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The probability of relatively prime polynomials in $\mathbb{Z}_{p^k}[x]$

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Let $P_R(m, n)$ denote the probability that two randomly chosen monic polynomials $f, g \in R[x]$ of degrees m and n , respectively, are relatively prime. Let $q = p^k$ be a prime power. We establish an explicit formula for $P_R(m, 2)$ when $R = \mathbb{Z}_q$, the ring of integers mod q .

1. Introduction

Given two polynomials $f(x), g(x)$ chosen at random, what is the probability that they are relatively prime? For a ring R , we say that two polynomials $f, g \in R[x]$ are relatively prime if there is no monic polynomial of positive degree that divides both f and g . Let $P_R(m, n)$ denote the probability that two randomly chosen monic polynomials $f, g \in R[x]$ of degrees m and n , respectively, are relatively prime. If R has an infinite number of elements, then $P_R(m, n) = 1$, so we restrict our attention to finite rings R . Let $R = \mathbb{F}_q$, the finite field with q elements. The formula, $P_{\mathbb{F}_q}(m, m) = 1 - 1/q$ was proved in [Cortee et al. 1998]. When $q = p = 2$, Reifergerste [2000] gave a combinatorial proof that $P_{\mathbb{F}_2}(m, m) = 1/2$. Benjamin and Bennett subsequently found a beautifully simple proof generalizing these results:

Theorem 1.1 [Benjamin and Bennett 2007]. *If $m, n \geq 1$, then $P_{\mathbb{F}_q}(m, n) = 1 - \frac{1}{q}$.*

This can be generalized in at least two ways. Hou and Mullen [2009] have generalized Theorem 1.1 by considering the problem of relatively prime polynomials in several variables over a finite field. In earlier work, Gao and Panario [2006] considered the probability distribution of the greatest common divisor of l randomly chosen monic single-variable polynomials in $\mathbb{F}_q[x]$ with degrees n_1, \dots, n_l as the $n_i \rightarrow \infty$. In this paper, we restrict ourselves to single-variable polynomials and explore a different perspective.

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As the formula in Theorem 1.1 only depends on the number of elements in the field \mathbb{F}_q , one can ask whether the same formula holds when R is another ring with q elements. For example, if $R = \mathbb{Z}_q$, the integers mod q , does the same formula hold? It does not, but the formula for $P_{\mathbb{F}_q}(m, n)$ can be viewed as a first approximation to the formula for $P_{\mathbb{Z}_q}(m, n)$. In this paper, we prove an explicit formula for $P_{\mathbb{Z}_{p^k}}(m, 2)$ for p odd.

For each positive integer k , we define a monic polynomial $f_k(x) \in \frac{1}{2}\mathbb{Z}[x]$ by

$$f_k(x) = x^{2k} + (1 - x) \sum_{i=0}^{(k-3)/2} x^{(k+3)/2+3i} + \frac{1}{2} \sum_{i=0}^{k-1} (-x)^i + \frac{1}{2} x^{(k-1)/2} - 1,$$

for k odd, and

$$f_k(x) = x^{2k} + (1 - x) \sum_{i=1}^{k/2-1} x^{2k-3i} - \frac{1}{2} \sum_{i=1}^{k-1} (-x)^i - x^{k/2+1} + \frac{3}{2} x^{k/2} - 1,$$

for k even. The polynomial $f_k(x)$ has degree $2k$ and its coefficients have absolute value at most 2.

Theorem 1.2. *Let p be an odd prime and let $m, k \geq 1$ be integers. The probability that two randomly chosen monic polynomials in $\mathbb{Z}_{p^k}[x]$ of degrees m and 2 , respectively, are relatively prime is*

$$P_{\mathbb{Z}_{p^k}}(m, 2) = 1 - \frac{1}{p^{3k}} f_k(p).$$

When $k = 1$, we rediscover $P_{\mathbb{F}_p}(m, 2) = 1 - 1/p$. For small values of k , we have

$$\begin{aligned} P_{\mathbb{Z}_{p^2}}(m, 2) &= 1 - \frac{1}{p^2} + \frac{1}{p^4} - \frac{2}{p^5} + \frac{1}{p^6}, \\ P_{\mathbb{Z}_{p^3}}(m, 2) &= 1 - \frac{1}{p^3} + \frac{1}{p^5} - \frac{1}{p^6} - \frac{1}{2p^7} + \frac{1}{2p^9}, \\ P_{\mathbb{Z}_{p^4}}(m, 2) &= 1 - \frac{1}{p^4} + \frac{1}{p^6} - \frac{1}{p^7} + \frac{1}{2p^9} - \frac{1}{p^{10}} - \frac{1}{2p^{11}} + \frac{1}{p^{12}}. \end{aligned}$$

