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Stephanie Edwards, Victoria Elandt, Nicholas James,
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Lights Out on finite graphs

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Lights Out is a one-player game played on a finite graph. In the standard game the vertices can be either on or off; pressing a vertex toggles its state and that of all adjacent vertices. The goal of the game is to turn off all of the lights. We study an extension of the game in which the state of a vertex may be one of a finite number of colors. We determine which graphs in certain families (spider graphs and generalized theta graphs) are winnable for every initial coloring. We also provide a construction that gives every always-winnable tree for any prime power number of colors.

1. Introduction

The Lights Out game was popularized as a hand-held electronic puzzle produced by Tiger Electronics in 1995. The puzzle consists of a 5×5 square grid of buttons, each of which can be either on or off. A move consists of pressing one of the buttons, which changes the state of that button and all vertical and horizontal neighbors. Given an initial configuration in which some subset of the lights are on, the goal of the solver is to turn off all the lights. An initial configuration of lights will be called *winnable* if the puzzle can be solved when starting from that configuration.

The mathematical study of this puzzle and its generalizations has produced interesting results in graph theory, some of which predate the electronic version of the game. The puzzle on 5×5 grids was studied by Anderson and Feil [1998], who used linear algebra over $\mathbb{Z}_2$ to classify winnable configurations. The analysis on $n \times m$ grids was done using Fibonacci polynomials in [Goldwasser and Klostermeyer 1997; Goldwasser et al. 1997; 2002]. Earlier, Sutner [1989] had shown that the winnability of configurations in Lights Out is also equivalent to a question on finite cellular automata.

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The puzzle has a natural generalization to any finite graph, in which each vertex of the graph starts as either on or off, and pressing a vertex toggles that vertex and all adjacent vertices. Amin and Slater [1996] have studied this generalization for some classes of finite graphs under the equivalent notion of parity domination theory. In particular, they classify paths, 'spider' graphs (i.e., generalized stars), and 'caterpillar' graphs for which every initial configuration is winnable — in their language, *all parity realizable* graphs — and they give a construction which produces all trees that are winnable from every initial configuration.

Giffen and Parker [2009] have further generalized the puzzle to the setting in which each vertex on the finite graph has k states, which are denoted by the elements of $\mathbb{Z}_k$. The state $0 \in \mathbb{Z}_k$ is considered off. Pressing a vertex increments that vertex and all adjacent vertices by 1 (mod k). A graph G is *always winnable* (AW) over $\mathbb{Z}_k$ if every initial configuration is winnable with the above assumptions. Giffen and Parker classify winnable configurations on paths and cycles, and also determine which paths, cycles and caterpillar graphs are AW over $\mathbb{Z}_k$. Moreover, they develop a notion of domination theory for finite graphs that is equivalent to the multicolored Lights Out puzzle.

This paper generalizes both the results in [Amin and Slater 1996; Giffen and Parker 2009] by studying the winnability of Lights Out over $\mathbb{Z}_k$ for spider graphs, (generalized) theta graphs, and trees. We establish our basic notation and prove some helpful technical results in Section 2. In Section 3, we study the winnability of spider graphs and determine which spider graphs are AW over $\mathbb{Z}_k$. We prove similar results for generalized theta graphs in Section 4. In Section 5, we generalize the construction of Amin and Slater to produce all AW trees over $\mathbb{Z}_{p^e}$, where p is prime and e is a positive integer.

2. Notation and basic results

The term *graph* will designate a finite multigraph (without loops). Given a graph G, we denote the vertex set by $V(G)$ and the edge set by $E(G)$. An edge will typically be denoted by the pair of incident vertices. Given an enumeration

$$V(G) = \{v_1, \ldots, v_n\}$$

of the vertices of G, we define the *neighborhood matrix* of G to be

$$N(G) = \operatorname{adj}(G) + I_n,$$

where $\operatorname{adj}(G)$ is the usual adjacency matrix of G and I_n is the $n \times n$ identity matrix. A *coloring* of the vertices will correspond to a column vector $\vec{b} \in \mathbb{Z}_k^n$ where b_i is the color of v_i. The act of 'pressing vertex v_i' adds the ith column of $N(G)$ to $\vec{b}$, with addition taking place in $\mathbb{Z}_k^n$.

Remark 2.1. We have allowed a graph G to have multiple edges, even though most of the graphs considered in this paper do not have multiple edges. This is due to the fact that the reduction described in Proposition 2.7 may result in multiple edges when used on generalized theta graphs in Section 4. If there are m edges between vertices v and w, then pressing v will increment the color on v by 1 and the color on w by m.

An initial coloring $\vec{b} \in \mathbb{Z}_k^n$ is called *winnable* if there exists a sequence of presses that transforms $\vec{b}$ to $\vec{0}$. As shown for two colors in Anderson and Feil [1998], $\vec{b}$ is winnable if and only if the equation

$$N(G)\vec{x} = -\vec{b}$$

has a solution vector $\vec{x} \in \mathbb{Z}_k^n$. In this case, the solution vector $\vec{x}$ is called the *winning strategy* for $\vec{b}$ and gives the vertices that should be pressed, and how many times, in order to convert $\vec{b}$ to $\vec{0}$. Thus, an initial coloring $\vec{b}$ is winnable if and only if $-\vec{b}$ (and hence $\vec{b}$) is in the column space of $N(G)$ over $\mathbb{Z}_k$. A graph G will be called *always winnable* (AW) over $\mathbb{Z}_k$ if every initial coloring $\vec{b} \in \mathbb{Z}_k^n$ can be won (i.e., if the column space of $N(G)$ is equal to $\mathbb{Z}_k^n$).

