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\mathbb{G} -planar abelian groups

Andrea DeWitt, Jillian Hamilton, Alys Rodriguez and Jennifer Daniel

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For a group G with generating set $S = \{s_1, s_2, \dots, s_k\}$, the \mathbb{G} -graph of G , denoted by $\Gamma(G, S)$, is the graph whose vertices are distinct cosets of $\langle s_i \rangle$ in G . Two distinct vertices are joined by an edge when the set intersection of the cosets is nonempty. In this paper, we explore the planarity of $\Gamma(G, S)$.

1. Introduction

Let G be a group with a generating set $S = \{s_1, \dots, s_k\}$. We say that the subset $T_{\langle s_i \rangle} \subset G$ is a *left transversal* for the subgroup $\langle s_i \rangle$ of G if $\{x\langle s_i \rangle \mid x \in T_{\langle s_i \rangle}\}$ is precisely the set of all left cosets of $\langle s_i \rangle$ in G . As in [Bauer et al. 2008], we associate with (G, S) a simple graph $\Gamma(G, S)$ with vertex set $V(\Gamma(G, S)) = \{x_j\langle s_i \rangle \mid x_j \in T_{\langle s_i \rangle}\}$. Two distinct vertices $x_j\langle s_i \rangle$ and $x_l\langle s_k \rangle$ in $V(\Gamma(G, S))$ are joined by an edge if $x_j\langle s_i \rangle \cap x_l\langle s_k \rangle$ is nonempty. The edge set, $E(\Gamma(G, S))$, consists of pairs $(x_j\langle s_i \rangle, x_l\langle s_k \rangle)$. $\Gamma(G, S)$ defined this way has no multiedge or loop.

Let $V_i = \{x_j\langle s_i \rangle \mid x_j \in T_{s_i}\}$. Then $V = \bigcup_{i=1}^k V_i$. The number of vertices in V_i is simply the order of G divided by the order of s_i which is the index of $\langle s_i \rangle$ in G , denoted $[G : \langle s_i \rangle]$. The minimum number of elements required to generate a finite group G is called the *rank of G* . A *minimal generating set for G* is a subset $S = \{s_1, \dots, s_k\}$ such that $G = \langle S \rangle$, where k is the rank of G . This concept is not to be confused with nonredundancy. A *nonredundant* set of generators is a set S such that S generates all of G , that is, $\langle S \rangle = G$, but no proper subset of S generates all of G .

The main object of this paper is to explore the planarity of $\Gamma(G, S)$.

Definition 1.1. A group G is \mathbb{G} -planar if there exists a generating set S such that the graph, $\Gamma(G, S)$, is a planar graph.

We recall a fundamental criterion for the \mathbb{G} -planarity of a group:

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Theorem 1.2 (Wagner). *A finite graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor.*

2. Examples of \mathbb{G} -planar groups

The next two theorems give us two classes of \mathbb{G} -planar groups.

Theorem 2.1. *All cyclic groups are \mathbb{G} -planar.*

Proof. Let G be a cyclic group. Since G is cyclic, there exists an element $b \in G$ such that $\langle b \rangle = G$. Let $S = \{b\}$ be the generating set of G . Then $\Gamma(G, S)$ contains only one vertex and $\Gamma(G, S)$ is a planar graph. Therefore G is a \mathbb{G} -planar group. \square

For the dihedral group, D_n , let r be a rotation of $360^\circ/n$ and let f be any reflection.

Proposition 2.2. *For $S = \{f, rf\}$, the graph $\Gamma(G, S)$ of the dihedral group D_n is the cycle of length $2n$, C_{2n} .*

Proof. Write

$$V_1 = \{\langle f \rangle, r\langle f \rangle, r^2\langle f \rangle, \dots, r^{n-1}\langle f \rangle\},$$

$$V_2 = \{\langle rf \rangle, r\langle rf \rangle, r^2\langle rf \rangle, \dots, r^{n-1}\langle rf \rangle\}.$$

Since f and rf are both reflections, their composition is a rotation. Denote this rotation by r^m .

