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We determine rank numbers for the prism graph $P_2 \times C_n$ (P_2 being the connected two-node graph and C_n a cycle of length n) and for the square of an even cycle.

1. Introduction

A k-ranking of a graph is a vertex labeling using integers between 1 and k inclusive such that any path between two vertices of the same rank contains a vertex of strictly larger rank. When the value of k is unimportant, we will refer to a k-ranking simply as a ranking. A ranking f is minimal if the reduction of any label violates the ranking property [Ghoshal et al. 1996]. Another definition of a minimal ranking is obtained by replacing the reduction of a label by the reduction of labels for any nonempty set of vertices. It was shown in [Jamison 2003] and [Isaak et al. 2009] that these two definitions of minimal rankings are equivalent. The *rank number* of a graph G, denoted $\chi_r(G)$ is the smallest k such that G has a minimal k-ranking.

Recall that a vertex coloring of a graph is a vertex labeling in which no two adjacent vertices have the same label. Hence a k-ranking is a restricted vertex coloring. Then the rank number is similar to the chromatic number. The *arank number of a graph* G, denoted $\psi_r(G)$, is the largest k such that G has a minimal k-ranking.

The study of the rank number was motivated by applications including the design of very large scale integration (VLSI) layout and Cholesky factorizations associated with parallel processing [de la Torre et al. 1992; Ghoshal et al. 1996; 1999;

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Leiserson 1980; Laskar and Pillone 2001; 2000; Sen et al. 1992]. Numerous related papers have since followed [Bodlaender et al. 1998; Hsieh 2002; Jamison 2003; Dereniowski 2006; 2004; Dereniowski and Nadolski 2006; Kostyuk and Narayan $\geq$ 2010; Kostyuk et al. 2006; Isaak et al. 2009; Novotny et al. 2009a]. Ghoshal, Laskar, and Pillone were the first to investigate minimal k-rankings [Ghoshal et al. 1999; 1996; Laskar and Pillone 2001; 2000]. The determination of the rank number and the arank number was shown to be NP-complete [Laskar and Pillone 2000]. The rank number was explored in [Bodlaender et al. 1998] where the authors showed that $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$. Rank numbers are known for a few other graph families such as cycles, wheels, complete bipartite graphs, and split graphs [Ghoshal et al. 1996; Dereniowski 2004]. The rank number for ladder graphs $P_2 \times P_n$ and the square of a path P_n^2 were determined in [Novotny et al. 2009b].

Throughout the paper P_n will denote the path on n vertices. We use $G \times H$ to denote the *Cartesian product* of G and H. The k-th power of a path, P_n^k, has vertices $v_1, v_2, \ldots, v_n$ and edges (v_i, v_j) for all i, j satisfying $|i - j| \leq k$. The k-th power of a cycle, C_n^k, is defined similarly.

In this paper we determine rank numbers for the prism graph $P_2 \times C_n$ and the square of an even cycle.

We begin by restating two elementary results from [Ghoshal et al. 1996].

Lemma 1. *In any minimal ranking of a connected graph G the highest label must be unique.*

Proof. Suppose there exist two vertices u and v that both have the highest label k. Then any path between u and v will not contain a vertex with a higher label. This is a contradiction. □

The following lemma gives a monotonicity result involving the rank number.

Lemma 2. *Let H be a subgraph of a graph G. Then $\chi_r(H) \leq \chi_r(G)$.*

Proof. The proof is straightforward. Suppose $\chi_r(H) > \chi_r(G)$. Then we could relabel the vertices of H using the corresponding labels used in the ranking of G. This produces a ranking with fewer labels, and hence a contradiction. □

1.1. *The ladder graph L_n.*

We next describe a family of graphs built using the *Cartesian product*.

Definition 3. The *Cartesian product* of G and H written $G \times H$ is the graph with vertex set $V(G) \times V(H)$ specified by putting $\{u, v\}$ adjacent to (u', v') if and only if $u = u'$ and $(v, v') \in E(H)$ or $v = v'$ and $(u, u') \in E(G)$.