We use $d(G)$ to denote $\det(N(G))$, computed over $\mathbb{Z}$, since this number occurs often. We adopt the convention that the determinant of the 'empty' matrix is 1. Thus, if G is the graph with no vertices and no edges, $d(G) = 1$ by convention. If G_1 and G_2 are the connected components of G, then $N(G)$ is block-diagonal, and $d(G) = d(G_1)d(G_2)$.

Proposition 2.2. *For any graph G and integer $k \geq 2$, the following are equivalent.*

(1) *G is AW over $\mathbb{Z}_k$.*

(2) *The column space of $N(G)$ is $\mathbb{Z}_k^n$.*

(3) *The null space of $N(G)$ is $\{\vec{0}\}$.*

(4) *$d(G)$ is relatively prime to k.*

Proof. When k is prime, this is immediate from the basic theory of vector spaces over fields. the general case involves the relationship between determinants and free modules over a commutative ring; see [Bourbaki 1974, III.8.2, Theorem 1]. □

The following corollary is a consequence of the equivalence of (1) and (4) in the preceding proposition.

Corollary 2.3. *Let G be any graph.*

(1) *For any integer $k \geq 2$, G is AW over $\mathbb{Z}_k$ if and only if G is AW over $\mathbb{Z}_p$ for every prime factor p of k.*

(2) *For any prime number p and any positive integer e, G is AW over $\mathbb{Z}_{p^e}$ if and only if G is AW over $\mathbb{Z}_p$.*

Remark 2.4. It is also immediate that a graph is AW if and only if each of its connected components is AW.

We now give some useful technical results that involve winnability of colorings on related graphs. Given a graph G and a subset $S \subset V(G)$, we define $G - S$ to be the graph obtained by deleting the vertices in S from G, along with any edges incident with vertices in S.

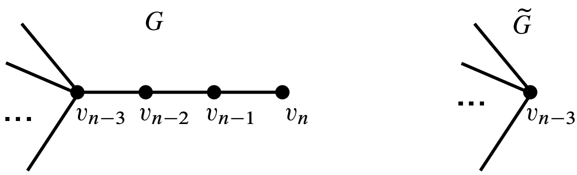
Proposition 2.5. *Suppose a graph G has a set of distinct vertices*

$$\{v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$$

with edges

$$\{v_{n-3}v_{n-2}, v_{n-2}v_{n-1}, v_{n-1}v_n\},$$

where $\deg v_{n-2} = \deg v_{n-1} = 2$ *and* $\deg v_n = 1$. (*Note that v_{n-3} can have any degree.*) *Let* $\widetilde{G} = G - \{v_{n-2}, v_{n-1}, v_n\}$.



(1) *The following are equivalent*:
   (a) *The initial coloring* $\vec{b} = \langle b_1, b_2, \ldots, b_{n-3}\rangle^T \in \mathbb{Z}_k^{n-3}$ *is winnable on* $\widetilde{G}$.
   (b) *The initial coloring*

$$\vec{b}' = \langle b_1, b_2, \ldots, b_{n-3} + c_1, c_1 + c_2, c_1 + c_2 + c_3, c_2 + c_3\rangle^T \in \mathbb{Z}_k^n$$

   is winnable on G for all choices of $c_1, c_2, c_3 \in \mathbb{Z}_k$.
   (c) *The initial coloring* $\vec{b}' = \langle b_1, b_2, \ldots, b_{n-3}, 0, 0, 0\rangle^T \in \mathbb{Z}_k^n$ *is winnable on G.*

(2) *G is AW over $\mathbb{Z}_k$ if and only if $\widetilde{G}$ is AW over $\mathbb{Z}_k$.*

Proof. Let J be the $(n-3) \times 3$ matrix such that $j_{(n-3),1} = 1$ and all other entries are 0. Then

$$N(G) = \left(\begin{array}{c|c} N(\widetilde{G}) & J \\ \hline J^T & \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{matrix} \end{array}\right).$$

(a) $\Rightarrow$ (b) Suppose $\vec{x} = \langle x_1, \ldots, x_{n-3}\rangle^T$ is a winning strategy for $\vec{b} \in \mathbb{Z}_k^{n-3}$ on $\widetilde{G}$. For any $c_1, c_2, c_3 \in \mathbb{Z}_k$, let

$$\vec{x}' = \langle x_1, \ldots, x_{n-3}, -c_1, -c_2 - x_{n-3}, -c_3 + x_{n-3}\rangle^T,$$
$$\vec{b}' = \langle b_1, b_2, \ldots, b_{n-3} + c_1, c_1 + c_2, c_1 + c_2 + c_3, c_2 + c_3\rangle^T.$$

Then $N(G)\vec{x}' = -\vec{b}'$, showing that $\vec{x}'$ is a winning strategy for $\vec{b}'$ on G.

(b) $\Rightarrow$ (c) Immediate.

