denote the number of incidences between P and L. Then bound*

$$I_{P,L} \lesssim |P|^{2/3}\,|L|^{2/3} + |P| + |L|.$$

Let $L = \{y = ax + c : a \in A,\ c \in C\}$ and $P = B \times A\cdot B + C$. Clearly, given any $a \in A$, $b \in B$, $c \in C$, the point $(b, ab + c)$ is incident to the line $y = ax + c$. Therefore, by Szemeredi–Trotter, $|A|\,|B|\,|C| \lesssim |A|^{2/3}\,|B|^{2/3}\,|C|^{2/3}\,|A\cdot B + C|^{2/3}$.

In the context of $\mathbb{F}_q$, the finite field containing q elements, similar questions have been explored as well. Bourgain [2005] showed that for $A \subseteq \mathbb{F}_q$ such that $|A| \gtrsim q^{3/4}$, one has $A\cdot A + A\cdot A + A\cdot A = \mathbb{F}_q$; in particular, if $|A| \approx q^{3/4}$, then $|A\cdot A + A\cdot A + A\cdot A| \gtrsim |A|^{4/3}$. In [Hart and Iosevich 2008] it was shown that if $|A| \gtrsim q^{3/4}$, then $A\cdot A + A\cdot A = \mathbb{F}_q^*$; in particular, if $|A| \approx q^{3/4}$, then $|A\cdot A + A\cdot A| \gtrsim |A|^{4/3}$. Due to the misbehavior of the zero element, it is not possible to guarantee that $A\cdot A + A\cdot A = \mathbb{F}_q$ unless A is a positive proportion of the elements of $\mathbb{F}_q$. Under the weaker conclusion that $|A \cdot A + A \cdot A| \gtrsim q$ it is shown in the same paper that one may take $|A| \gtrsim q^{2/3}$. Shparlinski [2008] applied multiplicative character sums to show that if $|A| \gtrsim q^{2/3}$, then $|A \cdot A + A| \gtrsim q$, implying that if $|A| \approx q^{2/3}$, then $|A \cdot A + A| \gtrsim |A|^{3/2}$.

Theorem 3 [Chapman et al. 2009, Theorem 2.10]. *Let A be a subset of $\mathbb{F}_q^*$. Then*

$$|A|^{-1} \sum_{a \in A} |aA + A| \gtrsim \min(q, |A|^3 q^{-1}).$$

In particular, if $|A| \approx q^{2/3}$, there exists a subset A' of A with $|A'| \gtrsim |A|$ such that

$$|aA + A| \gtrsim |A|^{3/2} \approx q,$$

for all $a \in A'$.

It is natural to ask whether a similar statement holds in the case that A is a finite subset of the real numbers. We show that this is in fact the case in Section 4.

2. Statement of results

Define the multiplicative energy of the finite subsets A, B, C, D of real numbers by

$$\mathrm{E}(A, B, C, D) = \left|\left\{(x_1, x_2, y_1, y_2) \in A \times B \times C \times D : x_1 y_2 = x_2 y_1\right\}\right|.$$

For A, B finite subsets of positive real numbers with $|A| \le |B|$, the argument of [Solymosi 2009] gives the bound

$$\mathrm{E}(A, B, A, B) \le 4\lceil \log |A| \rceil \, |A + A| \, |B + B|. \tag{2-1}$$

A short Cauchy–Schwarz argument gives that $\mathrm{E}(A, B, A, B) \ge |A|^2 |B|^2 / |A \cdot B|$, which in turn gives the sum-product inequality

$$|A|^2 |B|^2 \le 4\lceil \log |A| \rceil |A + A| \, |B + B| \, |A \cdot B|. \tag{2-2}$$

In the case that $A = B$, this immediately implies the Solymosi sum-product bound discussed in the introduction:

$$\max(|A + A|, |A \cdot A|) \ge (4\lceil \log |A| \rceil)^{-1/3} |A|^{4/3}. \tag{2-3}$$

We will use a slight variant of the argument of Solymosi to give a different bound on the multiplicative energy:

Theorem 4. *Let A, B, C, D be finite subsets of positive real numbers. Then*

$$\mathrm{E}(A, B, C, D) \le 4\lceil \log(\min(|A|, |C|)) \rceil \, \lceil \log(\min(|B|, |D|)) \rceil |A + B| \, |C + D|.$$

(Notice that the logarithmic loss is worse than what was obtained by Solymosi.)

