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# Minimal $k$ -rankings for prism graphs

Juan Ortiz, Andrew Zemke, Hala King, Darren Narayan and Mirko Horňák

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We determine rank numbers for the prism graph  $P_2 \times C_n$  ( $P_2$  being the connected two-node graph and  $C_n$  a cycle of length  $n$ ) and for the square of an even cycle.

## 1. Introduction

A  $k$ -ranking of a graph is a vertex labeling using integers between 1 and  $k$  inclusive such that any path between two vertices of the same rank contains a vertex of strictly larger rank. When the value of  $k$  is unimportant, we will refer to a  $k$ -ranking simply as a ranking. A ranking  $f$  is minimal if the reduction of any label violates the ranking property [Ghoshal et al. 1996]. Another definition of a minimal ranking is obtained by replacing the reduction of a label by the reduction of labels for any nonempty set of vertices. It was shown in [Jamison 2003] and [Isaak et al. 2009] that these two definitions of minimal rankings are equivalent. The *rank number* of a graph  $G$ , denoted  $\chi_r(G)$  is the smallest  $k$  such that  $G$  has a minimal  $k$ -ranking.

Recall that a vertex coloring of a graph is a vertex labeling in which no two adjacent vertices have the same label. Hence a  $k$ -ranking is a restricted vertex coloring. Then the rank number is similar to the chromatic number. The *arank number of a graph*  $G$ , denoted  $\psi_r(G)$ , is the largest  $k$  such that  $G$  has a minimal  $k$ -ranking.

The study of the rank number was motivated by applications including the design of very large scale integration (VLSI) layout and Cholesky factorizations associated with parallel processing [de la Torre et al. 1992; Ghoshal et al. 1996; 1999;

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Leiserson 1980; Laskar and Pillone 2001; 2000; Sen et al. 1992]. Numerous related papers have since followed [Bodlaender et al. 1998; Hsieh 2002; Jamison 2003; Dereniowski 2006; 2004; Dereniowski and Nadolski 2006; Kostyuk and Narayan  $\geq 2010$ ; Kostyuk et al. 2006; Isaak et al. 2009; Novotny et al. 2009a]. Ghoshal, Laskar, and Pillone were the first to investigate minimal  $k$ -rankings [Ghoshal et al. 1999; 1996; Laskar and Pillone 2001; 2000]. The determination of the rank number and the arank number was shown to be NP-complete [Laskar and Pillone 2000]. The rank number was explored in [Bodlaender et al. 1998] where the authors showed that  $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$ . Rank numbers are known for a few other graph families such as cycles, wheels, complete bipartite graphs, and split graphs [Ghoshal et al. 1996; Dereniowski 2004]. The rank number for ladder graphs  $P_2 \times P_n$  and the square of a path  $P_n^2$  were determined in [Novotny et al. 2009b].

Throughout the paper  $P_n$  will denote the path on  $n$  vertices. We use  $G \times H$  to denote the *Cartesian product* of  $G$  and  $H$ . The  $k$ -th power of a path,  $P_n^k$ , has vertices  $v_1, v_2, \dots, v_n$  and edges  $(v_i, v_j)$  for all  $i, j$  satisfying  $|i - j| \leq k$ . The  $k$ -th power of a cycle,  $C_n^k$ , is defined similarly.

In this paper we determine rank numbers for the prism graph  $P_2 \times C_n$  and the square of an even cycle.

We begin by restating two elementary results from [Ghoshal et al. 1996].

**Lemma 1.** *In any minimal ranking of a connected graph  $G$  the highest label must be unique.*

*Proof.* Suppose there exist two vertices  $u$  and  $v$  that both have the highest label  $k$ . Then any path between  $u$  and  $v$  will not contain a vertex with a higher label. This is a contradiction.  $\square$

The following lemma gives a monotonicity result involving the rank number.