(c) $\Rightarrow$ (a) For a given vector $\vec{b} \in \mathbb{Z}_k^{n-3}$, suppose that

$$\vec{b}' = \langle b_1, b_2, \ldots, b_{n-3}, 0, 0, 0\rangle^T$$

can be won on G with winning strategy $\vec{y} = \langle y_1, \ldots, y_n\rangle^T$. The last two entries of $N(G)\vec{y} = -\vec{b}'$ imply that $y_{n-2} + y_{n-1} + y_n = 0 \pmod{k}$ and $y_{n-1} + y_n = 0 \pmod{k}$. This implies that $y_{n-2} = 0 \pmod{k}$. This, combined with the fact that $N(G)\vec{y} = -\vec{b}'$, implies that

$$N(\widetilde{G})\begin{pmatrix} y_1 \\ \vdots \\ y_{n-3}\end{pmatrix} = -\begin{pmatrix} b_1 \\ \vdots \\ b_{n-3}\end{pmatrix}.$$

Thus, $\vec{b}$ is winnable on $\widetilde{G}$. $\square$

Corollary 2.6. *We retain the hypotheses and notation of Proposition 2.5. An initial coloring $\vec{a} = \langle a_1, a_2, \ldots, a_n\rangle^T \in \mathbb{Z}_k^n$ is winnable on G if and only if the initial coloring*

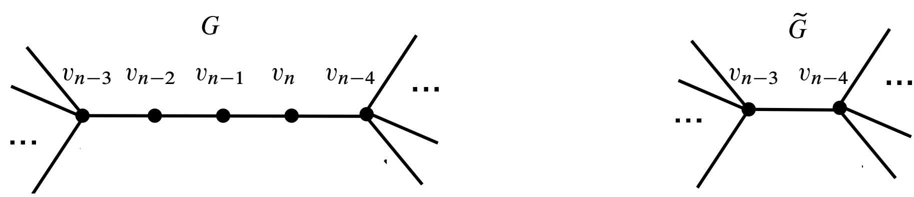
$$\vec{a}' = \langle a_1, a_2, \ldots, a_{n-4}, a_{n-3} - a_{n-1} + a_n\rangle^T \in \mathbb{Z}_k^{n-3}$$

is winnable on $\widetilde{G}$.

Proposition 2.7. *Suppose a graph G has a set of distinct vertices*

$$\{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\}$$

with edges $\{v_{n-3}v_{n-2}, v_{n-2}v_{n-1}, v_{n-1}v_n, v_nv_{n-4}\}$, where $\deg v_{n-2} = \deg v_{n-1} = \deg v_n = 2$. (Note that v_{n-3} and v_{n-4} can have any degree.) *Let $\widehat{G}$ be the union of $G - \{v_{n-2}, v_{n-1}, v_n\}$ with a new edge $v_{n-3}v_{n-4}$.*



(1) *The following are equivalent*:

(a) *The initial coloring*

$$\vec{b} = \langle b_1, b_2, \dots, b_{n-4}, b_{n-3} \rangle^T \in \mathbb{Z}_k^{n-3}$$

is winnable on $\widehat{G}$.

(b) *The initial coloring*

$$\vec{b}' = \langle b_1, b_2, \dots, b_{n-4} + c_3, b_{n-3} + c_1, c_1 + c_2, c_1 + c_2 + c_3, c_2 + c_3 \rangle^T \in \mathbb{Z}_k^n$$

is winnable on G *for all choices of* $c_1, c_2, c_3 \in \mathbb{Z}_k$.

(c) *The initial coloring*

$$\vec{b}' = \langle b_1, b_2, \dots, b_{n-4}, b_{n-3}, 0, 0, 0 \rangle^T \in \mathbb{Z}_k^n$$

is winnable on G.

(2) G *is AW over* $\mathbb{Z}_k$ *if and only if* $\widehat{G}$ *is AW over* $\mathbb{Z}_k$.

Proof. We proceed as in the proof of Proposition 2.5.

(a) $\Rightarrow$ (b): If $\vec{x} = \langle x_1, \dots, x_{n-3} \rangle^T$ is a winning strategy for $\vec{b} \in \mathbb{Z}_k^{n-3}$ on $\widehat{G}$, then, for any $c_1, c_2, c_3 \in \mathbb{Z}_k$, the vector

$$\vec{x}' = \langle x_1, \dots x_{n-4}, x_{n-3}, -c_1 + x_{n-4}, -c_2 - x_{n-3} - x_{n-4}, -c_3 + x_{n-3} \rangle^T$$

is a winning strategy on G for

$$\vec{b}' = \langle b_1, b_2, \dots, b_{n-4} + c_3, b_{n-3} + c_1, c_1 + c_2, c_1 + c_2 + c_3, c_2 + c_3 \rangle^T.$$

(b) $\Rightarrow$ (c): Immediate.

(c) $\Rightarrow$ (a): Given a winning strategy $\vec{y} \in \mathbb{Z}_k^n$ for

$$\vec{b}' = \langle b_1, b_2, \dots, b_{n-3}, 0, 0, 0 \rangle^T$$

on G, the fact that $N(G)\vec{y} = -\vec{b}'$ shows that $y_{n-4} = y_{n-2} \pmod k$ and $y_{n-3} = y_n \pmod k$. This implies that

$$N(\widehat{G}) \begin{pmatrix} y_1 \\ \vdots \\ y_{n-3} \end{pmatrix} = - \begin{pmatrix} b_1 \\ \vdots \\ b_{n-3} \end{pmatrix}.$$

Thus, $\vec{b}$ is winnable on $\widehat{G}$. $\square$

Corollary 2.8. *We retain the hypotheses and notation of Proposition 2.7. An initial coloring* $\vec{a} = \langle a_1, a_2, \dots, a_n \rangle^T \in \mathbb{Z}_k^n$ *is winnable on* G *if and only if the initial coloring*

$$\vec{a}' = \langle a_1, a_2, \dots, a_{n-5}, a_{n-4} - a_{n-1} + a_{n-2}, a_{n-3} - a_{n-1} + a_n \rangle^T \in \mathbb{Z}_k^{n-3}$$

is winnable on $\widehat{G}$.

