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Weakly viewing lattice points

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A specific rectangular array of lattice points is investigated. We say that the array is weakly visible from a lattice point not in the array if no point in the array lies on the line connecting the external point to any other point in the array. A lower bound is found for the distance from a weakly viewing point to the array, and a point of minimal distance is determined for arrays of a specific size. A secondary type of visibility is also discussed, and a closest point viewing the array in this style is completely determined.

1. Introduction

Laison and Schick [2007] describe the situation of a photographer attempting to photograph every person in a rectangular formation, with all persons, including the photographer, standing on lattice points. The photos must be taken from a fixed position and each member of the formation must have a straight-line view of the photographer, unobstructed by all other persons in the rectangle. They prove that there are positions for the photographer to stand, but these may be quite a long way from the formation. How can we minimize this distance?

The problem is turned into one involving lattice points in the plane (the persons to be photographed forming the rectangle), and such an unobstructed view between two points is termed *weak visibility*. Utilizing a result from [Herzog and Stewart 1971], Laison and Schick proceed to investigate a more complicated question: assume the lattice points outside the formation also form obstructions. They term this situation being *externally visible*. In this paper we only consider the simpler question of weak visibility. We begin in Section 2 with the necessary terminology and preliminary results. In Section 3 we prove our main results. Section 4 considers a more specific type of visibility which we call *weak integer visibility*. Lastly, natural questions for future research are provided in Section 5.

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Keywords: weak visibility, lattice point.

2. Definitions and basics

All points are assumed to be lattice points in the first quadrant. Let $\Delta_{r,s}$ be the $r \times s$ rectangle of points with its lower left corner placed at $(1, 1)$. Say $\Delta_{r,s}$ is *weakly visible* from a point P if no point in $\Delta_{r,s}$ lies on any line segment connecting P and any other point in $\Delta_{r,s}$. Laison and Schick immediately prove the following result.

Theorem 2.1 [Laison and Schick 2007]. *The points* $P_1 = (rs - s + r, s + 1)$ *and* $P_2 = (r + 1, rs - r + s)$ *weakly view* $\Delta_{r,s}$ $(r, s \in \mathbb{Z}^+)$.

It turns out that this point may be quite far from $\Delta_{r,s}$, with distance being measured to the point (r, s). In the next section, we will place a lower bound on this distance dependent upon only the greater dimension of $\Delta_{r,s}$.

Notice that if $s < r$, then the point P_1 (of Theorem 2.1) is of closer distance to $\Delta_{r,s}$ than P_2. If $r < s$, then P_2 is closer to $\Delta_{r,s}$. If $r = s$, then the points are equidistant from $\Delta_{r,s}$. Because of this we will assume $s \le r$ for the remainder of the paper. The following lemma provides maximal and minimal values for certain calculations.

Lemma 2.2. *The lines of maximal* (*resp. minimal*) *positive slope passing through at least two points of* $\Delta_{r,s}$ *have slope* $s - 1$ (*resp.* $1/(r - 1)$).

Our last two definitions are the main reference tools for placing bounds on the visibility distance. Let $\mathrm{Adj}_{r,s}$, the *adjacency square to* $\Delta_{r,s}$, be the square of points whose corners are the points (r, s), $(r, r+s-1)$, $(2r-1, r+s-1)$, and $(2r-1, s)$. Define the *adjacency flag of slope* m, where $m = m_y/m_x$, by

$$\mathrm{Adj}_{r,s}^{F(m)} = \big\{(x, y) \,\big|\, mx - (m(r - m_x) - 1) \le y \le mx - (m(1 + m_x) - s)\big\}.$$

Intuitively, this is the union of all points between the extremal lines of slope m passing through at least two points of $\Delta_{r,s}$. See Figures 1 and 2.

3. Bounding visibility distance

To begin the search for a point of minimal distance weakly viewing $\Delta_{r,s}$, we look to adjacency flags. For certain values of m, every point in the adjacency flag can be disregarded.

Lemma 3.1. *Suppose that*

$$(1)\ m \le s/2, m \in \mathbb{Z}^+ \qquad or \qquad (2)\ 2/r \le m, 1/m \in \mathbb{Z}.$$

Then every point in $\mathrm{Adj}_{r,s}^{F(m)}$ *does not weakly view* $\Delta_{r,s}$.