Using the fact that $\mathrm{E}(A, B, A, B) \geq |A|^2|B|^2/|A \cdot B|$, we obtain the following sum-product estimate.

Corollary 5. *Let A, B be finite subsets of positive real numbers. Then*

$$\max(|A+B|, |A \cdot B|) \geq (4\lceil \log |A| \rceil \lceil \log |B| \rceil)^{-1/3} |A|^{2/3} |B|^{2/3}. \tag{2-4}$$

One may compare this to the result of applying Plünnecke's inequality to (2-2):

$$\max(|A+B|, |A \cdot B|) \geq (4\lceil \log |A| \rceil)^{-1/5} |A|^{3/5} |B|^{3/5}. \tag{2-5}$$

We will also show this:

Theorem 6. *Let A, B, C be finite subsets of $\mathbb{R}$ such that $|B|^{1/2} |C|^{-1/2} \lesssim |A| \lesssim |B|^2|C|$. Then*

$$|A|^{-1} \sum_{a \in A} |aB + C| \gtrsim |A|^{1/3} |B|^{1/3} |C|^{2/3}. \tag{2-6}$$

In particular, there exists an $a \in A$ such that

$$|aB + C| \gtrsim |A|^{1/3} |B|^{1/3} |C|^{2/3}. \tag{2-7}$$

3. Proof of Theorem 4

We begin by writing

$$\begin{aligned}\mathrm{E}(A, B, C, D) &= \sum_{x_1 y_2 = x_2 y_1} A(x_1) B(x_2) C(y_1) D(y_2) \\ &= \sum_{t \neq 0} \sum_{\substack{x_1 = t x_2 \\ y_1 = t y_2}} (A \times C)(x_1, y_1)(B \times D)(x_2, y_2),\end{aligned}$$

where $A(\cdot)$ denotes the characteristic function of the set A and $\times$ denotes the Cartesian product. Summing in t we have

$$\mathrm{E}(A, B, C, D) = \sum_{y \in (B \times D)} |(A \times C) \cap l_{m_y}|,$$

where l_{m_y} is the line through the origin and the point y with slope m_y. Each $y \in (B \times D)$ lies on some line l_{m_y} with $m_y \in D \cdot B^{-1} = \{db^{-1} : d \in D, b \in B\}$. Since the quantity $|(A \times C) \cap l_{m_y}|$ is constant and nonzero for each y on l_{m_y} with slope m_y in $C \cdot A^{-1}$, we have

$$\mathrm{E}(A, B, C, D) = \sum_{m \in M} |(A \times C) \cap l_m| \, |(B \times D) \cap l_m|,$$

where $M = C \cdot A^{-1} \cap D \cdot B^{-1}$. We then take a dyadic decomposition

$$\mathrm{E}(A, B, C, D) = \sum_{\substack{0 \le i \le \lceil \log(\min(|A|,|C|)) \rceil \\ 0 \le j \le \lceil \log(\min(|B|,|D|)) \rceil}} \sum \sum_{m \in M_{i,j}} |(A \times C) \cap l_m| \, |(B \times D) \cap l_m|,$$

where $M_{i,j} = \{m \in M : 2^i \le |(A \times C) \cap l_m| < 2^{i+1}, 2^j \le |(B \times D) \cap l_m| < 2^{j+1}\}$. Therefore, for some i' and j',

$$\frac{\mathrm{E}(A, B, C, D)}{\lceil \log(\min(|A|, |C|)) \rceil \lceil \log(\min(|B|, |D|)) \rceil} \le \sum_{m \in M_{i',j'}} |(A \times C) \cap l_m| \, |(B \times D) \cap l_m|.$$