For any matrix A, let A_{ij} represent the minor obtained by deleting the ith row and jth column from A. Again, we assume that the determinant of an 'empty' matrix, formed by taking a minor of a matrix with only one row or column, is 1.

Lemma 2.9. *Let M be an $m \times m$ matrix and N an $n \times n$ matrix. Let J be the $m \times n$ matrix such that $j_{m,1} = 1$ and all other entries of J are* 0. *Then*

$$\det \begin{pmatrix} M & J \\ J^T & N \end{pmatrix} = \det(M)\det(N) - \det M_{mm} \det N_{11}.$$

Proof. This follows from a standard proof by induction. □

Proposition 2.10. Let G_1 and G_2 be graphs, let v be a vertex of G_1, and let w be a vertex of G_2. Let $H = H(G_1, G_2, v, w)$ be the graph formed by connecting G_1 and G_2 with an edge vw. Then

$$d(H) = d(G_1)d(G_2) - d(G_1 - \{v\})d(G_2 - \{w\}).$$

Proof. Assume G_1 has m vertices, of which v is the last, and G_2 has n vertices, of which w is the first. The result follows immediately from Lemma 2.9, since

$$N(H) = \begin{pmatrix} N(G_1) & J \\ J^T & N(G_2) \end{pmatrix}$$

where J is as in the previous result, $N(G_1 - \{v\}) = N(G_1)_{mm}$ and $N(G_2 - \{w\}) = N(G_2)_{11}$. □

Corollary 2.11. Suppose $k = p^e$ for some prime p and positive integer e. Let G_1 and G_2 be graphs that are AW over $\mathbb{Z}_k$. Let $v \in V(G_1)$ and $w \in V(G_2)$, and suppose that $G_1 - \{v\}$ is not AW over $\mathbb{Z}_k$. Then the graph $H(G_1, G_2, v, w)$ constructed in the previous result is AW over $\mathbb{Z}_k$.

Proof. Since G_1 and G_2 are AW over $\mathbb{Z}_k$, we have $p \not| \, d(G_1)$ and $p \not| \, d(G_2)$. Since $G_1 - \{v\}$ is not AW over $\mathbb{Z}_k$, we have $p | d(G_1 - \{v_1\})$. Therefore,

$$p \not| \, [d(G_1)d(G_2) - d(G_1 - \{v\})d(G_2 - \{w\})].$$ □

Proposition 2.12. Let G_i be any graphs for $1 \le i \le m$, and let $v_i \in V(G_i)$. Let $W = W(G_1, \ldots, G_m, v_1, \ldots, v_m)$ be the graph formed by all vertices and edges of the graphs G_i together with a new vertex w and edges $v_i w$ for $1 \le i \le m$. Then

$$d(W) = \prod_{j=1}^{m} d(G_j) - \sum_{i=1}^{m} \left(d(G_i - \{v_i\}) \prod_{j \neq i} d(G_j) \right).$$

Proof. This follows from Proposition 2.10 and induction. For the base case, attach the single vertex w to a graph G_1 by an edge wv_1. □

Corollary 2.13. Let $k = p^e$ for some prime number p and positive integer e, and suppose each G_i is AW over $\mathbb{Z}_k$. Then $W(G_1, \dots, G_m, v_1, \dots, v_m)$ is AW over $\mathbb{Z}_k$ if and only if

$$\sum_{i=1}^{m} d(G_i)^{-1} d(G_i - \{v_i\}) \neq 1 \pmod{p}.$$

Here, the inverse is taken mod p.

Proof. The graph W is AW over $\mathbb{Z}_k$ if and only if $d(W) \neq 0 \pmod{p}$. The result follows by applying Proposition 2.12 to expand $d(W) \neq 0 \pmod{p}$, then multiplying both sides of the resulting equation by $\prod_{j=1}^{m} d(G_j)^{-1}$. □

3. Spider graphs

In this section we study the winnability of spider graphs (also called *generalized stars*). Specifically, we define *reduced spider graph* and determine which initial colorings are winnable on reduced spiders. This is then used to determine which spider graphs are AW over $\mathbb{Z}_k$.

First we provide a formal definition of a spider graph.

Definition 3.1. Let $V(P_i) = \{v_{i,1}, v_{i,2}, \dots, v_{i,n_i}\}$ be the vertices of a path with edges $E(P_i) = \{v_{i,j}v_{i,j+1} : 1 \leq j \leq n_i - 1\}$. A *spider graph* G is defined as the union of paths $P_1, \dots, P_l$ for some $l > 2$ along with a new vertex v_0, with edges consisting of the original edges from each P_i together with edges $v_0 v_{i,1}$ for $1 \leq i \leq l$. We call the paths P_i the *legs* of the spider and leg i has *length* n_i. A *reduced spider graph* is a spider graph that has legs of lengths 1 and 2 only.

Notation 3.2. Throughout this section, we assume that a spider graph G has

- m legs of length 1 (mod 3), labeled $P_1, \dots, P_m$,
- t legs of length 2 (mod 3), labeled $P_{m+1}, \dots, P_{m+t}$, and
- $l - (m + t)$ legs of length 0 (mod 3), labeled $P_{m+t+1}, \dots, P_l$.