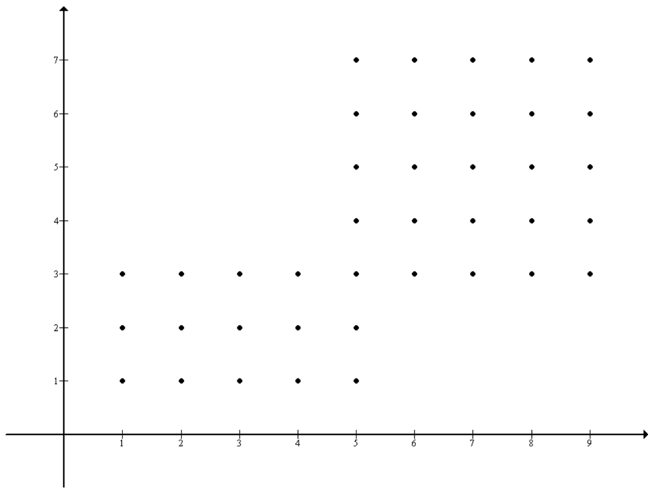

Figure 1. $\Delta_{5,3}$ and $\mathrm{Adj}_{5,3}$.

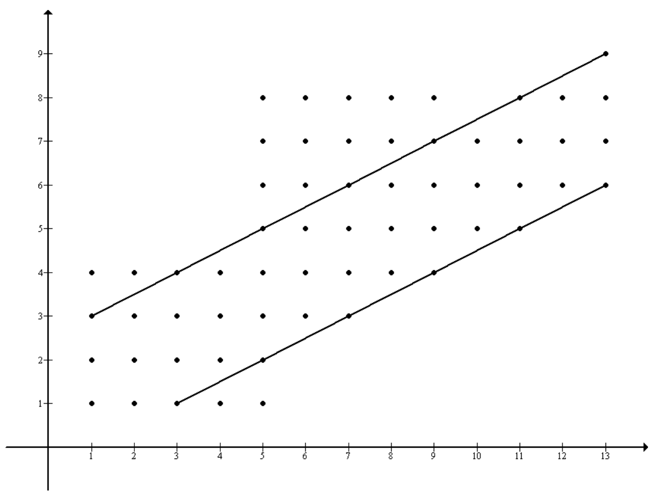

Figure 2. $\Delta_{5,4}$ with a portion of $\mathrm{Adj}_{5,4}^{F(1/2)}$.

Proof. We will prove the first case; the second is proven similarly. Suppose $m \in \mathbb{Z}^+$ and $m \leq s/2$. Take $(x_0, y_0) \in \mathrm{Adj}_{r,s}^{F(m)}$. Let L be the line

$$y - y_0 = m(x - x_0). \tag{1}$$

Let (a_0, b_0) be the point no in $\Delta_{r,s}$ on L closest to L. First, we claim that there is a point on L in $\Delta_{r,s}$. To prove this, we consider two possibilities:

Case 1: $s < b_0$. Since $(a_0, b_0) \notin \Delta_{r,s}$, then $2 \leq a_0$ and $b_0 \leq s + m$ by the choice of (a_0, b_0). Thus, since $m \in \mathbb{Z}^+$, $(a_0 - 1, b_0 - m) \in \Delta_{r,s}$.

Case 2: $b_0 \le s$. Since $m \in \mathbb{Z}^+$, we must have $a_0 = r + 1$. We need only show $1 \le b_0 - m \le s$. We know $b_0 - m \le s$ and since $(a_0, b_0) \in \mathrm{Adj}_{r,s}^{F(m)}$, we have:

$$\begin{aligned} ma_0 - mr + m + 1 &\le b_o, &(2)\\ ma_0 - mr + 1 &\le b_0 - m, &(3)\\ m(a_0 - r) + 1 &\le b_0 - m, &(4)\\ 1 &\le b_0 - m. &(5)\end{aligned}$$

Line (5) of the derivation follows from $r < a_0$. Both cases are proven, showing that there is indeed a point on L in $\Delta_{r,s}$. To finish the proof, we show that if $(a, b) \in \Delta_{r,s} \cap \mathrm{Adj}_{r,s}^{F(m)}$, then either $(a+1, b+m)$ or $(a-1, b-m) \in \Delta_{r,s}$. There are three cases to consider: $a = 1$, $a = r$, and $1 < a < r$.