An example is the ladder graph $L_n = P_2 \times P_n$, shown in Figure 1.

In this paper we investigate the family of prism graphs $P_2 \times C_n$. We will start with a ladder $P_2 \times P_n$ with n even, and insert either a $P_2 \times P_1$ or $P_2 \times P_2$ and

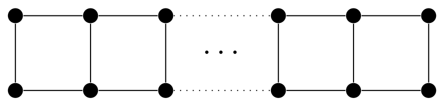

Figure 1. The ladder graph $L_n = P_2 \times P_n$.

"wrap" the ends to form a prism graph $P_2 \times C_{n+1}$ or $P_2 \times C_{n+2}$. In order for this construction to work, it is essential that in the labeling of the vertices labeled 1 of the ladder satisfies an "alternating 1's property": for each vertex v, either v is labeled 1 or all of its neighbors are labeled 1 (Figure 2). That is, the vertices labeled 1 form a particular dominating set of the graph. It was shown in [Novotny et al. 2009b] that in a minimal ranking of a ladder the 1's can be made to alternate.

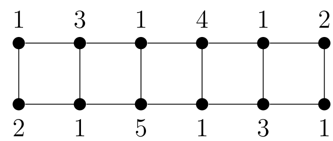

Figure 2. A graph with the alternating 1s property.

We can insert in $P_2 \times P_n$ either a 1-bridge (Figure 3, left) or a 2-bridge (Figure 3, right). In general, the bridges will contain the labels k and $k+1$ where $k-1$ is the rank of the original ladder. Our example shows the extension where $k = 6$.

In each case we insert four edges to connect the bridge to each end of the ladder. When n is even the wrapping of the ladder L_n creates a prism graph where the 1's alternate. When n is odd the 1's alternate except in one place where there are two vertices labeled 1 that are distance 3 apart (Figure 4).

Novotny et al. [2009b] determined the rank number of a ladder graph. This result is stated in our next lemma.

Lemma 4. $\chi_r(L_n) = \lfloor \log_2(n+1) \rfloor + \left\lfloor \log_2(n+1-2^{\lfloor \log_2 n \rfloor - 1}) \right\rfloor + 1$ *for* $n \geq 1$.

Applying our construction immediately gives an upper bound for the rank number of the prism graph $P_2 \times C_n$, as stated in our next theorem.

Theorem 5. *For* $k \geq 2$, *both* $\chi_r(P_2 \times C_{2k-1})$ *and* $\chi_r(P_2 \times C_{2k})$ *are bounded from above by* $r(2k-2)+2$.

We will show later that this bound is tight.

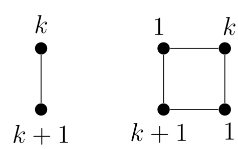

Figure 3. A 1-bridge (left) and 2-bridge (right).

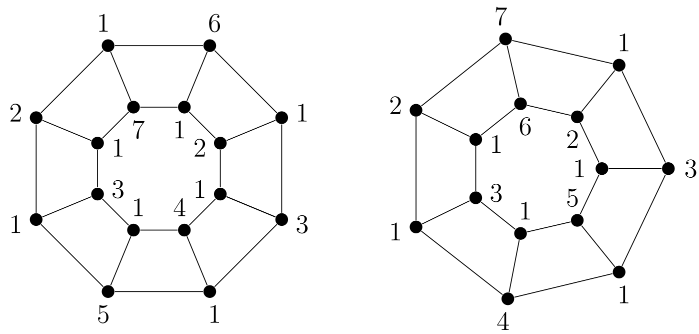

Figure 4. Prism graphs for n even (left) and n odd (even).