For a reduced spider, we will have no legs of length 0 (mod 3), and in that case, $l = m + t$. An initial coloring $\vec{b}$ on a spider graph G is a vector $\mathbb{Z}_k^N$ (where $N = 1 + \sum_{i=1}^{l} n_i$) such that $b_{i,j}$ is the initial color of $v_{i,j}$ and b_0 is the initial color of v_0.

The next result shows which initial colorings are winnable on reduced spider graphs. Corollary 2.6 can then be used inductively to determine whether any given initial coloring of a general spider is winnable.

Theorem 3.3. *Let G be a reduced spider graph labeled as in Notation 3.2. Let $\vec{b} \in \mathbb{Z}_k^N$ be an initial coloring of G.*

(1) *If* $t = 0$ *then* $\vec{b}$ *is winnable on* G *if and only if* $\gcd(m-1, k)$ *divides*

$$-b_0 + \sum_{i=1}^{m} b_{i,1}.$$

(2) *If* $t \neq 0$ *then* $\vec{b}$ *is winnable on* G *if and only if* $b_{i,2} - b_{i,1} = b_{j,2} - b_{j,1}$ *for all* i *and* j *such that* $m+1 \leq i, j \leq m+t = l$.

Proof. Let G and $\vec{b}$ be as in the hypotheses of the theorem. In order to win, we must press each vertex some number of times. Suppose v_0 is pressed d_0 times and $v_{i,j}$ is pressed $d_{i,j}$ times. The effects of pressing these vertices are:

- the color of v_0 is changed by $d_0 + \sum_{i=1}^{m+t} d_{i,1}$;
- for $1 \leq i \leq m$, the color of $v_{i,1}$ is changed by $d_0 + d_{i,1}$;
- for $m+1 \leq i \leq m+t$, the color of $v_{i,1}$ is changed by $d_0 + d_{i,1} + d_{i,2}$;
- for $m+1 \leq i \leq m+t$, the color of $v_{i,2}$ is changed by $d_{i,1} + d_{i,2}$.

To win, we must change the color of every vertex to 0, which yields the following system of equations mod k. These equations are equivalent to the matrix equation $N(G)\vec{d} = -\vec{b}$.

$$b_0 + d_0 + \sum_{i=1}^{m+t} d_{i,1} = 0 \tag{3-1}$$

$$b_{i,1} + d_{i,1} + d_0 = 0 \quad \text{for } 1 \leq i \leq m \tag{3-2}$$

$$b_{i,1} + d_{i,1} + d_{i,2} + d_0 = 0 \quad \text{for } m+1 \leq i \leq m+t \tag{3-3}$$

$$b_{i,2} + d_{i,1} + d_{i,2} = 0 \quad \text{for } m+1 \leq i \leq m+t \tag{3-4}$$

Equation (3-4) allows us to reduce (3-3) to:

$$d_0 = b_{i,2} - b_{i,1} \quad \text{for } i = m+1, \ldots, m+t. \tag{3-5}$$

(1) Suppose $t = 0$. The initial coloring $\vec{b}$ is winnable on G if and only if Equations (3-1) and (3-2) are consistent. Rewriting (3-2) as $d_{i,1} = -d_0 - b_{i,1}$ and substituting into (3-1) shows that $\vec{b}$ is winnable if and only if

$$d_0(1-m) = -b_0 + \sum_{i=1}^{m} b_{i,1}$$

has a solution for d_0 mod k. This is true if and only if $\gcd(m-1, k)$ divides $-b_0 + \sum_{i=1}^{m} b_{i,1}$.

(2) Suppose $t \neq 0$. If an initial coloring $\vec{b}$ is winnable on G, Equation (3-5) gives $b_{i,2} - b_{i,1} = b_{j,2} - b_{j,1}$ for all $m+1 \leq i, j \leq m+t$.

Conversely, if $b_{i,2}-b_{i,1}=b_{j,2}-b_{j,1}$ for all $m+1\le i,\, j\le m+t$, the value of d_0 is determined by (3-5). The values of $d_{i,1}$ for $1\le i\le m$ are then determined by (3-2). Now, we may choose any integers $d_{i,1}$ for $m+1\le i\le m+t$ so that (3-1) holds, and this is possible since $t>0$. Finally, the values of $d_{i,2}$ for $m+1\le i\le m+t$ are determined (consistently) by (3-3) and (3-4). Therefore, the system has a solution vector $\vec{d}$. □

Theorem 3.4 (Characterization of AW spider graphs). *Let G be a spider graph* (see 3.2 for notation). *Then G is AW over $\mathbb{Z}_k$ if and only if either*

(1) $t=0$ *and* $\gcd(m-1,k)=1$, *or*

(2) $t=1$.

Proof. Consider a spider graph G. By Proposition 2.5, G is AW over $\mathbb{Z}_k$ if and only if $\widetilde{G}$ is AW over $\mathbb{Z}_k$, where $\widetilde{G}$ is the reduced spider graph with m legs of length 1 and t legs of length 2. We assume that $\widetilde{G}$ is also labeled as in 3.2.

Suppose that $t=0$ and $\gcd(m-1,k)=1$. Then by Theorem 3.3(1), every initial coloring on $\widetilde{G}$ is winnable over $\mathbb{Z}_k$. Conversely, suppose that $t=0$ and that $\gcd(m-1,k)\neq 1$. Then $\gcd(m-1,k)$ does not divide $-b_0+\sum_{i=1}^{m} b_{i,1}$ when $b_{1,1}=1$, $b_0=0$, and $b_{i,1}=0$ for $i=2,\dots m$. This gives an example of an initial coloring $\vec{b}$ which is not winnable on $\widetilde{G}$.