As an immediate corollary to Theorem 1.2, we obtain:

Corollary 1.3. *Given $k \geq 1$, there exists a monic polynomial*

$$g_k(x) = \sum a_i x^i \in \frac{1}{2}\mathbb{Z}[x]$$

with degree $2k - 2$ and $|a_i| \leq 2$, such that

$$P_{\mathbb{Z}_{p^k}}(m, 2) = 1 - \frac{1}{p^k} + \frac{1}{p^{3k}} g_k(p) \quad \text{for all odd primes } p \text{ and all } m \geq 1.$$

We obtain Theorem 1.2 and its corollary by adapting the arguments of Benjamin and Bennett [2007], who proved Theorem 1.1 by a clever use of the Euclidean algorithm in $\mathbb{F}_q[x]$. While $\mathbb{Z}_{p^k}[x]$ does not have the Euclidean algorithm, due to the existence of noninvertible elements in \mathbb{Z}_{p^k} , it does have a division algorithm for monic polynomials. This division algorithm, together with some facts about polynomial factorization of quadratics in $\mathbb{Z}_{p^k}[x]$, suffices to prove Theorem 1.2 for odd primes p . It appears that our arguments can also be used to prove the formula for $P_{\mathbb{Z}_{p^k}}(m, 2)$ when $p = 2$, and also a formula for $P_{\mathbb{Z}_{p^k}}(m, 3)$, but the details are much more involved and have not yet been fully worked through. However, the present approach does not seem able to establish a formula for $P_{\mathbb{Z}_{p^k}}(m, n)$ for general $m, n \geq 4$ as the number of cases to consider in the proof grows as a function of $\min(m, n)$.

2. Arithmetic in $\mathbb{Z}_{p^k}[x]$

In this section, we establish some basic results on the rings \mathbb{Z}_{p^k} and $\mathbb{Z}_{p^k}[x]$. Recall that \mathbb{Z}_n denotes the ring of integers mod n . We will make use of Hensel's lemma [Gouvêa 1997, page 70] in the following form:

Lemma 2.1 (Hensel's lemma). *Let $f(x) \in \mathbb{Z}_{p^k}[x]$ be a polynomial and denote its reduction mod p by $\bar{f}(x) \in \mathbb{Z}_p[x]$. Suppose there exists $u_0 \in \mathbb{Z}_p$ with $\bar{f}(u_0) = 0$ in \mathbb{Z}_p and $\bar{f}'(u_0) \neq 0$ in \mathbb{Z}_p . Then there exists a unique $u \in \mathbb{Z}_{p^k}$, with $f(u) = 0$ in \mathbb{Z}_{p^k} and $u \equiv u_0 \pmod{p}$.*

We start by counting the squares in \mathbb{Z}_{p^k} and its unit subgroup $\mathbb{Z}_{p^k}^*$.

Lemma 2.2. *Let p be an odd prime and $k \geq 1$.*

- (a) $\mathbb{Z}_{p^k}^*$ has $\frac{1}{2}p^{k-1}(p-1)$ squares.
- (b) Let d be even, with $0 \leq d < k$. There are $\frac{1}{2}(p-1)p^{k-1-d}$ nonzero squares $x \in \mathbb{Z}_{p^k}$ with $x \in p^d\mathbb{Z}_{p^k} \setminus p^{d+1}\mathbb{Z}_{p^k}$.
- (c) There are $1 + \frac{1}{2(p+1)}(p^{k+1} - p^{1-k+2\lfloor k/2 \rfloor})$ squares in \mathbb{Z}_{p^k} .

Proof. (a) We first note that the $(p-1)/2$ squares $x = 1^2, \dots, (\frac{p-1}{2})^2$ are distinct nonzero squares in both \mathbb{Z}_p and \mathbb{Z}_{p^k} . Now consider a unit $u \in \mathbb{Z}_{p^k}$ satisfying $u \equiv 1 \pmod{p}$. Letting $f(x) = x^2 - u \in \mathbb{Z}_{p^k}[x]$, and $u_0 = 1$, by Lemma 2.1, u is a square in \mathbb{Z}_{p^k} . Thus the p^{k-1} units $u \in \mathbb{Z}_{p^k}$ with $u \equiv 1 \pmod{p}$ are squares. Hence, the $\frac{1}{2}p^{k-1}(p-1)$ distinct units xu are all squares and every unit square can be seen to be of this form.