2. Main results

Theorem 6. *Let $l = \chi_r(P_2 \times C_n)$ where $n \geq 3$. If f is a minimal l-ranking of $P_2 \times C_n$, then $l \geq 5$ and the largest four labels of f appear exactly once.*

Proof. In the minimal ranking $f : V(P_2 \times C_n) \to \{1, 2, \ldots, l\}$ every label appears at least once. Since $G = P_2 \times C_n$ is (vertex) 3-connected, any two distinct vertices of G are joined by three internally vertex disjoint paths. Hence each of the largest three labels appears exactly once in f.

Assume that $l - 3$ appears at least twice with $f(x) = f(y) = l - 3$, where $x \neq y$. We have $l \geq 5$ because the independence number of G is $2\lfloor n/2 \rfloor$ and $2\lfloor n/2 \rfloor + 3 < 2n = |V(G)|$.

Let S be a minimum-sized x, y vertex separating set. It is clear that $|V(S)| = 3$. It is well known that every 3-element separating set $\tilde{S}$ is a prism graph P is a neighborhood of a single vertex $\tilde{z} \in V(P)$ and the nontrivial component of $P - \tilde{S}$ is induced by $V(P) - (\tilde{S} \cup \{\tilde{z}\})$. Thus, there exists $z \in \{x, y\}$ such that S is the neighborhood of z. However if z has its neighbors labeled $l - 2$, $l - 1$, and l, then $f(x)$ can be reduced to 1, contradicting the minimality of f. □

For a positive integer n let

$$r(n) = \lfloor \log_2(n+1) \rfloor + \lfloor \log_2(n + 1 - (2^{\lfloor \log_2 n \rfloor - 1})) \rfloor + 1. \tag{1}$$

Then Lemma 4 states that $\chi_r(L_n) = \chi_r(P_2 \times P_n) = r(n)$ for $n \geq 1$.

Theorem 7. *For $k \geq 2$, we have*

$$\chi_r(P_2 \times C_{2k-1}) = \chi_r(P_2 \times C_{2k}) = \chi_r(P_2 \times P_{2k-2}) + 2 = r(2k-2) + 2.$$

Proof. By Theorem 5, both $\chi_r(P_2 \times C_{2k-1})$ and $\chi_r(P_2 \times C_{2k})$ are bounded from above by $r(2k - 2) + 2$. In other words, if $m = 2k - 1$ or $2k$, then

$$\chi_r(P_2 \times C_m) \leq \chi_r(P_2 \times P_{2\lceil m/2 \rceil - 2}) + 2 = r(2\lceil m/2 \rceil - 2) + 2.$$

To prove the theorem we will show that this last inequality is in fact equality. If $k = 2$ and $m = 2k - 1$ or $2k$, then $r(2\lceil m/2\rceil - 2) + 2 = 5$. So by Theorem 6, $\chi_r(P_2 \times C_m) = 5$.

Now assume that $m = 2k - 1$ or $2k$, $k \geq 3$, and

$$\chi_r(P_2 \times C_m) = l \leq r\Big(2\Big\lceil \frac{m}{2} \Big\rceil - 2\Big) + 1. \tag{2}$$

Let f be an l-minimal ranking of $G = P_2 \times C_m$. If $k = 3$, then $5 \leq l = r(4)+1 \leq 5$, $l = 5$, and by Theorem 6, the label 1 appears $2m - 4$ times in f. However the independence number of G equals $2\lfloor m/2\rfloor \leq m < 2m-4$, which is a contradiction.

