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The cardinality of the value sets modulo n of $x^2 + x^{-2}$ and $x^2 + y^2$

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Consider the modular circle $\mathcal{C}_{a,n} = \{(x, y) : x^2 + y^2 \equiv a \pmod{n}, 0 \leq x, y \leq n-1\}$ and the modular hyperbola $\mathcal{H}_n = \{(x, y) : xy \equiv 1 \pmod{n}, 0 \leq x, y \leq n-1\}$. We provide explicit formulas for the cardinality of the sets

$$\{a \pmod{n} : \mathcal{C}_{a,n} \cap \mathcal{H}_n \neq \emptyset\} \quad \text{and} \quad \{a \pmod{n} : \mathcal{C}_{a,n} \neq \emptyset\}.$$

Introduction

Let \mathcal{H}_n denote the *modular hyperbola*

$$\{(x, y) : xy \equiv 1 \pmod{n}, 0 \leq x, y \leq n-1\}.$$

This simply defined discrete set of points has connections to a variety of other mathematical topics including Kloosterman sums, consecutive Farey fractions, and quasirandomness. These connections have inspired a closer look at the distribution of the points of \mathcal{H}_n , and many questions remain open. For a discussion of recent results and open problems on modular hyperbolas, see [[Shparlinski 2007](#)].

The propensity of the points on \mathcal{H}_n to collect on lines of slope ± 1 was investigated in [[Eichhorn et al. 2009](#)]. In the course of that investigation, formulas for the cardinalities of the sets

$$\{(x - y) \pmod{n} : (x, y) \in \mathcal{H}_n\} \quad \text{and} \quad \{(x + y) \pmod{n} : (x, y) \in \mathcal{H}_n\},$$

were derived. The techniques used to determine these formulas are elementary—within the grasp of an undergraduate mathematics major who has had a course in number theory or abstract algebra.

In this article we investigate the intersection of \mathcal{H}_n with the modular circles

$$\mathcal{C}_{a,n} = \{(x, y) : x^2 + y^2 \equiv a \pmod{n}, 0 \leq x, y \leq n-1\},$$

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This work was done as part of Hanrahan's undergraduate honors thesis under Khan's supervision.

and in particular we determine the cardinality of the set

$$\{a \bmod n : \mathcal{C}_{a,n} \cap \mathcal{H}_n \neq \emptyset\} = \{(x^2 + y^2) \bmod n : (x, y) \in \mathcal{H}_n\}.$$

Figure 1 contrasts the modular circle $\mathcal{C}_{1,997}$ with the modular hyperbola \mathcal{H}_{997} . **Figure 2** shows the two superimposed, and the intersection $\mathcal{C}_{1,997} \cap \mathcal{H}_{997}$.

This short note is a concise version of SH's honors thesis. It is also a natural addendum to [Eichhorn et al. 2009], as we used the formulas found there to prove our results.

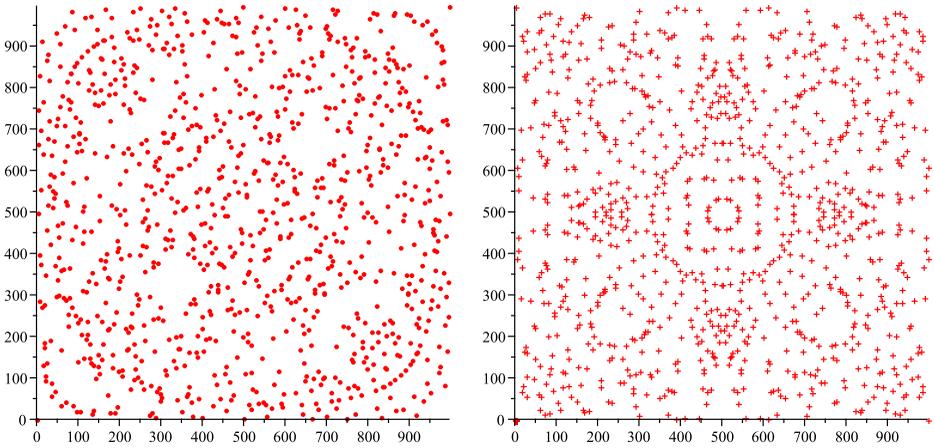


Figure 1. Left: The modular hyperbola \mathcal{H}_{997} . Right: The modular circle $\mathcal{C}_{1,997}$.