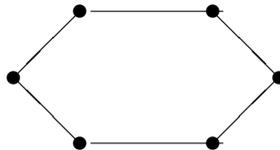
Choose a vertex from V_1 , $r^s\langle f \rangle$. Since

$$r^s \in r^s\langle f \rangle \cap r^s\langle rf \rangle,$$

the edge $(r^s\langle f \rangle, r^s\langle rf \rangle)$ is in E . Now we need to show that there is another edge between $r^s\langle f \rangle$ and V_2 . By simple calculation, we have $r^s f = r^{(s+m) \bmod n} rf$; moreover $(r^s\langle f \rangle, r^{(s+m) \bmod n}\langle rf \rangle)$ is in E .

Therefore the degree of each vertex in V_1 is 2. By similar arguments, the degree of each vertex in V_2 is 2 and $\Gamma(G, S)$ is a cycle. \square

Example 2.3. Let $G = D_3$ and $S = \{f, rf\}$. Then the \mathbb{G} -graph is the cycle C_6 :



Theorem 2.4. *All dihedral groups are \mathbb{G} -planar.*

Proof. Let $G = D_n$ and $S = \{f, rf\}$. Since $\Gamma(G, S)$ is a cycle, $\Gamma(G, S)$ is a planar graph and G is a \mathbb{G} -planar group. \square

From [DeWitt et al. \geq 2010], we have a few other examples of \mathbb{G} -planar groups.

Example 2.5. The modular group M has presentation

$$\langle s, t \mid s^8 = t^2 = e, st = ts^5 \rangle.$$

Let $S = \{s, ts\}$. From [DeWitt et al. \geq 2010], $\Gamma(M, S)$ is $K_{2,2}$. Therefore $\Gamma(M, S)$ is a planar graph and M is a \mathbb{G} -planar group.

Example 2.6. The quasihedral group QS has presentation

$$\langle s, t \mid s^8 = t^2 = e, st = ts^3 \rangle.$$

Let $S = \{s, ts\}$. From [DeWitt et al. \geq 2010], $\Gamma(QS, S)$ is $K_{2,4}$. Therefore $\Gamma(QS, S)$ is a planar graph and QS is a \mathbb{G} -planar group.

Recall that the generalized quaternion group Q_{2^n} has presentation

$$\langle s, t \mid s^{2^{n-1}} = e, s^{2^{n-2}} = t^2, tst^{-1} = s^{-1} \rangle.$$

Theorem 2.7. *The generalized quaternion group Q_{2^n} is \mathbb{G} -planar.*

Proof. Let $G = Q_{2^n}$ and $S = \{ts^k, ts^m\}$, where k is odd and m is even. $\Gamma(G, S)$ is a bipartite connected graph with every vertex of degree 2 [DeWitt et al. \geq 2010]. Therefore, $\Gamma(G, S)$ is a cycle and Q_{2^n} is \mathbb{G} -planar. \square

3. Finite abelian groups

The fundamental theorem of finite abelian groups tells us that every finite abelian group of rank k is isomorphic to a direct product of cyclic groups of prime-power order, that is, $G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \dots \times \mathbb{Z}_{m_k}$. A *standard generating set for G* is a subset $S = \{s_1, \dots, s_k\}$ such that $G = \langle s_1 \rangle \times \dots \times \langle s_k \rangle$. Let G be an abelian group with standard generating set $S = \{s_1, \dots, s_k\}$, then G is isomorphic to

$$\mathbb{Z}_{|s_1|} \times \mathbb{Z}_{|s_2|} \times \dots \times \mathbb{Z}_{|s_k|}.$$

From Theorem 2.1, we know that all finite abelian groups with 1 generator are \mathbb{G} -planar. We now consider three cases: finite abelian groups with 4 or more generators, 3 generators or 2 generators.

Let G be a group with generating set S . There exists a subset of S , S' , that is nonredundant and generates G . From [Bretto and Gillibert 2004], $\Gamma(G, S')$ is necessarily a subgraph of $\Gamma(G, S)$. If $\Gamma(G, S')$ is not a planar graph, then $\Gamma(G, S)$ is not planar. Therefore, it is only necessary to consider generating sets that are nonredundant.