Let $k \geq 4$. This implies $m \geq 7$. Let i be the maximum label used at least twice. Since $r(2\lceil m/2\rceil-2)+1 \leq r(m-1)+1 < 2m = |V(G)|$, such a label does exist, and $i \leq l-4$ by Theorem 6. Consider vertices $x_1, x_2 \in V(G)$ with $f(x_1) = i = f(x_2)$, and let y_j be the neighbor of x_j that is not on the "ring" containing x_j. We will refer to this vertex as the special neighbor of x_j for $j = 1, 2$. There are two distinct subgraphs G_1, G_2 of G that are ladders with corners x_1, x_2, y_1, y_2. The restriction $f|_{V(G_j)}$ is a ranking of G_j; hence there is a minimal separating set $S_j \subseteq V(G_j)$ such that $\min f(S_j) > i$ and x_1, x_2 are in distinct components of $G_j - S_j$, $j = 1, 2$. It is easy to see that any minimal separating set that separates two "distant" corners of a ladder on at least six vertices has two vertices and is of one of the two types shown in Figure 3 (consisting of the vertices labeled k and $k+1$). As all labels in $\{i+1, \ldots, l\}$ are used by f exactly once, any permutation of those labels yields a ranking of G. Therefore, we may suppose without loss of generality that $f(S_1) \cup f(S_2) = \{l-3, l-2, l-1, l\}$. Further, let $\bar{S}_j$ be the set consisting of the vertices of S_j together with their special neighbors (so that $|\bar{S}_j|$ is 2 or 4). The graph $G - (\bar{S}_1 \cup \bar{S}_2)$ is a union of two vertex disjoint ladders H_1 and H_2. Clearly if $|V(H_1)| \geq |V(H_2)|$, then $H_1 = P_2 \times P_q$, where $q \geq \lceil (m-4)/2\rceil$. Now $f|_{V(H_1)}$ uses only labels from the set $\{1, \ldots, l-4\}$; hence, by (2),

$$\chi_r(H_1) \leq l - 4 \leq r\left(2\left\lceil \frac{m}{2}\right\rceil - 2\right) - 3. \tag{3}$$

On the other hand if s, t are positive integers with $s \leq t$, then $P_2 \times P_s$ is a subgraph of $P_2 \times P_t$. Then by Lemma 2 we have $r(s) = \chi_r(P_2 \times P_s) \leq \chi_r(P_2 \times P_t) = r(t)$. Consequently,

$$\chi_r(H_1) = \chi_r(P_2 \times P_q) = r(q) \geq r\left(\left\lceil \frac{m-4}{2}\right\rceil\right). \tag{4}$$

If m is even, then it follows from Equations (3) and (4) that

$$r(m-2) = r\Big(2 \cdot \frac{m-4}{2} + 2\Big) \geq r\Big(\frac{m-4}{2}\Big) + 3.$$

If m is odd we have

$$r(m-1) = r\left(2 \cdot \frac{m-3}{2} + 2\right) \geq r\left(\frac{m-3}{2}\right) + 3.$$

However both cases lead to a contradiction. From (1) it is easy to see that

$$r(2n+2) - r(n) = 2$$

for any positive integer n. $\square$

Since $r(2k-3) = r(2k-2)$ for $n \geq 3$, we obtain from Theorem 7:

Theorem 8. $\chi_r(P_2 \times C_n) = \chi_r(L_{n-2}) + 2$ *for* $n \geq 4$.

3. Rankings for other classes of graphs

We now show that the rank number of a prism graph can be used to give the rank number of the square of an even cycle. We recall some earlier facts:

Definition 9 [Ghoshal et al. 1996]. For a graph G and a set $S \subseteq V(G)$ the *reduction* of G, denoted by $G_S^\flat$, is a subgraph of G induced by $V - S$ with an edge uv in $E(G_S^\flat)$ if and only if there exists a $u - v$ path in G with all internal vertices belonging to S.

Lemma 10 [Ghoshal et al. 1996]. *Let G be a graph and let f be a minimal k-ranking of G. If*

$$S_1 = \{x \in V(G) : f(x) = 1\} \quad \text{and} \quad f^\flat : V(G_{S_1}^\flat) \to \{1, \ldots, k-1\}$$

is defined by $f^\flat(x) = f(x) - 1$, *then* $f^\flat$ *is a minimal* $(k-1)$*-ranking of* $G_{S_1}^\flat$.