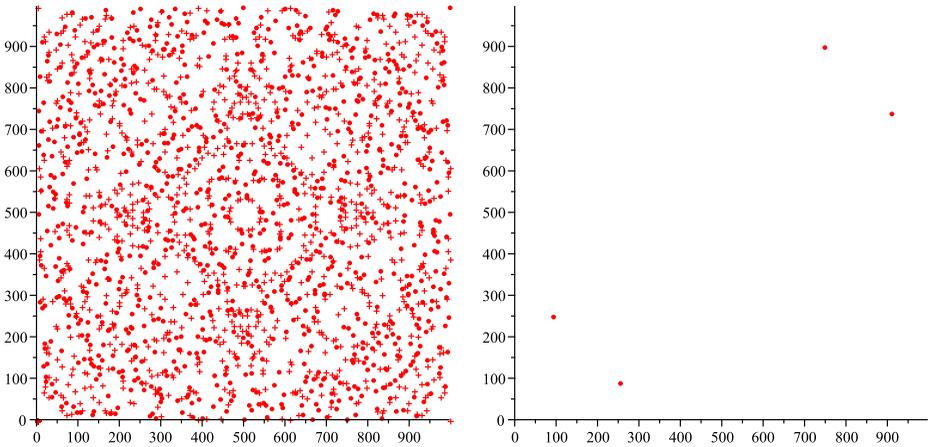


Figure 2. Left: Superposition of the preceding two sets. Points of the modular circle are represented by crosses; those of the modular hyperbola by solid circles. Right: The intersection $\mathcal{C}_{1,997} \cap \mathcal{H}_{997} = \{(91, 252), (252, 91), (745, 906), (906, 745)\}$.

1. Preliminary results

Let $f \in \mathbb{Z}[x_1, \dots, x_k]$ and let $S \subseteq \mathbb{Z}_n^k$ (where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the set of integers modulo n). Then $I(f, S)$ will denote the set

$$I(f, S) = \{f(x_1, \dots, x_k) \bmod n : (x_1, \dots, x_k) \in S\}.$$

We also define two subsets of $I(f, S)$:

$$\begin{aligned} I'(f, S) &= \{a : a \in I(f, S), \gcd(a, n) = 1\}, \\ I''(f, S) &= \{a : a \in I(f, S), \gcd(a, n) \neq 1\}. \end{aligned}$$

Our first result is that the quantity $\#I(f, \mathcal{H}_n)$ is a multiplicative function of n . Furthermore, by replacing each occurrence of \mathcal{H}_n with \mathbb{Z}_n^2 in the statement and proof of the theorem, we get that $\#I(f, \mathbb{Z}_n^2)$ is also a multiplicative function of n .

Proposition 1. *Let $f \in \mathbb{Z}[x, y]$ and define $f_n : \mathcal{H}_n \rightarrow \mathbb{Z}_n$ by*

$$f_n((x, y)) = f(x, y) \bmod n.$$

If $n = a \cdot b$ with $\gcd(a, b) = 1$, then

$$\#I(f, \mathcal{H}_n) = \#I(f, \mathcal{H}_a) \cdot \#I(f, \mathcal{H}_b).$$

It follows that if $n = \prod_{i=1}^m p_i^{e_i}$ is the canonical factorization of n , then

$$\#I(f, \mathcal{H}_n) = \prod_{i=1}^m \#I(f, \mathcal{H}_{p_i^{e_i}}). \tag{1}$$

Proof. The Chinese remainder theorem says that the map $r : \mathbb{Z}_n \rightarrow \mathbb{Z}_a \times \mathbb{Z}_b$ given by

$$r(x) = (x \bmod a, x \bmod b)$$

is an isomorphism of rings. Hence the map $R : \mathcal{H}_n \rightarrow \mathcal{H}_a \times \mathcal{H}_b$ defined by

$$R((x, y)) = ((x \bmod a, y \bmod a), (x \bmod b, y \bmod b))$$

is a bijection. The result now follows from the observation that the diagram

$$\begin{array}{ccc} \mathcal{H}_n & \xrightarrow{R} & \mathcal{H}_a \times \mathcal{H}_b \\ f_n \downarrow & & \downarrow f_a \times f_b \\ \mathbb{Z}_n & \xrightarrow{r} & \mathbb{Z}_a \times \mathbb{Z}_b. \end{array}$$