Example 3.1. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_6$ and $S = \{(1, 0), (0, 0), (0, 2), (0, 3), (0, 4)\}$. The subset $S' = \{(1, 0), (0, 2), (0, 3)\}$ of S is a nonredundant generating set of G . The set $S'' = \{(1, 0), (0, 1)\}$ is a minimal generating set of G that is also nonredundant.

Lemma 3.2. *Let G be a finite abelian group and let $S = \{s_1, s_2, s_3, \dots, s_k\}$ be a nonredundant generating set, then $|s_i| \geq 2$ for all i .*

Proof. Assume $|s_i| < 2$. Then $|s_i| = 1$ and $\langle s_i \rangle = \{e\}$. Therefore s_i is not needed to generate G and $S \setminus \{s_i\}$ generates G . This is a contradiction. Therefore, $|s_i| \geq 2$. \square

Finite abelian groups G with 4 or more generators.

Lemma 3.3. *Let G be a finite abelian group and let $S = \{s_1, s_2, s_3, s_4, \dots, s_k\}$ be a nonredundant generating set of G with $k \geq 4$. Consider the subgroup H of G that is generated by $S' = \{s_1, s_2, s_3, s_4\}$. The vertices $\langle s_1 \rangle, \langle s_2 \rangle, \langle s_3 \rangle, \langle s_4 \rangle, s_1 \langle s_2 \rangle, s_2 \langle s_1 \rangle, s_2 \langle s_3 \rangle, s_3 \langle s_2 \rangle, s_3 \langle s_4 \rangle, s_4 \langle s_3 \rangle$ of $\Gamma(H, S')$ are all unique.*

Proof. To see that each of these vertices is unique, assume $\langle s_1 \rangle, s_2 \langle s_1 \rangle \in V_1$ are not distinct, that is, $\langle s_1 \rangle = s_2 \langle s_1 \rangle$. So there exists $k \in \mathbb{Z}^+$ such that $s_2 = s_1^k$ which contradicts the fact that S is a nonredundant generating set of G . The proofs of the other cases are similar. \square

Theorem 3.4. *Let G be a finite abelian group and let $S = \{s_1, s_2, s_3, s_4, \dots, s_k\}$ be a nonredundant generating set of G with $k \geq 4$. Then $\Gamma(G, S)$ is not a planar graph.*

Proof. Consider the subgroup H of G generated by $S' = \{s_1, s_2, s_3, s_4\}$. Define a contraction Γ of $\Gamma(H, S')$ in this way: Let $\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4, \bar{V}_5 \in V(\Gamma)$ with

$$\begin{aligned} \{\langle s_1 \rangle\} &= \bar{V}_1, & \{\langle s_2 \rangle\} &= \bar{V}_2, & \{\langle s_3 \rangle\} &= \bar{V}_3, & \{\langle s_4 \rangle\} &= \bar{V}_4, \\ \{s_1 \langle s_2 \rangle, s_2 \langle s_1 \rangle, s_2 \langle s_3 \rangle, s_3 \langle s_2 \rangle, s_3 \langle s_4 \rangle, s_4 \langle s_3 \rangle\} &= \bar{V}_5. \end{aligned}$$

Then, $e \in (\bar{V}_1 \cap \bar{V}_2), e \in (\bar{V}_1 \cap \bar{V}_3), e \in (\bar{V}_1 \cap \bar{V}_4), s_1 \in (\bar{V}_1 \cap \bar{V}_5), e \in (\bar{V}_2 \cap \bar{V}_3), e \in (\bar{V}_2 \cap \bar{V}_4), s_2 \in (\bar{V}_2 \cap \bar{V}_5), e \in (\bar{V}_3 \cap \bar{V}_4), s_3 \in (\bar{V}_3 \cap \bar{V}_5),$ and $s_4 \in (\bar{V}_4 \cap \bar{V}_5)$. Then $(\bar{V}_i, \bar{V}_j) \in E(\Gamma)$ for all $i \neq j$ and $\Gamma = K_5$. So, $\Gamma(H, S')$ has K_5 as a minor and $\Gamma(H, S')$ is not planar. From [Bretto et al. 2005], $\Gamma(H, S')$ is a subgraph of $\Gamma(G, S)$. Therefore, $\Gamma(G, S)$ is not a planar graph. \square

Corollary 3.5. *Let G be a finite abelian group of rank 4 or more. Then G is not \mathbb{G} -planar.*

Finite abelian groups G with 3 generators.