**Lemma 2.** *Let  $H$  be a subgraph of a graph  $G$ . Then  $\chi_r(H) \leq \chi_r(G)$ .*

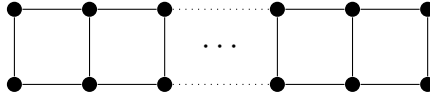
*Proof.* The proof is straightforward. Suppose  $\chi_r(H) > \chi_r(G)$ . Then we could relabel the vertices of  $H$  using the corresponding labels used in the ranking of  $G$ . This produces a ranking with fewer labels, and hence a contradiction.  $\square$

**1.1. The ladder graph  $L_n$ .** We next describe a family of graphs built using the *Cartesian product*.

**Definition 3.** The *Cartesian product* of  $G$  and  $H$  written  $G \times H$  is the graph with vertex set  $V(G) \times V(H)$  specified by putting  $\{u, v\}$  adjacent to  $(u', v')$  if and only if  $u = u'$  and  $(v, v') \in E(H)$  or  $v = v'$  and  $(u, u') \in E(G)$ .

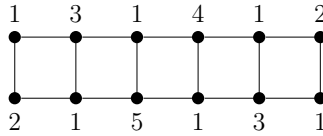
An example is the ladder graph  $L_n = P_2 \times P_n$ , shown in Figure 1.

In this paper we investigate the family of prism graphs  $P_2 \times C_n$ . We will start with a ladder  $P_2 \times P_n$  with  $n$  even, and insert either a  $P_2 \times P_1$  or  $P_2 \times P_2$  and



**Figure 1.** The ladder graph  $L_n = P_2 \times P_n$ .

“wrap” the ends to form a prism graph  $P_2 \times C_{n+1}$  or  $P_2 \times C_{n+2}$ . In order for this construction to work, it is essential that in the labeling of the vertices labeled 1 of the ladder satisfies an “alternating 1’s property”: for each vertex  $v$ , either  $v$  is labeled 1 or all of its neighbors are labeled 1 (Figure 2). That is, the vertices labeled 1 form a particular dominating set of the graph. It was shown in [Novotny et al. 2009b] that in a minimal ranking of a ladder the 1’s can be made to alternate.



**Figure 2.** A graph with the alternating 1s property.

We can insert in  $P_2 \times P_n$  either a 1-bridge (Figure 3, left) or a 2-bridge (Figure 3, right). In general, the bridges will contain the labels  $k$  and  $k + 1$  where  $k - 1$  is the rank of the original ladder. Our example shows the extension where  $k = 6$ .

In each case we insert four edges to connect the bridge to each end of the ladder. When  $n$  is even the wrapping of the ladder  $L_n$  creates a prism graph where the 1’s alternate. When  $n$  is odd the 1’s alternate except in one place where there are two vertices labeled 1 that are distance 3 apart (Figure 4).

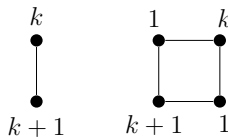
Novotny et al. [2009b] determined the rank number of a ladder graph. This result is stated in our next lemma.

**Lemma 4.**  $\chi_r(L_n) = \lfloor \log_2(n + 1) \rfloor + \lfloor \log_2(n + 1 - 2^{\lfloor \log_2 n \rfloor - 1}) \rfloor + 1$  for  $n \geq 1$ .

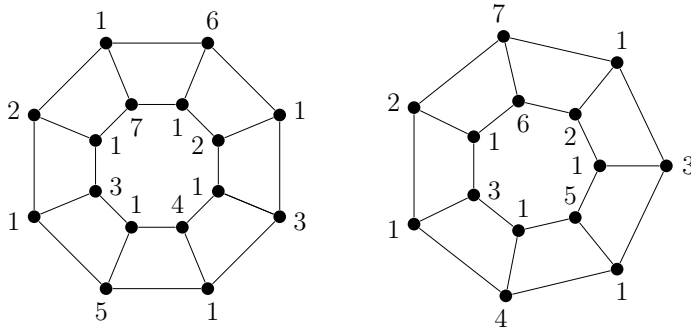
Applying our construction immediately gives an upper bound for the rank number of the prism graph  $P_2 \times C_n$ , as stated in our next theorem.

**Theorem 5.** For  $k \geq 2$ , both  $\chi_r(P_2 \times C_{2k-1})$  and  $\chi_r(P_2 \times C_{2k})$  are bounded from above by  $r(2k-2) + 2$ .

We will show later that this bound is tight.