commutes. □

Thus we have reduced the problem of determining formulas for $\#I(x^2 + y^2, \mathcal{H}_n)$ (or $\#I(x^2 + y^2, \mathbb{Z}_n^2)$) to determining them for prime powers. From this point, we shall refer to the set $I(x^2 + y^2, \mathcal{H}_n)$ as $I(x^2 + x^{-2}, \mathbb{Z}_n)$. All of our formulas were

discovered through extensive numerical experimentation with Maple. Maple was the most valuable research tool at our disposal — only in discovering the formulas, but also in the *proving* stage. In the remainder of this section, we list the mathematical results we need to prove these formulas.

It is more convenient to work with the value set $I((x + x^{-1})^2, \mathbb{Z}_n)$ than with $I(x^2 + x^{-2}, \mathbb{Z}_n)$. The following lemma justifies the change.

Lemma 2. *For any positive integer n ,*

$$\#I(x^2 + x^{-2}, \mathbb{Z}_n) = \#I((x + x^{-1})^2, \mathbb{Z}_n). \quad (2)$$

Proof. The map $z \mapsto (z + 2) \bmod n$ defines a bijection between $I(x^2 + x^{-2}, \mathbb{Z}_n)$ and $I((x + x^{-1})^2, \mathbb{Z}_n)$. \square

We next state a basic criterion on the solvability of quadratic congruences modulo prime powers: $x^2 \equiv a \pmod{p^t}$.

Proposition 3 [Ireland and Rosen 1982, Propositions 4.2.3, 4.2.4, p. 46]. *Let p be prime and let a be an integer such that $\gcd(a, p) = 1$.*

- (1) *Suppose $p > 2$. If the congruence $x^2 \equiv a \pmod{p}$ is solvable, then for every $t \geq 2$ the congruence $x^2 \equiv a \pmod{p^t}$ is solvable with precisely 2 distinct solutions.*
- (2) *Suppose $p = 2$. If the congruence $x^2 \equiv a \pmod{2^3}$ is solvable, then for every $t \geq 3$ the congruence $x^2 \equiv a \pmod{2^t}$ is solvable with precisely 4 distinct solutions.*

Proposition 4 [Stangl 1996]. *Let p be an odd prime. Then*

$$\#I(x^2, \mathbb{Z}_{p^t}) = \frac{p^{t+1}}{2(p+1)} + (-1)^{t-1} \frac{p-1}{4(p+1)} + \frac{3}{4}. \quad (3)$$

For the special case $p = 2$ we have

$$\#I(x^2, \mathbb{Z}_{2^t}) = \frac{2^{t-1}}{3} + \frac{(-1)^{t-1}}{6} + \frac{3}{2}, \quad t \geq 2. \quad (4)$$

Proposition 5 [Eichhorn et al. 2009].

$$\#I(x + x^{-1}, \mathbb{Z}_{p^t}) = \frac{(p-3)p^{t-1}}{2} + \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{2(p+1)} + \frac{3}{2}. \quad (5)$$

2. The formulas for $\#I((x + x^{-1})^2, \mathbb{Z}_{p^t})$

The central result of this paper is as follows.

Theorem 6. For $p = 2$ and $t \geq 7$,

$$\#I((x + x^{-1})^2, \mathbb{Z}_{2^t}) = \frac{2^{t-7}}{3} + \frac{(-1)^{t-1}}{6} + \frac{3}{2}. \quad (6)$$

If $p \equiv 1 \pmod{4}$ then

$$\#I((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \frac{(p-5)p^{t-1}}{4} + \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{2(p+1)} + \frac{3}{2}. \quad (7)$$

If $p \equiv 3 \pmod{4}$ then

$$\#I((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \frac{(p-3)p^{t-1}}{4} + \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{4(p+1)} + \frac{3}{4}. \quad (8)$$

The proof occupies most of this section.