If $a = 1$, then $b \le s - m$, giving $(a + 1, b + m) \in \Delta_{r,s}$. If $a = r$, then $m \le b$, yielding $(a - 1, b - m) \in \Delta_{r,s}$. If $1 < a < r$ but $(a - 1, b - m) \notin \Delta_{r,s}$, then

$$b - m \le 0, \quad b \le m, \quad b + m \le s.$$

Thus, $(a+1, b+m) \in \Delta_{r,s}$. Together with the first claim this shows that there are two points in $\Delta_{r,s}$ on L. Hence, (x_0, y_0) does not weakly view $\Delta_{r,s}$. □

Though it may seem restricted in its usefulness, this lemma is the main tool in proving our main result, Theorem 3.2. To prove it, we need one additional definition. Notice that the flag $\mathrm{Adj}_{r,s}^{F(1)}$ partitions $\mathrm{Adj}_{r,s}$ into two regions: those points that lie in $\mathrm{Adj}_{r,s}^{F(1)}$ and those that do not. We will refer to those points of $\mathrm{Adj}_{r,s}$ not in $\mathrm{Adj}_{r,s}^{F(1)}$ as the *lower triangle of* $\mathrm{Adj}_{r,s}$, denoted $\mathrm{Adj}_{r,s}^{LT}$. In particular, it is the triangle of points whose vertices are $(s+r, s)$, $(2r-1, s)$, and $(2r-1, r)$.

Theorem 3.2. *For $r, s > 1$, no point in $\mathrm{Adj}_{r,s}$ weakly views $\Delta_{r,s}$.*

Proof. First note that no point on $y = s$ weakly views $\Delta_{r,s}$. Via the following claims we will show the remainder of $\mathrm{Adj}_{r,s}^{LT}$ is contained in the union of adjacency flags satisfying the hypotheses of Lemma 3.1 (in the case of $s = 2$ we will need one additional observation).

Claim 1. *The upper edge of $\mathrm{Adj}_{r,s}^{LT}$ is fully contained in $\mathrm{Adj}_{r,s}^{1/2}$.*

The upper and lower boundaries of $\mathrm{Adj}_{r,s}^{1/2}$ are, respectively,

$$y = \tfrac{1}{2}x - \tfrac{3}{2} + s \tag{6}$$

and

$$y = \tfrac{1}{2}x - \tfrac{1}{2}r. \tag{7}$$

Line (6) intersects $x = 2r - 1$ at $(2r - 1, r + s - 2)$. Thus, the upper corner of $\mathrm{Adj}_{r,s}^{LT}$ (the point $(2r - 1, r)$) lies within $\mathrm{Adj}_{r,s}^{F(1/2)}$. The line

$$y = x - r + 1 \tag{8}$$

forms the upper boundary of $\mathrm{Adj}_{r,s}^{LT}$ and it intersects (7) at $x = r-2$, which is to the left of $(r+s, s+1)$. Therefore Claim 1 holds.

Claim 2. *The lower edge of* $\mathrm{Adj}_{r,s}^{LT}$ *is fully contained in* $\mathrm{Adj}_{r,s}^{F(2/r)}$ $(s \geq 3)$.

As in Claim 1, the segment connecting

$$(s+r, s+1) \quad \text{and} \quad (2r-1, s+1)$$

(the lower edge of $\mathrm{Adj}_{r,s}^{LT}$ satisfying $y > s$) lies on or between the lines forming the boundary of $\mathrm{Adj}_{r,s}^{F(2/r)}$. The details are left to the reader.

Claim 3. $\mathrm{Adj}_{r,s}^{LT} \subseteq \bigcup_m \mathrm{Adj}_{r,s}^{F(1/m)} (2 \leq m \leq r/2)$.