3.1. ***The square of a cycle.*** Next we reduce even prism graphs to squares of cycles.

Theorem 11. $\chi_r(C_n^2) = \chi_r(P_2 \times C_n)$ *for even* $n \geq 4$.

Proof. (Illustrated in Figure 5.) If $n = 2$, the result follows from Theorem 7 which states that $\chi_r(P_2 \times C_4) = 5$ and from the fact that $\chi_r(C_4^2) = \chi_r(K_4) = 4$.

Henceforth suppose that $n \geq 3$. Let $k = \chi_r(P_2 \times C_{2n})$ and let $l = \chi_r(C_{2n}^2)$. Let f be a k-ranking of $P_2 \times C_{2n}$ in which the 1's alternate. It is straightforward to see that then $(P_2 \times C_{2n})_{S_1}^\flat$ is isomorphic to C_{2n}^2. Therefore, by Lemma 10 $\chi_r(C_{2n}^2) \leq k-1$.

Now let g be an l-ranking of C_{2n}^2. One can easily see that C_{2n}^2 is isomorphic to an n-sided antiprism A_n. Pick a new vertex inside each of the $2n$ triangles of A_n, join it to all three vertices of "its" triangle and delete all edges of A_n. The result is a graph G_n that is isomorphic to $P_2 \times C_{2n}$. Consider the mapping $\tilde{g} : V(G_n) \to \{1, \ldots, l+1\}$ defined as follows: $\tilde{g}(x) = g(x) + 1$ if $x \in V(C_{2n}^2)$ and $\tilde{g}(x) = 1$ if $x \in V(G_n) - V(C_{2n}^2)$. Since $\tilde{g}$ is a ranking of G_n (a simple exercise left to the reader), we have $\chi_r(P_2 \times C_{2n}) \leq l + 1$.

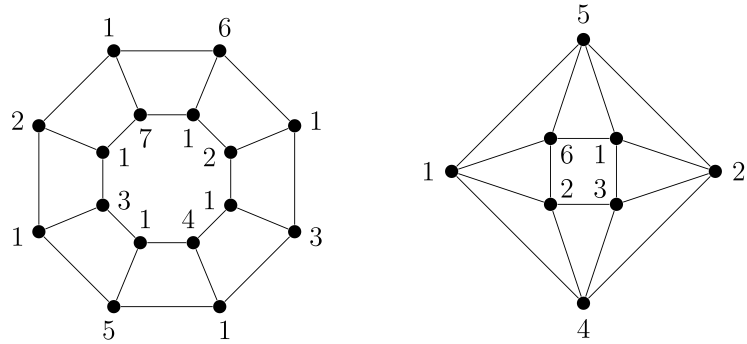

Figure 5. A minimal 7-ranking of $P_2 \times C_8$ (left) and a minimal 6-ranking of A_4 (right).

Thus $l = \chi_r(C_{2n}^2) \le k - 1 = \chi_r(P_2 \times C_{2n}) - 1 \le (l+1) - 1 = l$, and since both inequalities turn into equalities, we are done. □

Combining Theorems 8 and 11 gives:

Corollary 12. *Let $n \ge 4$ be even. Then*

$$\chi_r(C_n^2) = \chi_r(P_2 \times C_n) - 1 = \lfloor \log_2(n-1) \rfloor + \left\lfloor \log_2\left(n - 1 - (2^{\lfloor \log_2(n-2) \rfloor - 1})\right) \right\rfloor + 2.$$

4. Conclusion

We conclude by posing some problems for future research. In this paper we determined the rank number of $P_2 \times C_n$ using known results for the rank number of $P_2 \times P_n$. It would be interesting to determine the rank numbers for grid graphs $P_m \times P_n$ and cylinders $P_m \times C_n$. We found out recently that [Alpert ≥ 2010] gives rank numbers for $P_3 \times P_n$, among other results including an alternate proof of our Theorem 7.

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