Proof of Theorem 6, case $p > 2$. We will use the squaring map modulo p^t :

$$Q : I(x + x^{-1}, \mathbb{Z}_{p^t}) \rightarrow I((x + x^{-1})^2, \mathbb{Z}_{p^t}), \quad Q(z) = z^2 \pmod{p^t}.$$

We note that it preserves coprimeness with p :

$$\begin{aligned} Q(I'(x + x^{-1}, \mathbb{Z}_{p^t})) &= I'((x + x^{-1})^2, \mathbb{Z}_{p^t}), \\ Q(I''(x + x^{-1}, \mathbb{Z}_{p^t})) &= I''((x + x^{-1})^2, \mathbb{Z}_{p^t}). \end{aligned}$$

Proposition 7. Let p be an odd prime. For any $a \in I'((x + x^{-1})^2, \mathbb{Z}_{p^t})$, we have $\#Q^{-1}(\{a\}) = 2$, and consequently

$$\#I'((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \#I'(x + x^{-1}, \mathbb{Z}_{p^t})/2. \quad (9)$$

Proof. Let a be an arbitrary element of $I'((x + x^{-1})^2, \mathbb{Z}_{p^t})$. There exists a point $(x_1, y_1) \in \mathcal{H}_{p^t}$ such that

$$(x_1 + y_1)^2 \equiv a \pmod{p^t}.$$

Since $\gcd(x_1 + y_1, p) = 1$,

$$x_1 + y_1 \not\equiv -(x_1 + y_1) \pmod{p^t};$$

hence the two distinct elements of $I'((x + x^{-1})^2, \mathbb{Z}_{p^t})$ that Q maps to a are

$$(x_1 + y_1) \pmod{p^t} \quad \text{and} \quad -(x_1 + y_1) \pmod{p^t}.$$

By Proposition 3, the congruence $x^2 \equiv a \pmod{p^t}$ has at most two solutions and we conclude that $\#Q^{-1}(\{a\}) = 2$. \square

Proposition 8.

$$\#I''(x + x^{-1}, \mathbb{Z}_{p^t}) = \begin{cases} p^{t-1} & \text{if } p \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases} \quad (10)$$

Consequently, when $p \equiv 1 \pmod{4}$,

$$I''(x + x^{-1}, \mathbb{Z}_{p^t}) = \{kp : k = 0, 1, \dots, p^{t-1} - 1\}.$$

Proof. Define $s_{p^t} : \mathcal{H}_{p^t} \rightarrow \mathbb{Z}_{p^t}$ by $s_{p^t}((x, y)) = (x + y) \pmod{p^t}$ and let

$$\mathcal{H}''_{p^t} = \{(x, y) : (x, y) \in \mathcal{H}_{p^t} \text{ with } s_{p^t}((x, y)) \in I''(x + x^{-1}, \mathbb{Z}_{p^t})\}.$$

If $(x, y) \in \mathcal{H}''_{p^t}$, then $x + y = 0 \pmod{p}$ and consequently $x^2 = -1 \pmod{p}$. Since -1 is a quadratic residue modulo p if and only if $p \equiv 1 \pmod{4}$, we obtain the second part of (10).

We now restrict our attention to primes p that are congruent to 1 modulo 4. Since $s_{p^t}(\mathcal{H}''_{p^t}) = I''(x + x^{-1}, \mathbb{Z}_{p^t})$, we prove the first part of (10) by proving the following two assertions:

- (i) $\#s_{p^t}^{-1}(\{a\}) = 2$ for any $a \in I''(x + x^{-1}, \mathbb{Z}_{p^t})$.
- (ii) $\#\mathcal{H}''_{p^t} = 2p^{t-1}$.

The proof of (i) is as follows. Let $(r, s) \in s_{p^t}^{-1}(\{a\})$. Then $(2r - a)$ and $(2s - a)$ are two distinct roots of the congruence

$$x^2 \equiv (a^2 - 4) \pmod{p^t}.$$

Since $p \mid a$, we have $\gcd(a^2 - 4, p) = 1$. Hence by Proposition 3

$$x^2 \equiv (a^2 - 4) \pmod{p^t}$$

cannot have more than two roots. Consequently $s_{p^t}^{-1}(\{a\}) = \{(r, s), (s, r)\}$.