Due to the previous two claims, it is necessary only to consider

$$\mathrm{Adj}_{r,s}^{F(1/n)} \cap \mathrm{Adj}_{r,s}^{F(1/(n+1))}.$$

The upper boundaries of these flags intersect at $(1, s-1)$ while the lower boundaries intersect at $(r, 2)$. We need only show the intersection of the lower boundary of $\mathrm{Adj}_{r,s}^{F(1/n)}$ and the upper boundary of $\mathrm{Adj}_{r,s}^{F(1/(n+1))}$ occurs at or to the right of $x = 2r-1$, the right edge of $\mathrm{Adj}_{r,s}^{LT}$. This intersection occurs at

$$x = (n+1)r - 3n(n+1) - n + sn(n+1). \tag{9}$$

For $s = 3$,

$$2r-1 < \tfrac{5}{2}r+1 \leq 3r-n \leq (n+1)r-n = (s-3)n(n+1)-n+r(n+1),$$

while for $s \geq 4$, both

$$-1 < (s-3)n(n+1-n) \quad \text{and} \quad 2r < (n+1)r. \tag{10}$$

For $s = 2$, the lower edge of $\mathrm{Adj}_{r,s}^{F(2/r)}$ intersects $x = 2r-1$ at $y = 4$, leaving numerous points of $\mathrm{Adj}_{r,s}^{LT}$ (those with $y = 3$) unaccounted for in the above claims. However, consider the points $(x, 3)$ $(r \leq x \leq 2r-1)$. Take the line through $(x, 3)$ and $(r, 2)$. This line also passes through $(2r-x, 1)$, which lies in $\Delta_{r,s}$. Moreover, the overlapping flags of Claim 3 contain all (x, y) of $\mathrm{Adj}_{r,s}^{LT}$ when $y \geq 4$.

In all cases, we have shown that Claim 3 holds; that is,

$$\mathrm{Adj}_{r,s}^{LT} \subseteq \bigcup_m \mathrm{Adj}_{r,s}^{F(1/m)} (2 \leq m \leq r/2).$$

Since $r \geq s$, we know

$$\mathrm{Adj}_{r,s} - (\mathrm{Adj}_{r,s}^{LT} \cup (2r-1, s)) \subseteq \mathrm{Adj}_{r,s}^{F(1)}. \tag{11}$$

Because $(2r-1, s)$ lies on $x = 2r-1$ and $s \geq 2$, the result holds. □

Considering that every point in the adjacency square does not weakly view $\Delta_{r,s}$, we place a lower bound on the distance any point weakly viewing $\Delta_{r,s}$ must be from our $\Delta_{r,s}$:

Corollary 3.3. *If a point P weakly views $\Delta_{r,s}$, then P is at least $\sqrt{r^2+1}$ units away from $\Delta_{r,s}$.*

Proof. The closest possible P would be $(2r, s+1)$, which is of distance $\sqrt{r^2+1}$ from $\Delta_{r,s}$. □

We conclude our discussion on weak visibility with a complete determination of the specific case $s = 2$. This result is an improvement upon the initial point P_1 of in Theorem 2.1.

Corollary 3.4. *A point of minimal distance from $\Delta_{r,2}$ weakly viewing $\Delta_{r,2}$ is $(2r,3)$, for $r \geq 2$.*

Proof. The point $P = (2r, 3)$ lies below the line of minimal slope (from Lemma 2.2) passing through at least two points of $\Delta_{r,s}$ and P realizes the lower bound of Corollary 3.3. □

4. Weak integer visibility

Lemma 3.1 induces a different though significantly weaker version of viewing $\Delta_{r,s}$. Let $m \in \mathbb{Z}$. Say a point P *weakly integer views* $\Delta_{r,s}$ if no line of slope m or $1/m$ passes through P and two or more points of $\Delta_{r,s}$. The original question of weak visibility can now be posed in terms of weak integer visibility and its solution is completely determined. We begin by considering adjacency flags of integral (resp. integer reciprocal) slopes. Lemma 4.1 is a generalization of Claim 3 in the proof of Theorem 3.2. Its computational proof is left to the reader.

Lemma 4.1. *Let $n \in \mathbb{Z}^+$.*

(1) *The upper boundary of $\mathrm{Adj}_{r,s}^{F(n)}$ and the lower boundary of $\mathrm{Adj}_{r,s}^{F(n+1)}$ intersect at $((r-3)n+r+s-2, (r-3)n^2+(r+s-4)n+s)$.*

(2) *The lower boundary of $\mathrm{Adj}_{r,s}^{F(1/n)}$ and the upper boundary of $\mathrm{Adj}_{r,s}^{F(1/(n+1))}$ intersect at $((s-3)n^2+(s+r-4)n+r, (s-3)n+s+r-2)$.*

Each pair of adjacency flags creates a region of lattice points that weakly integer views $\Delta_{r,s}$: all points that are both below the higher sloped flag and above the lower sloped flag. Within each region there is a point closest to $\Delta_{r,s}$ weakly integer viewing $\Delta_{r,s}$, as described below. By comparing these points, we can find the point of minimal distance weakly integer viewing $\Delta_{r,s}$.