Set $n = |M_{i',j'}|$ and order the elements of $M_{i',j'}$: $m_1 < m_2 < \ldots < m_n$. This gives

$$\frac{\mathrm{E}(A, B, C, D)}{\lceil \log(\min(|A|, |C|)) \rceil \lceil \log(\min(|B|, |D|)) \rceil} \le 4n2^{i'+j'}.$$

Given that $|(A \times C) \cap l_{m_l} + (B \times D) \cap l_{m_{l+1}}| = |(A \times C) \cap l_{m_l}| \, |(B \times D) \cap l_{m_{l+1}}|$, noting that any two sum sets $(A \times C) \cap l_{m_l} + (B \times D) \cap l_{m_{l+1}}$ and $(A \times C) \cap l_{m_k} + (B \times D) \cap l_{m_{k+1}}$ are disjoint for any $l \ne k$ gives

$$n2^{i'+j'} \le \left| \bigcup_{l=1}^{n} ((A \times C) \cap l_{m_l} + (B \times D) \cap l_{m_{l+1}}) \right| \le |A + B| \, |C + D|.$$

Here, in an abuse of notation, $(B \times D) \cap l_{m_{n+1}}$ is the orthogonal projection of $(B \times D) \cap l_{m_n}$ onto the vertical line running through the minimal element of B. We may without loss of generality assume that the minimal element of B is also the minimal element of $A \cup B$.

4. Proof of Theorem 6

We will need a lemma, whose proof we will delay until the end of the section.

Lemma 7. *Let A, B, C be finite subsets of $\mathbb{R}$ such that $|B|^{1/2} |C|^{-1/2} \lesssim |A| \lesssim |B|^2 |C|$. Then*

$$\left| \left\{ (a, b, c, d, e) \in A \times B \times C \times B \times C : ab + c = ad + e \right\} \right| \lesssim |A|^{2/3} |B|^{5/3} |C|^{4/3}.$$

With this lemma in hand one may then apply the Cauchy–Schwarz inequality:

$$|A| \, |B|^2 \, |C|^2 = |A|^{-1} \Bigg(\sum_{\substack{t \in aB+C \\ a \in A}} \sum_{ab+c=t} B(b) C(c) \Bigg)^2$$

$$\le \Bigg(|A|^{-1} \sum_{a \in A} |aB + C| \Bigg) \sum_{\substack{t \in aB+C \\ a \in A}} \Bigg(\sum_{ab+c=t} B(b) C(c) \Bigg)^2.$$

Noting that

$$\sum_{\substack{t\in aB+C\\ a\in A}}\Big(\sum_{ab+c=t} B(b)C(c)\Big)^2 = \Big|\big\{(a,b,c,d,e)\in A\times B\times C\times B\times C : ab+c=ad+e\big\}\Big|$$

completes the proof of Theorem 6.

Proof of Lemma 7. We will apply the Szemerédi–Trotter incidence theorem. For a fixed $b\in B$, consider the set of lines $L_b=\{y=(b-d)x+c : c\in C, d\in B\}$. Also consider the set of points $P=\{(a,e)\in(A\times C)\}$. Then $|\{(a,b,c,d,e)\in A\times B\times C\times B\times C : ab+c=ad+e\}|\le|B|\max_{b\in B} I_{P,L_b}$. Noting that $|L_b|=|B|\,|C|$ and $|P|=|A|\,|C|$ and applying the Szemerédi–Trotter theorem gives

$$\Big|\big\{(a,b,c,d,e)\in A\times B\times C\times B\times C : ab+c=ad+e\big\}\Big| \lesssim |A|^{2/3}\,|B|^{5/3}\,|C|^{4/3},$$

as long as $|B|^{1/2}\,|C|^{-1/2}\lesssim|A|\lesssim|B|^2|C|$. □

5. Remarks

The argument of Elekes [1997] actually gives a more general bound for finite subsets A, B, C of positive real numbers:

$$\max(|A+B|, |A\cdot C|) \gtrsim |A|^{3/4}\,|B|^{1/4}\,|C|^{1/4}.$$

A direct application of Plünnecke's inequality [Tao and Vu 2006, Corollary 6.26] to (2-3) gives

$$\max(|A+B|, |A\cdot C|) \ge (4\lceil\log|A|\rceil)^{-1/6}|A|^{2/3}\,|B|^{1/3}\,|C|^{1/6}.$$