We now prove (ii). Let (r, s) be an arbitrary element of \mathcal{H}''_{p^t} and let

$$r = d_0 + d_1 p + d_2 p^2 + \dots + d_{t-1} p^{t-1}$$

be the expansion of r in base p . There are only two possible choices for d_0 , specifically, the two roots of $x^2 \equiv -1 \pmod{p}$, and for each of the other d_i 's there are p possible choices: $0, 1, \dots, p - 1$. So there are $2p^{t-1}$ possible r 's. Since s is completely determined by the choice of r , we conclude that $\#\mathcal{H}''_{p^t} = 2p^{t-1}$. \square

Proposition 9. *If $p \equiv 1 \pmod{4}$ then*

$$\#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \frac{2p^{t-1} + (-1)^{t-1}(p - 1)}{4(p + 1)} + \frac{3}{4}. \quad (11)$$

Proof. By Proposition 8

$$I''(x + x^{-1}, \mathbb{Z}_{p^t}) = \{kp : 0 \leq k \leq p^{t-1} - 1\}.$$

Consequently,

$$\begin{aligned} I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) &= Q(I''(x + x^{-1}, \mathbb{Z}_{p^t})) \\ &= Q(\{kp : 0 \leq k \leq p^{t-1} - 1\}) = \{j^2 \bmod p^t : p \mid j\}. \end{aligned}$$

Therefore,

$$\#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \#\{k^2 \bmod p^t\} - \#\{k^2 \bmod p^t : \gcd(k, p) = 1\}.$$

Combining Stangl's formula (3) with the standard result that the number of quadratic residues modulo p^t is $(p^t - p^{t-1})/2$, we obtain

$$\#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{4(p+1)} + \frac{3}{4},$$

which proves Proposition 9. \square

We are now ready to prove formulas (7) and (8). We have

$$\begin{aligned} \#I((x + x^{-1})^2, \mathbb{Z}_{p^t}) &= \#I'((x + x^{-1})^2, \mathbb{Z}_{p^t}) + \#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) \\ &= \frac{\#I'(x + x^{-1}, \mathbb{Z}_{p^t})}{2} + \#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) \\ &= \frac{\#I(x + x^{-1}, \mathbb{Z}_{p^t})}{2} - \frac{\#I''(x + x^{-1}, \mathbb{Z}_{p^t})}{2} + \#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}). \end{aligned}$$

Formula (5) is

$$\#I(x + x^{-1}, \mathbb{Z}_{p^t}) = \frac{(p-3)p^{t-1}}{2} + \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{2(p+1)} + \frac{3}{2}.$$

If $p \equiv 3 \pmod{4}$, then $\#I''(x + x^{-1}, \mathbb{Z}_{p^t}) = \#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) = 0$ by (10). If $p \equiv 1 \pmod{4}$, then

$$\#I''(x + x^{-1}, \mathbb{Z}_{p^t}) = p^{t-1}$$

and

$$\#I''((x + x^{-1})^2, \mathbb{Z}_{p^t}) = \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{4(p+1)} + \frac{3}{4},$$

by (10) and (11). We complete the proof with simple algebraic computations. \square

Proof of Theorem 6, case $p = 2$. Interestingly this was the most difficult and time consuming part. It was only through experimenting with Maple that we discovered the map f (defined below) that allowed us to prove the formula for powers of 2.

Proposition 10. *Let $t \geq 3$. The image of the map*

$$f : I(x^2, \mathbb{Z}_{2^t}) \rightarrow \{0, 1, \dots, 2^{t+6} - 1\}$$

given by

$$f(k^2) = (64k^2 + 4) \pmod{2^{t+6}}$$

is $I((x + x^{-1})^2, \mathbb{Z}_{2^{t+6}})$. Since f is injective we conclude that

$$\#I((x + x^{-1})^2, \mathbb{Z}_{2^{t+6}}) = \#I(x^2, \mathbb{Z}_{2^t}). \quad (12)$$

Proof. First we show that $I((x + x^{-1})^2, \mathbb{Z}_{2^{t+6}}) \subseteq \text{Image}(f)$. Let $(x, y) \in \mathcal{H}_{2^{t+6}}$. We can write

$$x = 8x_1 + a \quad \text{and} \quad y = 8y_1 + a,$$

with $0 \leq x_1, y_1 < 2^{t+3}$ and $a = 1, 3, 5$ or 7 . (We are using the fact that each element in \mathbb{Z}_8^* is its own inverse.) The following calculation now shows that $(x + y)^2 \pmod{2^{t+6}} \in \text{Image}(f)$.

$$\begin{aligned} (x + y)^2 &= (8x_1 + 8y_1 + 2a)^2 \\ &= 64x_1^2 - 128x_1y_1 + 64y_1^2 + 256x_1y_1 + 32x_1a + 32y_1a + 4a^2 \\ &= 64(x_1 - y_1)^2 + 4(64x_1y_1 + 8x_1a + 8y_1a + a^2) \\ &= 64(x_1 - y_1)^2 + 4xy \\ &\equiv (64(x_1 - y_1)^2 + 4) \pmod{2^{t+6}}. \end{aligned}$$