(b) Let $x \in \mathbb{Z}_{p^k}$ satisfy $x \in p^d\mathbb{Z}_{p^k} \setminus p^{d+1}\mathbb{Z}_{p^k}$. Let $x = (p^t u)^2 = p^{2t} u^2$, where u is a unit. To satisfy the given conditions, $t = d/2$, u^2 is a unit square in $\mathbb{Z}_{p^k}^*$,

and $u^2 \equiv u_1^2 \pmod{p^{k-d}}$. Hence, the number of distinct x equals the number of unit squares in $\mathbb{Z}_{p^{k-d}}$, which is given by (a).

(c) Every nonzero square can be written as $p^{2d}u$, where u is a unit square and $0 \leq 2d < n$. Counting the square 0, the total sum is, thanks to (b),

$$1 + \frac{1}{2}(p-1) \sum_{d=0}^{\lfloor (k-1)/2 \rfloor} p^{k-1-2d}.$$

This expression simplifies to the claimed formula. □

For $g(x) = x^2 + bx + c \in \mathbb{Z}_{p^k}[x]$, define the discriminant $\Delta_g = b^2 - 4c$. As when $k = 1$, we can describe the number of roots of $g(x) \in \mathbb{Z}_{p^k}[x]$ using Δ_g .

Lemma 2.3. *Let p be an odd prime, $k \geq 1$, and $g(x) = x^2 + bx + c \in \mathbb{Z}_{p^k}[x]$.*

- (a) Δ is a square mod p^k if and only if g is reducible.
- (b) If $\Delta \equiv 0 \pmod{p^k}$, then g has the $p^{\lfloor k/2 \rfloor}$ roots given by $\frac{-b}{2} + p^{\lfloor (k+1)/2 \rfloor} t \pmod{p^k}$, where $t = 1, \dots, p^{\lfloor k/2 \rfloor}$.
- (c) Suppose $\Delta \equiv p^d u \pmod{p^k}$ is a nonzero square with $0 \leq d < k$, d even, $u \in \mathbb{Z}_{p^k}^*$ a square. Choose a such that $u \equiv a^2 \pmod{p^k}$. Then g has the $2p^{d/2}$ roots

$$-\frac{1}{2}b \pm \frac{1}{2}ap^{d/2} + tp^{k-d/2} \pmod{p^k}, \quad \text{where } t = 1, \dots, p^{d/2}.$$

Proof. Since p is odd, we have $g(x) = (x + b/2)^2 - \Delta/4$. Hence $r = -(b+z)/2$ is a root of $g(x)$ if and only if z is a solution of the equation $z^2 \equiv \Delta \pmod{p^k}$. Condition (a) is thus proved. Condition (b) follows as well as the roots of the equation $z^2 \equiv 0 \pmod{p^k}$ are $z \equiv p^{\lfloor (k+1)/2 \rfloor} t \pmod{p^k}$, for $t = 1, \dots, p^{\lfloor k/2 \rfloor}$, or equivalently, $z \equiv 2p^{\lfloor (k+1)/2 \rfloor} t \pmod{p^k}$, for $t = 1, \dots, p^{\lfloor k/2 \rfloor}$. (c) By the hypothesis, d is even and $a \not\equiv 0 \pmod{p}$. The solutions to the equation $z^2 \equiv p^d a^2 \pmod{p^k}$ have the form $z \equiv p^{d/2} w \pmod{p^k}$, where $w \in \mathbb{Z}_{p^k}$ is a solution of $x^2 \equiv a^2 \pmod{p^{k-d}}$. Hensel's lemma (using the polynomial $f(x) = x^2 - a^2$), shows that the solutions to this latter equation are the $w \in \mathbb{Z}_{p^k}$ satisfying $w \equiv \pm a \pmod{p^{k-d}}$. Thus $w = \pm a + tp^{k-d}$, for $t = 1, \dots, p^d$, or equivalently, as 2 is a unit mod p^d , $w = \pm a + 2tp^{k-d}$ for $t = 1, \dots, p^d$. Now two roots $z = p^{d/2}w$ and $z_1 = p^{d/2}w_1$ are equal precisely when the signs in the expressions for w and w_1 agree and the respective parameters t and t_1 satisfy $t \equiv t_1 \pmod{p^{d/2}}$. Hence we have shown that the original equation $z^2 \equiv p^d a^2 \pmod{p^k}$ has the $2p^{d/2}$ distinct roots given by $z = \pm ap^{d/2} + 2tp^{k-d/2}$, for $t = 1, \dots, p^{d/2}$. □