This bound is preferable if $|B|$ is much larger than $|A|\,|C|$. We do not currently know of a way to use Solymosi's argument to obtain an improved bound for the case that the three sets are close together in size.

References

[Alon and Spencer 2000] N. Alon and J. H. Spencer, *The probabilistic method*, 2nd ed., Wiley-Interscience, New York, 2000. MR 2003f:60003 Zbl 0996.05001

[Bourgain 2005] J. Bourgain, "Mordell's exponential sum estimate revisited", *J. Amer. Math. Soc.* **18**:2 (2005), 477–499. MR 2006b:11099 Zbl 1072.11063

[Chapman et al. 2009] J. Chapman, M. B. Erdoğan, D. Hart, A. Iosevich, and D. Koh, "Pinned distance sets, k-simplices, Wolff's exponent in finite fields and sum-product estimates", preprint, 2009. arXiv 0903.4218

[Elekes 1997] G. Elekes, "On the number of sums and products", *Acta Arith.* **81**:4 (1997), 365–367. MR 98h:11026 Zbl 0887.11012

[Erdős and Szemerédi 1983] P. Erdős and E. Szemerédi, "On sums and products of integers", pp. 213–218 in *Studies in pure mathematics*, Birkhäuser, Basel, 1983. MR 86m:11011 Zbl 0526.10011

[Ford 1998] K. Ford, "Sums and products from a finite set of real numbers", *Ramanujan J.* **2**:1-2 (1998), 59–66. MR 99i:11014 Zbl 0908.11008

[Hart and Iosevich 2008] D. Hart and A. Iosevich, "Sums and products in finite fields: an integral geometric viewpoint", pp. 129–135 in *Radon transforms, geometry, and wavelets*, Contemp. Math. **464**, Amer. Math. Soc., Providence, RI, 2008. MR 2009m:11032

[Nathanson 1997] M. B. Nathanson, "On sums and products of integers", *Proc. Amer. Math. Soc.* **125**:1 (1997), 9–16. MR 97c:11010 Zbl 0869.11010

[Shparlinski 2008] I. E. Shparlinski, "On the solvability of bilinear equations in finite fields", *Glasg. Math. J.* **50**:3 (2008), 523–529. MR 2009j:11189 Zbl 05350503

[Solymosi 2005] J. Solymosi, "On the number of sums and products", *Bull. London Math. Soc.* **37**:4 (2005), 491–494. MR 2006c:11021 Zbl 1092.11018

[Solymosi 2009] J. Solymosi, "Bounding multiplicative energy by the sumset", *Adv. Math.* **222**:2 (2009), 402–408. MR 2538014 Zbl 05597906

[Szemerédi and Trotter 1983] E. Szemerédi and W. T. Trotter, Jr., "Extremal problems in discrete geometry", *Combinatorica* **3**:3-4 (1983), 381–392. MR 85j:52014 Zbl 0541.05012

[Tao and Vu 2006] T. Tao and V. Vu, *Additive combinatorics*, vol. 105, Cambridge University Press, Cambridge, 2006. MR 2008a:11002 Zbl 1127.11002

Received: 2009-09-07 Accepted: 2009-11-12

dnhart@math.rutgers.edu *Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, United States*
http://www.math.rutgers.edu/~dnhart/

alexnizi@eden.rutgers.edu *School of Engineering, Rutgers University, Piscataway, NJ 08854-8019, United States*