**Figure 3.** A 1-bridge (left) and 2-bridge (right).



**Figure 4.** Prism graphs for  $n$  even (left) and  $n$  odd (even).

**2. Main results**

**Theorem 6.** *Let  $l = \chi_r(P_2 \times C_n)$  where  $n \geq 3$ . If  $f$  is a minimal  $l$ -ranking of  $P_2 \times C_n$ , then  $l \geq 5$  and the largest four labels of  $f$  appear exactly once.*

*Proof.* In the minimal ranking  $f : V(P_2 \times C_n) \rightarrow \{1, 2, \dots, l\}$  every label appears at least once. Since  $G = P_2 \times C_n$  is (vertex) 3-connected, any two distinct vertices of  $G$  are joined by three internally vertex disjoint paths. Hence each of the largest three labels appears exactly once in  $f$ .

Assume that  $l - 3$  appears at least twice with  $f(x) = f(y) = l - 3$ , where  $x \neq y$ . We have  $l \geq 5$  because the independence number of  $G$  is  $2\lfloor n/2 \rfloor$  and  $2\lfloor n/2 \rfloor + 3 < 2n = |V(G)|$ .

Let  $S$  be a minimum-sized  $x, y$  vertex separating set. It is clear that  $|V(S)| = 3$ . It is well known that every 3-element separating set  $\tilde{S}$  is a prism graph  $P$  is a neighborhood of a single vertex  $\tilde{z} \in V(P)$  and the nontrivial component of  $P - \tilde{S}$  is induced by  $V(P) - (\tilde{S} \cup \{\tilde{z}\})$ . Thus, there exists  $z \in \{x, y\}$  such that  $S$  is the neighborhood of  $z$ . However if  $z$  has its neighbors labeled  $l - 2, l - 1$ , and  $l$ , then  $f(x)$  can be reduced to 1, contradicting the minimality of  $f$ . □

For a positive integer  $n$  let

$$r(n) = \lfloor \log_2(n + 1) \rfloor + \lfloor \log_2(n + 1 - (2^{\lfloor \log_2 n \rfloor - 1})) \rfloor + 1. \tag{1}$$

Then Lemma 4 states that  $\chi_r(L_n) = \chi_r(P_2 \times P_n) = r(n)$  for  $n \geq 1$ .

**Theorem 7.** *For  $k \geq 2$ , we have*

$$\chi_r(P_2 \times C_{2k-1}) = \chi_r(P_2 \times C_{2k}) = \chi_r(P_2 \times P_{2k-2}) + 2 = r(2k - 2) + 2.$$

*Proof.* By Theorem 5, both  $\chi_r(P_2 \times C_{2k-1})$  and  $\chi_r(P_2 \times C_{2k})$  are bounded from above by  $r(2k - 2) + 2$ . In other words, if  $m = 2k - 1$  or  $2k$ , then

$$\chi_r(P_2 \times C_m) \leq \chi_r(P_2 \times P_{\lfloor m/2 \rfloor - 2}) + 2 = r(2\lfloor m/2 \rfloor - 2) + 2.$$

To prove the theorem we will show that this last inequality is in fact equality. If  $k = 2$  and  $m = 2k - 1$  or  $2k$ , then  $r(2\lceil m/2 \rceil - 2) + 2 = 5$ . So by Theorem 6,  $\chi_r(P_2 \times C_m) = 5$ .

Now assume that  $m = 2k - 1$  or  $2k$ ,  $k \geq 3$ , and

$$\chi_r(P_2 \times C_m) = l \leq r\left(2\left\lceil \frac{m}{2} \right\rceil - 2\right) + 1. \tag{2}$$

Let  $f$  be an  $l$ -minimal ranking of  $G = P_2 \times C_m$ . If  $k = 3$ , then  $5 \leq l = r(4) + 1 \leq 5$ ,  $l = 5$ , and by Theorem 6, the label 1 appears  $2m - 4$  times in  $f$ . However the independence number of  $G$  equals  $2\lfloor m/2 \rfloor \leq m < 2m - 4$ , which is a contradiction.