Example 3.6. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with standard generating set

$$S = \{s_1, s_2, s_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

The graph $\Gamma(G, S)$, illustrated in Figure 1, is a planar graph; hence G is a \mathbb{G} -planar group.

Next we show that this example is the only abelian group of rank three that is \mathbb{G} -planar.

Then

$$\begin{aligned}
 s_1 &\in (\bar{V}_1 \cap \bar{V}_2), & s_1 &\in (\bar{V}_1 \cap \bar{V}_3), \\
 e &\in (\bar{V}_1 \cap \bar{V}_4), & s_2 &\in (\bar{V}_1 \cap \bar{V}_5), \\
 s_1 &\in (\bar{V}_2 \cap \bar{V}_3), & s_1s_2s_3 &\in (\bar{V}_2 \cap \bar{V}_4), \\
 s_1s_2 &\in (\bar{V}_2 \cap \bar{V}_5), & s_3^2 &\in (\bar{V}_3 \cap \bar{V}_4), \\
 s_3^2s_2 &\in (\bar{V}_3 \cap \bar{V}_5), & s_2s_3 &\in (\bar{V}_4 \cap \bar{V}_5).
 \end{aligned}$$

It follows that $(\bar{V}_i, \bar{V}_j) \in E(\Gamma)$ for all $i \neq j$ and $\Gamma = K_5$. So, $\Gamma(G, S)$ has K_5 as a minor and is not a planar graph. □

Corollary 3.9. *Let G be a finite abelian group of rank 3 such that $G \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then G is not a \mathbb{G} -planar group.*

Finite abelian groups G with 2 generators. Since we have results for groups of rank 1 and for groups of rank 3 or more, the only case left to consider is that of groups of rank 2. Notice that any finite abelian group of rank 2 is isomorphic to the direct product $\mathbb{Z}_m \times \mathbb{Z}_n$ with $\gcd(m, n) \neq 1$.

Lemma 3.10. *Let G be a finite abelian group of rank 2 and let S be a nonredundant generating set of G . If $|S| \geq 3$, then $\Gamma(G, S)$ is not a planar graph.*

Proof. If $|S| > 3$, then $\Gamma(G, S)$ is not planar by [Theorem 3.4](#). Assume that $|S| = 3$, that is, $S = \{s_1, s_2, s_3\}$ and that $|s_i| < 3$ for $i = 1, 2, 3$. Since S is nonredundant $|s_i| > 1$ and therefore $|s_i| = 2$ for $i = 1, 2, 3$. Consider the subset

$$H = \langle s_1 \rangle \langle s_2 \rangle = \{hk \mid h \in \langle s_1 \rangle, k \in \langle s_2 \rangle\} = \{e, s_1, s_2, s_1s_2\}$$

of G . Since G is abelian, this subset is a subgroup. Now consider the subset

$$K = H \langle s_3 \rangle = \{hk \mid h \in H, k \in \langle s_3 \rangle\} = \{e, s_1, s_2, s_1s_2, s_3, s_1s_3, s_2s_3, s_1s_2s_3\}$$

of G . Again K is necessarily a subgroup of G .

Now assume that $g \in G$. Since S generates G , there exists n, m, l such that $g = s_1^n s_2^m s_3^l$. Since the order of each generator is 2, n, m, l are congruent to 0 or 1 modulo 2 and $g \in K$. Therefore $G = K$. Since the order of each element in G is two, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. This is a contradiction since G is a group of rank 2. Therefore, $|s_i| \geq 3$ for at least one i and by [Theorem 3.8](#) the graph, $\Gamma(G, S)$, is not planar. □

Theorem 3.11. *Let G be a finite abelian group of rank 2. G is \mathbb{G} -planar if and only if $G \cong \mathbb{Z}_2 \times \mathbb{Z}_k$, for some $k \in \mathbb{N}$.*