To show the reverse inclusion, let $k^2 \in I(x^2, \mathbb{Z}_{2^t})$. By [Proposition 3](#) the congruence

$$x^2 \equiv 16k^2 + 1 \pmod{2^n}$$

has a solution for all values of n . Let l be any integer such that $l^2 = 16k^2 + 1 \pmod{2^{t+6}}$, and let

$$x = (l - 4k) \pmod{2^{t+6}}, \quad y = (l + 4k) \pmod{2^{t+6}}.$$

The immediate observations that $(x, y) \in \mathcal{H}_{2^{t+6}}$ and

$$(x + y)^2 \equiv 4l^2 \equiv 64k^2 + 4 \pmod{2^{t+6}}$$

complete the proof. □

Now the formula [\(6\)](#) for $\#I((x + x^{-1})^2, \mathbb{Z}_{2^t})$ is obtained by combining [\(2\)](#), [\(12\)](#) and [\(16\)](#). This concludes the proof of [Theorem 6](#). □

We can also derive the formula for $\#I(x^2 + x^{-2}, \mathbb{Z}_p)$ as a special case of an old formula for pairs of quadratic residues.

Theorem 11 [[Berndt et al. 1998](#), Theorem 6.3.1, page 197]. *Let p be an odd prime and let c be an integer relatively prime to p . Let $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$. Then*

$$\begin{aligned} \#\left\{n : 0 \leq n < p, \left(\frac{n}{p}\right) = \epsilon_1, \left(\frac{n+c}{p}\right) = \epsilon_2\right\} \\ = \frac{1}{4} \left\{ p - 2\epsilon_1 \left(\frac{-c}{p}\right) - \epsilon_2 \left(\frac{c}{p}\right) - \epsilon_1 \epsilon_2 \right\}. \quad (13) \end{aligned}$$

The special case of this formula with $\epsilon_1 = \epsilon_2 = c = 1$ was first published by Aladov in 1896. The connection between (13) and $\#I(x^2 + x^{-2}, \mathbb{Z}_p)$ is as follows.

Theorem 12. *Let $a \in \mathbb{Z}$ with $\gcd(a^2 - 4, n) = 1$. Then $\mathcal{C}_{a,n} \cap \mathcal{H}_n \neq \emptyset$ if and only if for every prime, p , in the canonical factorization of n we have*

$$\left(\frac{a-2}{p}\right) = \left(\frac{a+2}{p}\right) = 1. \tag{14}$$

Consequently,

$$\#I(x^2 + x^{-2}, \mathbb{Z}_p) = \#\left\{a : 0 \leq a < p, \left(\frac{a-2}{p}\right) = \left(\frac{a+2}{p}\right) = 1\right\} + 1.$$

Proof. For the “only if” part, let $(r, s) \in \mathcal{C}_{a,n} \cap \mathcal{H}_n$ and let p be an arbitrary prime divisor of n . So, $(r - s)^2 \equiv a - 2 \pmod{p}$ and $(r + s)^2 \equiv a + 2 \pmod{p}$, which leads immediately to (14).

To prove the converse, let $n = \prod_{i=1}^t p_i^{e_i}$ be the canonical factorization of n . By Proposition 3, we can lift the square roots (modulo p) of $(a - 2)$ and $(a + 2)$ to the e_i th power, $p_i^{e_i}$. Let $s_i = \sqrt{a - 2} \pmod{p_i^{e_i}}$, and $r_i = \sqrt{a + 2} \pmod{p_i^{e_i}}$. Then

$$2^{-1} \cdot (r_i + s_i, r_i - s_i) \in \mathcal{C}_{p_i^{e_i}} \cap \mathcal{H}_{p_i^{e_i}},$$

where 2^{-1} denotes the inverse of 2 modulo $p_i^{e_i}$. Now invoke the Chinese remainder theorem to determine integers r and s such that

$$r \equiv r_i \pmod{p_i^{e_i}} \text{ and } s \equiv s_i \pmod{p_i^{e_i}} \quad \text{for } i = 1, \dots, t.$$