Let  $k \geq 4$ . This implies  $m \geq 7$ . Let  $i$  be the maximum label used at least twice. Since  $r(2\lceil m/2 \rceil - 2) + 1 \leq r(m - 1) + 1 < 2m = |V(G)|$ , such a label does exist, and  $i \leq l - 4$  by Theorem 6. Consider vertices  $x_1, x_2 \in V(G)$  with  $f(x_1) = i = f(x_2)$ , and let  $y_j$  be the neighbor of  $x_j$  that is not on the “ring” containing  $x_j$ . We will refer to this vertex as the special neighbor of  $x_j$  for  $j = 1, 2$ . There are two distinct subgraphs  $G_1, G_2$  of  $G$  that are ladders with corners  $x_1, x_2, y_1, y_2$ . The restriction  $f|_{V(G_j)}$  is a ranking of  $G_j$ ; hence there is a minimal separating set  $S_j \subseteq V(G_j)$  such that  $\min f(S_j) > i$  and  $x_1, x_2$  are in distinct components of  $G_j - S_j$ ,  $j = 1, 2$ . It is easy to see that any minimal separating set that separates two “distant” corners of a ladder on at least six vertices has two vertices and is of one of the two types shown in Figure 3 (consisting of the vertices labeled  $k$  and  $k + 1$ ). As all labels in  $\{i + 1, \dots, l\}$  are used by  $f$  exactly once, any permutation of those labels yields a ranking of  $G$ . Therefore, we may suppose without loss of generality that  $f(S_1) \cup f(S_2) = \{l - 3, l - 2, l - 1, l\}$ . Further, let  $\bar{S}_j$  be the set consisting of the vertices of  $S_j$  together with their special neighbors (so that  $|\bar{S}_j|$  is 2 or 4). The graph  $G - (\bar{S}_1 \cup \bar{S}_2)$  is a union of two vertex disjoint ladders  $H_1$  and  $H_2$ . Clearly if  $|V(H_1)| \geq |V(H_2)|$ , then  $H_1 = P_2 \times P_q$ , where  $q \geq \lceil (m - 4)/2 \rceil$ . Now  $f|_{V(H_1)}$  uses only labels from the set  $\{1, \dots, l - 4\}$ ; hence, by (2),

$$\chi_r(H_1) \leq l - 4 \leq r\left(2\left\lceil \frac{m}{2} \right\rceil - 2\right) - 3. \tag{3}$$

On the other hand if  $s, t$  are positive integers with  $s \leq t$ , then  $P_2 \times P_s$  is a subgraph of  $P_2 \times P_t$ . Then by Lemma 2 we have  $r(s) = \chi_r(P_2 \times P_s) \leq \chi_r(P_2 \times P_t) = r(t)$ . Consequently,

$$\chi_r(H_1) = \chi_r(P_2 \times P_q) = r(q) \geq r\left(\left\lceil \frac{m-4}{2} \right\rceil\right). \tag{4}$$

If  $m$  is even, then it follows from Equations (3) and (4) that

$$r(m - 2) = r\left(2 \cdot \frac{m-4}{2} + 2\right) \geq r\left(\frac{m-4}{2}\right) + 3.$$

If  $m$  is odd we have

$$r(m-1) = r\left(2 \cdot \frac{m-3}{2} + 2\right) \geq r\left(\frac{m-3}{2}\right) + 3.$$

However both cases lead to a contradiction. From (1) it is easy to see that

$$r(2n+2) - r(n) = 2$$

for any positive integer  $n$ . □

Since  $r(2k-3) = r(2k-2)$  for  $n \geq 3$ , we obtain from Theorem 7:

**Theorem 8.**  $\chi_r(P_2 \times C_n) = \chi_r(L_{n-2}) + 2$  for  $n \geq 4$ .

### 3. Rankings for other classes of graphs

We now show that the rank number of a prism graph can be used to give the rank number of the square of an even cycle. We recall some earlier facts:

**Definition 9** [Ghoshal et al. 1996]. For a graph  $G$  and a set  $S \subseteq V(G)$  the *reduction* of  $G$ , denoted by  $G_S^b$ , is a subgraph of  $G$  induced by  $V - S$  with an edge  $uv$  in  $E(G_S^b)$  if and only if there exists a  $u - v$  path in  $G$  with all internal vertices belonging to  $S$ .