Suppose $t=1$. The condition in Theorem 3.3(2) is automatically satisfied for every initial coloring $\vec{b}$, and therefore $\widetilde{G}$ is AW over $\mathbb{Z}_k$.

Finally, suppose that $t>1$. In this case, there are clearly initial colorings on $\widetilde{G}$ that do not satisfy the condition in Theorem 3.3(2), showing that $\widetilde{G}$ is not AW. □

4. Generalized theta graphs

In this section we study the winnability of (generalized) theta graphs. We define a notion of *reduced* theta graph, and determine which initial colorings on reduced theta graphs are winnable. This information is then used to determine which theta graphs are AW over $\mathbb{Z}_k$.

Definition 4.1. Let $V(P_i)=\{v_{i,1},v_{i,2},\dots,v_{i,n_i}\}$ be the vertices of a path with edge set $E(P_i)=\{v_{i,j}v_{i,j+1}: 1\le j\le n_i-1\}$. A (*generalized*) *theta graph* G is defined as the union of disjoint paths $P_1,\dots,P_l$ for some $l>2$ along with two new vertices v_0 and v_n, with edges given by

- the original edges from each P_i,
- edges $v_0v_{i,1}$ and $v_{i,n_i}v_n$ for $1\le i\le l$, and
- possibly one or more edges of the form v_0v_n (i.e., there may or may not be edges of the form v_0v_n).

We call the paths P_i, where $1 \le i \le l$, *paths* of the theta graph and path i has *length* n_i. We will refer to each of the edges $v_0 v_n$ as a *path of length* 0 in the theta graph. A *reduced theta graph* is a theta graph which only has paths of lengths 0, 1, and 2.

The winnability of generalized theta graphs modulo 2 has been studied in the literature. In [Amin and Slater 1992], the graphs we have called *generalized theta graphs* occur as a particular case of *series parallel graphs*, and a linear time algorithm is given for determining their winnability modulo 2.

Notation 4.2. Throughout this section, we assume that a theta graph G has

- m paths of length 1 (mod 3), labeled $P_1, \dots, P_m$,
- t paths of length 2 (mod 3), labeled $P_{m+1}, \dots, P_{m+t}$, and
- $l-(m+t)$ paths of length 0 (mod 3), labeled $P_{m+t+1}, \dots, P_l$.

(Note: some of $P_{m+t+1}, \dots, P_l$ could be 'empty paths' corresponding to edges $v_0 v_n$.) An initial coloring $\vec{b}$ on a theta graph G is a vector $\mathbb{Z}_k^N$ (where $N = 2+\sum_{i=1}^{l} n_i$) such that $b_{i,j}$ is the initial color of $v_{i,j}$, b_0 is the initial color of v_0, and b_n is the color of v_n.

The next result shows which initial colorings are winnable on reduced theta graphs. Corollary 2.8 can then be used inductively to determine whether any given initial coloring of a general theta graph is winnable.

Theorem 4.3. *Let G be a reduced theta graph labeled as in Notation 4.2.*

(1) *If $t = 0$ then an initial coloring $\vec{b} \in \mathbb{Z}_k^N$ is winnable over $\mathbb{Z}_k$ if and only if the linear system*
$$\begin{cases} (1-m)d_0 + (l-2m)d_n = -b_0 + \sum_{i=1}^{m} b_{i,1}, \\ (l-2m)d_0 + (1-m)d_n = -b_n + \sum_{i=1}^{m} b_{i,1}, \end{cases}$$
has a solution for (d_0, d_n) mod k.

(2) *If $t \neq 0$ then an initial coloring $\vec{b} \in \mathbb{Z}_k^N$ is winnable over $\mathbb{Z}_k$ if and only if $b_{i,2}-b_{i,1} = b_{j,2}-b_{j,1}$ for all $m+1 \le i, j \le m+t$ and $\gcd(2-6m-3t+2l, k)$ divides*
$$-b_0 - b_n - (l-3m-t+1)(b_{m+1,1} - b_{m+1,2}) + \sum_{i=1}^{m} b_{i,1} + \sum_{i=1}^{m+t} b_{i,1}.$$

Proof. Let G and $\vec{b}$ be as in the hypotheses of the theorem. In order to win, we must press each vertex some number of times. Suppose v_0 is pressed d_0 times, v_n is pressed d_n times, and $v_{i,j}$ is pressed $d_{i,j}$ times. The effects of pressing these vertices are:

- the color of v_0 is changed by $d_0+\sum_{i=1}^{m+t} d_{i,1}+(l-m-t)d_n$;
- for $1\le i\le m$, the color of $v_{i,1}$ is changed by $d_0+d_{i,1}+d_n$;
- for $m+1\le i\le m+t$, the color of $v_{i,1}$ is changed by $d_0+d_{i,1}+d_{i,2}$;
- for $m+1\le i\le m+t$, the color of $v_{i,2}$ is changed by $d_{i,1}+d_{i,2}+d_n$;
- the color of v_n is changed by $d_n+\sum_{i=1}^{m} d_{i,1}+\sum_{i=m+1}^{m+t} d_{i,2}+(l-m-t)d_0$.