Lemma 2.4. *Let p be an odd prime and $k \geq 1$.*

- (a) Given $\Delta \in \mathbb{Z}_{p^k}$, there are p^k monic, quadratic polynomials $g \in \mathbb{Z}_{p^k}[x]$ with $\Delta_g \equiv \Delta \pmod{p^k}$.

(b) *There are*

$$\frac{p^k}{2(p+1)}(p^{k+1} + 2p^k - p - p^{k-2\lfloor k/2\rfloor} - 1)$$

monic, irreducible, quadratic polynomials $g \in \mathbb{Z}_{p^k}[x]$.

Proof. If $g = x^2 + bx + c$, then $\Delta_g = b^2 - 4c$. Since 4 is invertible mod p^k , for every Δ , $b \in \mathbb{Z}_{p^k}$, there is a unique choice of c such that $\Delta_g \equiv \Delta \pmod{p^k}$. Since there are p^k choices for b , (a) is proved. Now g is irreducible precisely when Δ_g is not a square. Let S be the number of squares in \mathbb{Z}_{p^k} . Then for each $b \in \mathbb{Z}_{p^k}$, there are $p^k - S$ choices for c such that $b^2 - 4c$ is not a square. Thus, using the formula for S given by Lemma 2.2(c), there are

$$p^k(p^k - S) = \frac{p^k}{2(p+1)}(p^{k+1} + 2p^k - 2p + p^{1-k+2\lfloor k/2\rfloor} - 2)$$

irreducible polynomials g . Simplification gives (b). \square

Given a monic, quadratic polynomial $g \in \mathbb{Z}_{p^k}[x]$, we define the set

$$A_g = \{h \in \mathbb{Z}_{p^k}[x] : \deg h \leq 1 \text{ and } g, h \text{ are not relatively prime}\},$$

and let $|A_g|$ denote its cardinality. We note that in the definition of A_g , we allow nonmonic polynomials h .

Lemma 2.5. *Let p be an odd prime and $g(x)$ be a monic quadratic polynomial in $\mathbb{Z}_{p^k}[x]$.*

(a) *If $\Delta_g \equiv 0 \pmod{p^k}$, then*

$$|A_g| = p^{k-\lfloor k/2\rfloor} \left(\frac{p^{2\lfloor k/2\rfloor+1} + 1}{p+1} \right).$$

(b) *Assume $\Delta_g \in \mathbb{Z}_{p^k}$ is a nonzero square. Let $\Delta_g \equiv p^d v \pmod{p^k}$, where d is even, $0 \leq d < k$, and $v \in (\mathbb{Z}_{p^k}^*)^2$. Then*

$$|A_g| = 2p^{k-d/2} \left(\frac{p^{d+1} + 1}{p+1} \right) - p^{d/2}.$$

Proof. We first note that a linear factor of $g(x)$ must have the form $u(x-r)$, where $u, r \in \mathbb{Z}_{p^k}$, u is a unit, and r is a root of g . Therefore, the elements $h(x) \in A_g$ are exactly the polynomials $h(x) = \alpha(x-r)$, for some $\alpha \in \mathbb{Z}_{p^k}$ and some root $r \in \mathbb{Z}_{p^k}$ of g . Hence, to calculate $|A_g|$, we need to count the number of distinct $h(x)$ of this form.

Suppose r_1 and r_2 are two roots of g and $\alpha(x-r_1) \equiv \beta(x-r_2) \pmod{p^k}$. Then $\beta \equiv \alpha \pmod{p^k}$ and $\alpha(r_1-r_2) \equiv 0 \pmod{p^k}$. Let $\alpha = p^s u$, with $u \in \mathbb{Z}_{p^k}^*$. If $s = k$, then $\alpha = 0$ is the only choice. Now suppose $s < k$. Then there are $p^{k-s-1}(p-1)$ distinct choices for u giving rise to distinct α . For each such α , we need to calculate the

number of roots of g in $\mathbb{Z}_{p^{k-s}}$. To proceed further, we need to have a description of the roots.