Clearly $(r, s) \in \mathcal{C}_n \cap \mathcal{H}_n$. □

3. The formulas for $\#I(x^2 + y^2, \mathbb{Z}_{p^t}^2)$

We now determine the formulas for $\#I(x^2 + y^2, \mathbb{Z}_{p^t}^2)$ to contrast them to

$$\#I(x^2 + x^{-2}, \mathbb{Z}_{p^t}).$$

Theorem 13. *Let p be an odd prime. Then*

$$\#I(x^2 + y^2, \mathbb{Z}_{p^t}^2) = \begin{cases} p^t & \text{if } p \equiv 1 \pmod{4}, \\ p & \text{if } p \equiv 3 \pmod{4} \text{ and } t = 1, \\ p^t - \sum_{j=0}^{\lfloor t/2 \rfloor - 1} \varphi(p^{t-1-2j}) & \text{if } p \equiv 3 \pmod{4} \text{ and } t > 1, \end{cases} \tag{15}$$

When $p = 2$ we have

$$\#I(x^2 + y^2, \mathbb{Z}_{2^t}^2) = \varphi(2^t) + 1. \tag{16}$$

As is typically the case, the formula for powers of two, 2^t , will require a separate argument. We first prove (15).

Proof of formula (15). We treat each case separately.

- $p \equiv 1 \pmod{4}$. Let $a \in \{0, 1, \dots, p^t - 1\}$. The simultaneous congruences

$$x - y \equiv 1 \pmod{p^t} \quad \text{and} \quad x + y \equiv a \pmod{p^t}$$

have the solutions

$$\begin{aligned} x &= ((a + 1) \cdot (2^{-1} \bmod p^t)) \bmod p^t, \\ y &= ((a - 1) \cdot (2^{-1} \bmod p^t)) \bmod p^t. \end{aligned}$$

It immediately follows that $x^2 + (i_{p^t} y)^2 \equiv a \pmod{p^t}$, where

$$i_{p^t}^2 \equiv -1 \pmod{p^t}.$$

- $p \equiv 3 \pmod{4}$, $t = 1$. Let $a \in \{0, 1, \dots, p - 1\}$. By (3), $\#I(x^2, \mathbb{Z}_p) = (p + 1)/2$ and therefore $\#(a - I(x^2, \mathbb{Z}_p)) = (p + 1)/2$. Since

$$\#I(x^2, \mathbb{Z}_p) + \#(a - I(x^2, \mathbb{Z}_p)) = p + 1,$$

it follows that there is an element $(a - x_1^2) \in (a - I(x^2, \mathbb{Z}_p))$ and an element $x_2^2 \in I(x^2, \mathbb{Z}_p)$ such that $(a - x_1^2) \equiv x_2^2 \pmod{p}$.

- $p \equiv 3 \pmod{4}$, $t \geq 2$. The key is to prove that an element $a \in \{0, 1, 2, \dots, p^t - 1\}$ satisfies $a \equiv x^2 + y^2 \pmod{p^t}$ if and only if $a = p^k b$, with $\gcd(p, b) = 1$ and k even.

(\Leftarrow) Since p^k is a square in \mathbb{Z} , it is sufficient to prove this for integers a that are relatively prime to p . We argue by induction. The previous case shows that the result holds for $t = 1$. Let us assume it is true for t . So

$$a \equiv (x^2 + y^2) \pmod{p^t}.$$

If $p^{t+1} \mid (a - x^2 - y^2)$, there is nothing to prove. So let us assume that $(a - x^2 - y^2) = p^l l$, with $\gcd(l, p) = 1$. Since $\gcd(a, p) = 1$ either $\gcd(x, p) = 1$ or $\gcd(y, p) = 1$. Without loss of generality we assume the former. We now define $s \in \mathbb{Z}$, with $1 \leq s < p$, to be the solution of the congruence

$$2xs \equiv l \pmod{p}.$$

An immediate calculation shows that

$$a \equiv (x + sp^t)^2 + y^2 \pmod{p^{t+1}}.$$

(\Rightarrow) We argue by contradiction. Suppose $a = p^k b$, with $a < p^t$, $\gcd(b, p) = 1$, and k odd, be the sum of two squares modulo p^t . So there are integers $x = p^{e_1} x_1$, $y = p^{e_2} y_1$, with $\gcd(x_1 y_1, p) = 1$, such that