To win, we must change the color of every vertex to 0, which yields the following system of equations mod k. As before, these equations are equivalent to the matrix equation $N(G)\vec{d}=-\vec{b}$.

$$b_0+d_0+\sum_{i=1}^{m+t} d_{i,1}+(l-m-t)d_n=0, \tag{4-1}$$

$$b_{i,1}+d_0+d_{i,1}+d_n=0 \quad \text{for } 1\le i\le m, \tag{4-2}$$

$$b_{i,1}+d_0+d_{i,1}+d_{i,2}=0 \quad \text{for } m+1\le i\le m+t, \tag{4-3}$$

$$b_{i,2}+d_{i,1}+d_{i,2}+d_n=0 \quad \text{for } m+1\le i\le m+t, \tag{4-4}$$

$$b_n+d_n+\sum_{i=1}^{m} d_{i,1}+\sum_{i=m+1}^{m+t} d_{i,2}+(l-m-t)d_0=0. \tag{4-5}$$

(1) Suppose $t=0$. The system in the statement of part (1) arises from a straightforward substitution using (4-2) to eliminate $d_{i,1}$ from (4-1) and (4-5).

(2) Suppose $t\ne 0$ and the system given by (4-1) through (4-5) has a solution. Then (4-3) and (4-4) combine to show that $d_n-d_0=b_{i,1}-b_{i,2}$ for $m+1\le i\le m+t$, which in turn shows that $b_{i,1}-b_{i,2}=b_{j,1}-b_{j,2}$ for $m+1\le i,\ j\le m+t$. Equations (4-2) and (4-3) can then be solved for $d_{i,1}$ for $1\le i\le m+t$. Using the expressions for $d_{i,1}$ to eliminate all occurrences of $d_{i,1}$ from (4-1) and (4-5) and simplifying gives

$$b_0+(1-3m-2t+l)d_0+(l-2m-t)(b_{m+1,1}-b_{m+1,2}) \\ -\sum_{i=1}^{m+t} b_{i,1}-\sum_{i=m+1}^{m+t} d_{i,2}=0, \tag{4-6}$$

$$b_n+(1-3m-t+l)d_0+(1-m)(b_{m+1,1}-b_{m+1,2})-\sum_{i=1}^{m} b_{i,1}+\sum_{i=m+1}^{m+t} d_{i,2}=0, \tag{4-7}$$

Adding (4-6) and (4-7), we find that

$$(2-6m-3t+2l)d_0=-b_0-b_n-(l-3m-t+1)(b_{m+1,1}-b_{m+1,2}) \\ +\sum_{i=1}^{m} b_{i,1}+\sum_{i=1}^{m+t} b_{i,1}. \tag{4-8}$$

This implies that $\gcd(2-6m-3t+2l, k)$ divides the right-hand side of (4-8), as required.

Conversely, if $\gcd(2-6m-3t+2l, k)$ divides the right-hand side of (4-8) and $b_{i,1}-b_{i,2} = b_{j,1}-b_{j,2}$ for $m+1 \le i, j \le m+t$, there exists d_0 such that (4-8) is satisfied. Since $t > 0$, values of $d_{m+1,2}, \dots, d_{m+t,2}$ can be chosen freely so that (4-6) is satisfied, and it follows that (4-7) is satisfied as well. Finally, values of $d_{i,1}$ can be determined for $1 \le i \le m+t$ from (4-2), (4-3), and (4-4), with the last two equations being consistent since $b_{i,1}-b_{i,2} = b_{j,1}-b_{j,2}$ for $m+1 \le i, j \le m+t$. □

Theorem 4.4 (Characterization of AW theta graphs). *Let G be a theta graph labeled as in Notation 4.2. Then G is AW over $\mathbb{Z}_k$ if and only if either*

(1) $t = 0$ *and* $\gcd((l-2m)^2-(m-1)^2, k) = 1$, *or*

(2) $t = 1$ *and* $\gcd(-1-6m+2l, k) = 1$.

Proof. Let G be a theta graph labeled as in Notation 4.2. By Proposition 2.7, G is AW over $\mathbb{Z}_k$ if and only if $\widehat{G}$ is AW over $\mathbb{Z}_k$, where $\widehat{G}$ is the reduced theta graph with m paths of length 1, t paths of length 2, and $l-m-t$ paths of length 0. We assume that $\widehat{G}$ is also labeled as in Notation 4.2.

If $t = 0$, then by Theorem 4.3, $\widehat{G}$ is AW over $\mathbb{Z}_k$ if and only if

$$A\begin{pmatrix} d_0 \\ d_n \end{pmatrix} = \vec{y}$$

has a solution mod k for all $\vec{y} \in \mathbb{Z}_k^2$, where

$$A = \begin{pmatrix} 1-m & l-2m \\ l-2m & 1-m \end{pmatrix}.$$

This is true if and only if $\det A$ is a unit in $\mathbb{Z}_k$ [Bourbaki 1974, III.8.7, Proposition 13]. Finally, $\det A$ is a unit in $\mathbb{Z}_k$ if and only if $\gcd((l-2m)^2-(m-1)^2, k) = 1$.