Writing $g(x) = x^2 + bx + c$, in case (a), the roots of g are $r = -b/2 + p^{[(k+1)/2]}t$, for $t = 1, \dots, p^{[k/2]}$ by Lemma 2.3. If $[k/2] \leq s < k$, for each choice of $\alpha = p^s u$, there is exactly one factor $\alpha(x - r) \bmod p^k$. As there are $p^{k-s-1}(p - 1)$ choices for u , and hence α , we obtain the same number of distinct factors $\alpha(x - r)$ for each s . If $0 \leq s \leq [k/2]$, then for each choice of $\alpha = p^s u$, there are $p^{[k/2]-s}$ distinct factors $\alpha(x - r) \bmod p^k$. Hence there are $p^{k+[k/2]-2s-1}(p - 1)$ distinct factors $\alpha(x - r) \bmod p^k$ for each s . In total then, we have

$$\begin{aligned} |A_g| &= \sum_{s=0}^{[k/2]} (p-1)p^{k+[k/2]-2s-1} + \left(\sum_{s=[k/2]+1}^{k-1} (p-1)p^{k-s-1} + 1 \right) \\ &= \sum_{s=0}^{[k/2]} (p-1)p^{k+[k/2]-2s-1} + p^{k-[k/2]-1} = p^{k-[k/2]} \left(\frac{p^{2[k/2]+1} + 1}{p+1} \right), \end{aligned}$$

where the last equality is obtained by evaluating a geometric sum. We thus obtain the desired formula for case (a). In case (b), by Lemma 2.3, the roots of g are $-\frac{1}{2}b \pm \frac{1}{2}ap^{d/2} + tp^{k-d/2} \bmod p^k$, where $a^2 \equiv v \bmod p^k$, $t = 1, \dots, p^{d/2}$. As in case (a), we let $\alpha = p^s u$, and consider the number of distinct factors $h(x) = \alpha(x - r)$ for each choice of s . When $s = k$, $h(x) = \alpha = 0$ is the only factor. There are three additional cases:

- (1) Suppose $k > s \geq k - d/2$. Then $k - s \leq d/2$ and all the roots of g are equivalent mod p^{k-s} . Since there are $p^{k-s-1}(p - 1)$ distinct choices for α , there are the same number of distinct factors $\alpha(x - r)$.
- (2) Suppose $k - d/2 > s \geq d/2$. Then $d/2 < k - s \leq k - d/2$ and the roots of g determine two equivalence classes mod p^{k-s} . Thus for each s , there are a total of $2p^{k-s-1}(p - 1)$ distinct factors $\alpha(x - r)$.
- (3) Suppose $d/2 \geq s \geq 0$. Then the roots of g determine $2p^{d/2-s}$ equivalence classes mod p^{k-s} for each α . Thus there are a total of $2p^{k+d/2-2s-1}(p - 1)$ distinct factors $\alpha(x - r)$, for each s .

In total, when $d < k - 1$, we have for $|A_g|$ the value

$$\begin{aligned} \sum_{s=0}^{d/2} 2(p-1)p^{k+d/2-2s-1} + \left(\sum_{s=d/2+1}^{k-d/2-1} 2(p-1)p^{k-s-1} + \sum_{s=k-d/2}^{k-1} (p-1)p^{k-s-1} + 1 \right) \\ = \sum_{s=0}^{d/2} 2(p-1)p^{k+d/2-2s-1} + 2p^{k-d/2-1} - p^{d/2}, \end{aligned}$$

which simplifies to the formula stated in (b). When $d = k - 1$, the second summation does not appear, and

$$\begin{aligned} |A_g| &= \sum_{s=0}^{d/2} 2(p-1)p^{k+d/2-2s-1} + \left(\sum_{s=k-d/2}^{k-1} (p-1)p^{k-s-1} + 1 \right) \\ &= \sum_{s=0}^{d/2} 2(p-1)p^{k+d/2-2s-1} + p^{d/2}, \end{aligned}$$