**Lemma 10** [Ghoshal et al. 1996]. *Let  $G$  be a graph and let  $f$  be a minimal  $k$ -ranking of  $G$ . If*

$$S_1 = \{x \in V(G) : f(x) = 1\} \quad \text{and} \quad f^b : V(G_{S_1}^b) \rightarrow \{1, \dots, k-1\}$$

*is defined by  $f^b(x) = f(x) - 1$ , then  $f^b$  is a minimal  $(k-1)$ -ranking of  $G_{S_1}^b$ .*

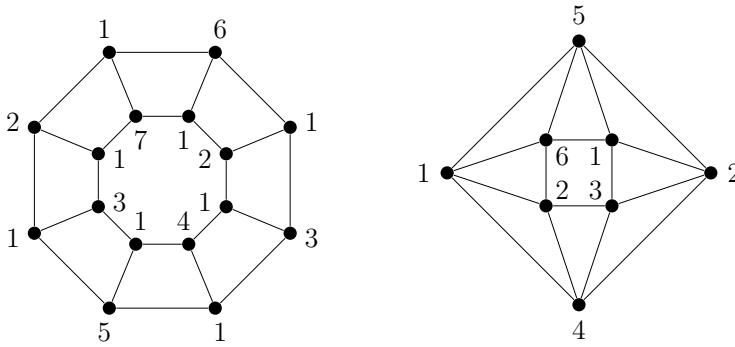
**3.1. The square of a cycle.** Next we reduce even prism graphs to squares of cycles.

**Theorem 11.**  $\chi_r(C_n^2) = \chi_r(P_2 \times C_n)$  for even  $n \geq 4$ .

*Proof.* (Illustrated in Figure 5.) If  $n = 2$ , the result follows from Theorem 7 which states that  $\chi_r(P_2 \times C_4) = 5$  and from the fact that  $\chi_r(C_4^2) = \chi_r(K_4) = 4$ .

Henceforth suppose that  $n \geq 3$ . Let  $k = \chi_r(P_2 \times C_{2n})$  and let  $l = \chi_r(C_{2n}^2)$ . Let  $f$  be a  $k$ -ranking of  $P_2 \times C_{2n}$  in which the 1's alternate. It is straightforward to see that then  $(P_2 \times C_{2n})_{S_1}^b$  is isomorphic to  $C_{2n}^2$ . Therefore, by Lemma 10  $\chi_r(C_{2n}^2) \leq k - 1$ .

Now let  $g$  be an  $l$ -ranking of  $C_{2n}^2$ . One can easily see that  $C_{2n}^2$  is isomorphic to an  $n$ -sided antiprism  $A_n$ . Pick a new vertex inside each of the  $2n$  triangles of  $A_n$ , join it to all three vertices of "its" triangle and delete all edges of  $A_n$ . The result is a graph  $G_n$  that is isomorphic to  $P_2 \times C_{2n}$ . Consider the mapping  $\tilde{g} : V(G_n) \rightarrow \{1, \dots, l+1\}$  defined as follows:  $\tilde{g}(x) = g(x) + 1$  if  $x \in V(C_{2n}^2)$  and  $\tilde{g}(x) = 1$  if  $x \in V(G_n) - V(C_{2n}^2)$ . Since  $\tilde{g}$  is a ranking of  $G_n$  (a simple exercise left to the reader), we have  $\chi_r(P_2 \times C_{2n}) \leq l + 1$ .



**Figure 5.** A minimal 7-ranking of  $P_2 \times C_8$  (left) and a minimal 6-ranking of  $A_4$  (right).

Thus  $l = \chi_r(C_{2n}^2) \leq k - 1 = \chi_r(P_2 \times C_{2n}) - 1 \leq (l + 1) - 1 = l$ , and since both inequalities turn into equalities, we are done.  $\square$

Combining Theorems 8 and 11 gives:

**Corollary 12.** *Let  $n \geq 4$  be even. Then*

$$\chi_r(C_n^2) = \chi_r(P_2 \times C_n) - 1 = \lfloor \log_2(n - 1) \rfloor + \lfloor \log_2(n - 1 - (2^{\lfloor \log_2(n-2) \rfloor - 1})) \rfloor + 2.$$

#### 4. Conclusion

We conclude by posing some problems for future research. In this paper we determined the rank number of  $P_2 \times C_n$  using known results for the rank number of  $P_2 \times P_n$ . It would be interesting to determine the rank numbers for grid graphs  $P_m \times P_n$  and cylinders  $P_m \times C_n$ . We found out recently that [Alpert  $\geq$  2010] gives rank numbers for  $P_3 \times P_n$ , among other results including an alternate proof of our Theorem 7.

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
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