If $t = 1$, then by Theorem 4.3, $\widehat{G}$ is AW over $\mathbb{Z}_k$ if and only if

$$\gcd(-1-6m+2l, k) = 1.$$

Finally, if $t > 1$, $\widehat{G}$ cannot be AW over $\mathbb{Z}_k$, since the condition

$$b_{i,1}-b_{i,2} = b_{j,1}-b_{j,2}$$

for $m+1 \le i, j \le m+t$ will not be satisfied for all $\vec{b}$. □

5. Always winnable trees

In this section, we give a construction describing all AW trees over $\mathbb{Z}_p$ where p is prime. By Corollary 2.3(2), this construction also gives all AW trees over $\mathbb{Z}_{p^e}$ for positive integers e. We follow the outline of the process used in [Amin and

Slater 1996], although the transition to p colors requires some changes to the main argument. *From this point on, 'AW' will mean 'AW over $\mathbb{Z}_p$'.*

Definition 5.1. Let G_1 and G_2 be AW graphs, and let $v_i \in V(G_i)$ such that $G_1 - \{v_1\}$ is not AW. The process of forming the AW graph H defined in Proposition 2.10 is called a *type-1 operation*.

Definition 5.2. Let $G_1, \ldots, G_m$ be AW graphs, and let $v_i \in V(G_i)$ such that $G_i - \{v_i\}$ is AW for all i. If

$$\sum_{i=1}^{m} d(G_i)^{-1} d(G_i - \{v_i\}) \neq 1 \pmod{p}$$

then the process of forming the AW graph W as in Proposition 2.12 is called a *type-2 operation centered at w*.

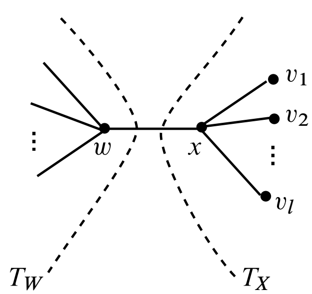
The main theorem in this section characterizes AW trees.

Theorem 5.3. *A tree T is AW if and only if T can be formed by starting with copies of K_1 and using only type-1 and type-2 operations.*

Proof. Corollaries 2.11 and 2.13 show that if one begins with copies of a single vertex K_1 and applies a series of type-1 and type-2 operations, an AW tree will always result.

Conversely, let T be an AW tree. If T has diameter 0, then $T = K_1$. It is not possible for T to have diameter 1, since P_2 is not AW for any value of k. If T has diameter 2 (i.e., if T is an AW star with l leaves for some $l \geq 2$), then T can be formed from copies of K_1 using one type-2 operation. (The summation condition on the type-2 operation is true because T is AW. This implies that $l \neq 1 \pmod{p}$, as in [Giffen and Parker 2009, Corollary 4.6].)

Therefore, we assume T has diameter at least 3. Let $x \in V(T)$ such that $\deg x = l + 1$ and x is adjacent to l leaves, which we denote $v_1, v_2, \ldots, v_l$. Let w be the nonleaf vertex of T adjacent to x. Let T_x be the component of $T - \{wx\}$ containing x, and let T_w be the component of $T - \{wx\}$ containing w.



Suppose first that T_w is not AW, so that $p|d(T_w)$. Proposition 2.10 implies that

$$d(T) = d(T - \{v_1\}) - d(T - \{x, v_1\}) = d(T - \{v_1\}) - d(T_w).$$

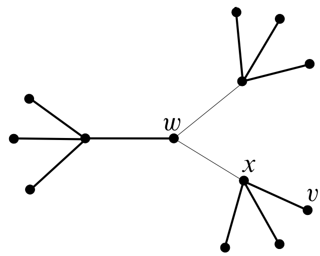
The fact that $p|d(T_w)$ and $p \nmid d(T)$ implies that $p \nmid d(T - \{v_1\})$, showing that $T - \{v_1\}$ is AW. This shows that T can be formed via a type-1 operation in which edge xv_1 is added to join $T - \{v_1\}$ to $\{v_1\}$.

From now on, we will assume that T_w is AW. If $T_w - \{w\}$ is also AW, we may construct T via a type-2 operation centered at x. Thus, we may assume that T_w is AW while $T_w - \{w\}$ is not. Proposition 2.10 implies that

$$d(T) = d(T_x)d(T_w) - d(T_w - \{w\}).$$

Since $p|d(T_w - \{w\})$ and $p \nmid d(T)$, we see that $p \nmid d(T_x)$. Thus, T can be formed by a type-1 operation in which edge wx is added to join T_x to T_w. □

Example 5.4. We show the necessity of the type-2 operation for forming trees. Consider the following tree T over $\mathbb{Z}_3$.



One can check that $d(T) = -20$, showing that T is AW over $\mathbb{Z}_3$. For any leaf v, the graph $T - \{v\}$ is not AW over $\mathbb{Z}_3$, since $d(T - \{v\}) = -12$. The two graphs T_w and T_x formed by deleting an edge wx incident with the center vertex w are both AW modulo 3. However, $T_w - \{w\}$ and $T_x - \{x\}$ are also AW. Thus, T cannot be formed from smaller trees using a type-1 operation. This tree can be formed via a type-2 operation centered at w.

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sedwards@hope.edu — *Department of Mathematics, Hope College, Holland, MI 49423, United States*

victoria.l.elandt@uwsp.edu — *Department of Mathematical Sciences, University of Wisconsin, Stevens Point, WI 54481-3897, United States*

nickjames8806@gmail.com — *Department of Mathematics, University of Florida, Gainesville, FL 32611-8105, United States*

kathryn.johnson@hope.edu — *Department of Mathematics, Hope College, Holland, MI 49423, United States*

zachary.mitchell@hope.edu — *Department of Mathematics, Hope College, Holland, MI 49423, United States*

stephenson@hope.edu — *Department of Mathematics, Hope College, Holland, MI 49423, United States*