which again simplifies to the stated formula for (b). □

3. Proof of the main theorem

In this section, we let $q = p^k$. To prove Theorem 1.2, we will count the number of polynomial pairs (f, g) , where $f, g \in \mathbb{Z}_q[x]$ are not relatively prime. Let $f(x), g(x)$ be monic polynomials. Then by the division algorithm, there is a unique choice of polynomials $q(x), r(x) \in \mathbb{Z}_q[x]$, with $q(x)$ monic, satisfying

$$f(x) = g(x)q(x) + r(x), \tag{1}$$

where $r(x) = 0$ or $\deg r(x) < \deg g(x)$. Thus the pair (f, g) is uniquely determined by the triple $(g, q(x), r(x))$. From (1), any common divisor of f and g is a common divisor of g and r and vice-versa. We define

$$\begin{aligned} S_{m,d,q} &= \{(f, g) : f, g \in \mathbb{Z}_q[x] \text{ monic with } \deg f = m, \deg g = d, \\ &\quad f \text{ and } g \text{ not relatively prime}\}, \\ T_{m,q} &= \{(g, r) : g, r \in \mathbb{Z}_q[x] \text{ with } g \text{ monic of degree } m, \deg r < m, \\ &\quad g \text{ and } r \text{ not relatively prime}\}. \end{aligned}$$

Lemma 3.1. *If $m \geq d$, then $|S_{m,d,q}| = q^{m-d}|T_{d,q}|$.*

Proof. Let $(g, r) \in T_{d,q}$. Then each of the q^{m-d} monic polynomials $q(x)$ with degree $m - d$ gives rise via (1) to a unique pair $(f, g) \in S_{m,d,q}$. Conversely, the inverse map

$$(f, g) \mapsto (g, q, r) \mapsto (g, r)$$

is a q^{m-d} -to-1 map from $S_{m,d,q}$ to $T_{d,q}$. □

Thus, proving Theorem 1.2 is reduced to calculating $|T_{2,q}|$. We begin with:

Proposition 3.2. $|T_{1,q}| = q$.

Proof. If $(g, r) \in T_{1,q}$, then $g(x) = x - c$. For g and r to have a common factor, $r = 0$. Hence $T_{1,q}$ consists of the q pairs $(x - c, 0)$. □

We now determine $|T_{2,q}|$. By Lemma 2.3, we have $|T_{2,q}| = B_1 + B_2 + B_3$, where the B_i are defined by

$$B_1 = |\{(g, r) \in T_{2,q} : g \text{ is irreducible}\}|,$$

$$B_2 = |\{(g, r) \in T_{2,q} : \Delta_g \equiv 0 \pmod{p^k}\}|,$$

$$B_3 = |\{(g, r) \in T_{2,q} : \Delta_g \pmod{p^k} \text{ is a square, and, for each } d < k, \\ \Delta_g \equiv 0 \pmod{p^d} \text{ and } \Delta_g \not\equiv 0 \pmod{p^{d+1}}\}|.$$

Lemma 3.3. (a) $B_1 = \frac{p^k}{2(p+1)}(p^{k+1} + 2p^k - p - p^{k-2\lfloor k/2\rfloor} - 1)$.

(b) $B_2 = p^{2k-\lfloor k/2\rfloor} \left(\frac{p^{2\lfloor k/2\rfloor+1} + 1}{p+1} \right)$.

(c) $B_3 = \frac{p^{2k-1-\lfloor (k-1)/2\rfloor} - p^{2k}}{2(p+1)(p^2 + p + 1)} \alpha$, where

$$\alpha = (p+1)(p^2 + p + 1) - 2p^{k+1}(p+1)^2 - 2p^{k-\lfloor (k-1)/2\rfloor}(p + p^{-\lfloor (k-1)/2\rfloor}).$$

Proof. (a) Assume $g \in \mathbb{Z}_{p^k}[x]$ is a monic, irreducible, quadratic polynomial. Since g has no factors, $(g, r) \in T_{2,q}$ only when $r = 0$. Hence, B_1 equals the number of monic, irreducible quadratic polynomials, which is given by Lemma 2.4.