$$p^k b \equiv (x^2 + y^2) \pmod{p^t},$$

that is,

$$p^k b \equiv (p^{2e_1} x_1^2 + p^{2e_2} y_1^2) \pmod{p^t}.$$

Since $b \not\equiv 0 \pmod{p}$ and k is odd we have $\min\{2e_1, 2e_2\} < k$. Without loss of generality we may assume that $e_1 \leq e_2$. We can reduce the congruence

$$p^k b \equiv (x^2 + y^2) \pmod{p^t}$$

to $p^{k-2e_1} b \equiv x_1^2 + p^{2(e_2-e_1)} y_1^2 \pmod{p^{k-2e_1}}$, which in turns reduces to

$$x_1^2 + p^{2(e_2-e_1)} y_1^2 \equiv 0 \pmod{p}.$$

Since $x_1 \not\equiv 0 \pmod{p}$ we must have $p^{2(e_2-e_1)} y_1^2 \not\equiv 0 \pmod{p}$, that is $e_2 = e_1$, and consequently $(x_1^2 + y_1^2) \equiv 0 \pmod{p}$, with $\gcd(x_1 y_1, p) = 1$. But this gives us the contradiction that $x^2 \equiv -1 \pmod{p}$ is solvable for a prime p with $p \equiv 3 \pmod{4}$. This concludes the proof of (15). \square

Proposition 14. *Let $t \geq 3$ and $0 < m < 2^t$. Then $m \in I(x^2 + y^2, \mathbb{Z}_{2^t}^2)$ if and only if $m = 2^j \cdot a$, with $j < t$ and $a \equiv 1 \pmod{4}$.*

Proof. (\Leftarrow) Let $a \equiv 1 \pmod{4}$. Since 2^j is a sum of squares (in \mathbb{Z}) we only need to show that a is a sum of two squares modulo 2^t . If $a \equiv 1 \pmod{8}$ then a is a square modulo 2^t by Proposition 3. If $a \equiv 5 \pmod{8}$, then $a - 4 \equiv 1 \pmod{8}$ and is therefore a square modulo 2^t . Consequently a is a sum of two squares modulo 2^t .

(\Rightarrow) We now assume that $a \equiv 3 \pmod{4}$ and argue by contradiction. Let

$$x^2 + y^2 \equiv m \pmod{2^t}.$$

We look at four possible cases.

(1) $j = 0$: We obtain the contradiction that

$$x^2 + y^2 \equiv 3 \pmod{4}.$$

(2) $j = 1$: We obtain the contradiction that

$$x^2 + y^2 \equiv 6 \pmod{8}.$$

(3) $j \geq 2, j \leq (t-2)$: We have $x = 2^{e_1} \cdot x_1$ and $y = 2^{e_2} \cdot y_1$, with x_1, y_1 odd and $j = \min\{2e_1, 2e_2\}$. Without loss of generality we may assume that $e_1 \leq e_2$. We now obtain the contradiction

$$x_1^2 + 4^{e_2-e_1} y_1^2 \equiv a \equiv 3 \pmod{4}.$$

(4) $j = t-1$: Then

$$m = 2^{t-1} \cdot a \geq 2^{t-1} \cdot 3 > 2^t,$$

contradicting the fact that the elements of $I(x^2 + y^2, \mathbb{Z}_{2^t}^2)$ are less than 2^t . \square

Proof of formula (16). Let M_t denote the set

$$M_t = \{m : 0 < m < 2^t, m = 2^j \cdot a, j < t, a \equiv 1 \pmod{4}\}.$$

In our previous proposition we proved that

$$I(x^2 + y^2, \mathbb{Z}_{2^t}^2) \setminus \{0\} = M_t.$$

We now make the following two observations about elements in M_t :

- (i) If $m \in M_t$, then $(m + 2^t) \in M_{t+1}$ provided $m \neq 2^{t-1}$.
- (ii) If $m \in M_{t+1}$ with $m > 2^t$, then $(m - 2^t) \in M_t$.

From these two observations we conclude that

$$M_{t+1} \setminus \{2^t\} = M_t \cup \{m + 2^t : m \in M_t \setminus \{2^{t-1}\}\},$$

and consequently $\#M_{t+1} = 2 \cdot \#M_t$. An inductive argument now proves that $\#M_t = \varphi(2^t)$ and therefore $\#I(x^2 + y^2, \mathbb{Z}_{2^t}^2) = \varphi(2^t) + 1$. \square

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