Notice where Lemma 4.1 places the intersection of flags of slopes n and $n+1$, and of those of slopes $1/n$ and $1/(n+1)$. For fixed values of r and s, the coordinate functions of the intersection point are strictly increasing with respect to n. Within these regions of points weakly integer viewing $\Delta_{r,s}$ there is a point closest to $\Delta_{r,s}$.

Lemma 4.2. *Let $n \in \mathbb{Z}$.*

(1) *The point of minimal distance to $\Delta_{r,s}$ between the upper boundary of $\mathrm{Adj}_{r,s}^{F(n)}$ and the lower boundary of $\mathrm{Adj}_{r,s}^{F(n+1)}$ is*

$$\big((r-3)n + r + s,\ (r-3)n^2 + (r+s-2)n + s + 1\big). \tag{12}$$

(2) *The point of minimal distance to $\Delta_{r,s}$ between the lower boundary of $\mathrm{Adj}_{r,s}^{F(1/n)}$ and the upper boundary of $\mathrm{Adj}_{r,s}^{F(1/(n+1))}$ is*

$$\big((s-3)n^2 + (s+r-2)n + r,\ (s-3)n + s + r\big). \tag{13}$$

Proof. Suppose two lines L_1 and L_2 of positive integral slopes n and $n+1$, respectively, intersect at (a, b). In the positive direction, the next lattice points to lie on L_1 and L_2 are $(a+1, b+n)$ and $(a+1, b+n+1)$. The triangle created by these two points and (a, b) contains no lattice points on its interior or its boundary (other than its vertices). Similarly, the quadrilateral whose vertices are $(a+1, b+n)$, $(a+1, b+n+1)$, $(a+2, b+2n)$, and $(a+2, b+2n+2)$ contains no lattice points on its interior. However, there is a single nonvertex lattice point on its exterior: $(a+2, b+2n+1)$.

The second case follows mutatis mutandis. □

With regards to the comments preceding Lemma 4.2, to prove the following theorem we need only consider the two pairs of flags of slopes 1 and 2 and slopes 1 and $1/2$.

Theorem 4.3. *The point of minimal distance weakly integer viewing $\Delta_{r,s}$ is*

$$(2(r+s)-4,\ r+2s-3).$$

Proof. Consider the two points weakly integer viewing $\Delta_{r,s}$ in the regions formed by the pair $\mathrm{Adj}_{r,s}^{F(1)}$ and $\mathrm{Adj}_{r,s}^{F(2)}$ and by the pair $\mathrm{Adj}_{r,s}^{F(1)}$ and $\mathrm{Adj}_{r,s}^{F(1/2)}$. We assume $s < r$. Because of this, the point of Lemma 4.2 falling between the flags of slope 1 and $1/2$ is nearer to $\Delta_{r,s}$. Moreover, this point is closer to $\Delta_{r,s}$ than the nearest point found outside the boundaries of considered flags, the point of Theorem 2.1: $(rs - s + r, s + 1)$. □

5. Further questions

The foremost natural question is still that posed in [Laison and Schick 2007]: is there a formula dependent only upon r and s giving the point closest to $\Delta_{r,s}$ weakly viewing $\Delta_{r,s}$? If not, how strong can the bounds be made? The flags mentioned can be broken into two types: integral (or reciprocal of) and fractional slopes. We did not discuss fractional sloped flags at all, but a deeper discussion of them may lead to more precise answers.

In terms of the original question, what if the formation is not rectangular? What can be said about triangular, pentagonal, or other simple geometric shapes? Another way of making the situation more realistic is by considering each lattice point to have some sort of weight attached to it.

Finally, following Laison and Schick's thoughts, what happens when we attempt to weakly view similar structures in higher dimensions? The problem becomes much more realistic by attaching weight functions corresponding to persons' heights to lattice points on the xy-plane in $\mathbb{R}^3$.

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