(b) Assume $g \in \mathbb{Z}_{p^k}[x]$ is a monic quadratic with $\Delta_g \equiv 0 \pmod{p^k}$. By Lemma 2.4, there are p^k such g . For each g , $|A_g|$ is given by Lemma 2.5(a). Thus

$$B_2 = p^k |A_g|.$$

(c) If $(g, r) \in T_{2,q}$ is included in the pairs counted for B_3 , then $\Delta_g = p^d u$, where $0 \leq d < k$, d even, and $u \in \mathbb{Z}_{p^k}^*$ is a square. For a fixed d , u , satisfying these conditions, there are p^k polynomials g with $\Delta_g = p^d u$ by Lemma 2.4(a). And for any such g , $|A_g|$ is given by Lemma 2.5(b). Now, for a fixed d , there are

$$\frac{1}{2}(p-1)p^{k-d-1}$$

choices for u that give distinct values for $p^d u$. Putting these results together, and replacing d by $2d$, we have

$$\begin{aligned} B_3 &= \sum_{d=0}^{\lfloor (k-1)/2\rfloor} \frac{1}{2}(p-1)p^{2k-d-1} \left(2p^{k-2d} \left(\frac{p^{2d+1} + 1}{p+1} \right) - 1 \right) \\ &= \frac{p^{2k-1}(p-1)}{2(p+1)} \sum_{d=0}^{\lfloor (k-1)/2\rfloor} p^{-d} (2p^{k-2d} (p^{2d+1} + 1) - p - 1). \end{aligned} \quad (2)$$

Summing the geometric sequences, we have

$$\begin{aligned} \sum_{d=0}^{[(k-1)/2]} p^{-d}(-p-1) &= -(p+1)p^{-[(k-1)/2]} \left(\frac{p^{[(k-1)/2]+1} - 1}{p-1} \right), \\ \sum_{d=0}^{[(k-1)/2]} p^{-d}(2p^{k-2d}(p^{2d+1} + 1)) &= 2p^{k-[(k-1)/2]+1} \left(\frac{p^{[(k-1)/2]+1} - 1}{p-1} \right) \\ &\quad + 2p^{k-3[(k-1)/2]} \left(\frac{p^{3[(k-1)/2]+3} - 1}{p^3 - 1} \right). \end{aligned}$$

Substituting these equations in (2) and simplifying with the help of a computer algebra system, we obtain the desired expression. \square

Proof of Theorem 1.2. There are q^m monic polynomials in $\mathbb{Z}_q[x]$ with degree m . Hence there are q^{m+2} pairs of monic polynomials (f, g) with $\deg f = m$, $\deg g = 2$. By Lemma 3.1, the probability that a pair of these polynomials is relatively prime is

$$1 - \frac{|S_{m,2,q}|}{q^{m+2}} = 1 - \frac{|T_{2,q}|}{q^4}.$$

Now $|T_{2,q}| = B_1 + B_2 + B_3$, with the values of B_i given by Lemma 2.5. Manipulating this expression with the help of a computer algebra system, one obtains

$$|T_{2,q}| = \frac{p^k}{2(p+1)} D,$$

where D equals the expression

$$2p^{2k+1} + 2p^{2+k/2}(p-1) \left(\frac{p^{3k/2} - 1}{p^3 - 1} \right) + p^{1+k/2} + 3p^{k/2} + p^k - p - 2$$

when k is even, and D equals

$$2p^{2k+1} + 2(p-1) \left(\frac{p^{2(k+1)} - p^{(k+1)/2}}{p^3 - 1} \right) + 3p^{(k+1)/2} + p^{(k-1)/2} + p^k - 2p - 1,$$

when k is odd. When k is even, algebraic manipulation shows

$$\begin{aligned} 2p^{2k+1} &= 2(p+1)p^{2k} - 2p^{2k}, \\ 2p^{2+k/2}(p-1) \left(\frac{p^{3k/2} - 1}{p^3 - 1} \right) &= 2p^{2k} - 2p^{2+k/2} + 2(1-p^2) \sum_{i=1}^{k/2-1} p^{2k-3i}, \\ p^{1+k/2} + 3p^{k/2} &= (p+1)(-2p^{1+k/2} + 3p^{k/2}) + 2p^{2+k/2}, \\ p^k - p - 2 &= -(p+1) \sum_{i=1}^{k-1} (-p)^i - 2(p+1). \end{aligned}$$

Adding both sides, the left hand side sums to D . With $f_k(x)$ defined as in the introduction, we then have

$$\frac{1}{2(p+1)}D = f_k(p).$$

Theorem 1.2 follows immediately for k even. Similar calculations establish it for k odd. \square

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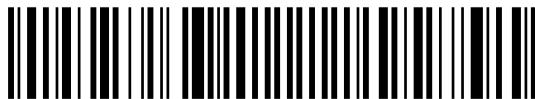
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