Proof. (\Leftarrow) Let $G \cong \mathbb{Z}_2 \times \mathbb{Z}_k$ and let $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_k, S)$ be the associated \mathbb{G} -graph of $\mathbb{Z}_2 \times \mathbb{Z}_k$ with $S = \{(1, 0), (0, 1)\}$. There exist an isomorphism $\phi : \mathbb{Z}_2 \times \mathbb{Z}_k \rightarrow G$. Let $(x, y) \in \mathbb{Z}_2 \times \mathbb{Z}_k$. There exists a, b such that $(x, y) = a(1, 0) + b(0, 1)$. Then $\phi(x, y) = \phi(a(1, 0) + b(0, 1)) = a\phi(1, 0) \oplus b\phi(0, 1)$. So $\phi(S) = \{\phi(1, 0), \phi(0, 1)\}$

Rank	Group	Planarity
1	all G	planar
2	$G \cong \mathbb{Z}_2 \times \mathbb{Z}_k$ $G \not\cong \mathbb{Z}_2 \times \mathbb{Z}_k$	planar not planar
3	$G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $G \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	planar not planar
4 or more	all G	not planar

Table 1. ℚ-planarity of finite abelian groups.

generates G . $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_k, S)$ is $K_{k,2}$, so $K_{k,2} \cong \Gamma(G, \phi(S))$. Since $K_{k,2}$ is planar, $\Gamma(G, \phi(S))$ is planar. Therefore G is ℚ-planar.

(\Rightarrow) Let G be a finite abelian ℚ-planar group of rank 2 and let S be a generating set such that $\Gamma(G, S)$ is a planar graph. From [Lemma 3.10](#), $|S| = 2$, that is, $S = \{s_1, s_2\}$.

Case 1. Assume that $|s_1| = 2$. Let $|G| = n$, $|V_1| = [G : \langle s_1 \rangle] = n/2$. So

$$V_1 = \{\langle s_1 \rangle, s_2 \langle s_1 \rangle, s_2^2 \langle s_1 \rangle, \dots, s_2^{n/2-1} \langle s_1 \rangle\},$$

and the elements of G are of the form

$$s_2, s_2^2, \dots, s_2^{n/2-1}, e \quad \text{and} \quad s_1 s_2, s_1 s_2^2, \dots, s_1 s_2^{n/2-1}, s_1.$$

Therefore $|s_2| = n/2$ and G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{n/2}$.

Case 2. Assume that $|s_1|, |s_2| > 2$. Consider the vertex induced subgraph generated by the six vertices $\langle s_1 \rangle, s_2 \langle s_1 \rangle, s_2^2 \langle s_1 \rangle, \langle s_2 \rangle, s_1 \langle s_2 \rangle, s_1^2 \langle s_2 \rangle$. This graph is $K_{3,3}$. Since this subgraph is not planar, $\Gamma(G, S)$ is not planar. This contradicts the supposition that S is a generating set such that $\Gamma(G, S)$ is a planar graph. Therefore, if G is ℚ-planar, then $G \cong \mathbb{Z}_2 \times \mathbb{Z}_k$. □

[Table 1](#) summarizes the results for all finite abelian groups.

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vol. 3

no. 2

Recursive sequences and polynomial congruences	129
J. LARRY LEHMAN AND CHRISTOPHER TRIOLA	
The Gram determinant for plane curves	149
JÓZEF H. PRZYTYSKI AND XIAOQI ZHU	
The cardinality of the value sets modulo n of $x^2 + x^{-2}$ and $x^2 + y^2$	171
SARA HANRAHAN AND MIZAN KHAN	
Minimal k -rankings for prism graphs	183
JUAN ORTIZ, ANDREW ZEMKE, HALA KING, DARREN NARAYAN AND MIRKO HORŇÁK	
An unresolved analogue of the Littlewood Conjecture	191
CLARICE FEROLITO	
Mapping the discrete logarithm	197
DANIEL CLOUTIER AND JOSHUA HOLDEN	
Linear dependency for the difference in exponential regression	215
INDIKA SATHISH AND DIAWARA NOROU	
The probability of relatively prime polynomials in $\mathbb{Z}_{p^k}[x]$	223
THOMAS R. HAGEDORN AND JEFFREY HATLEY	
\mathbb{G} -planar abelian groups	233
ANDREA DEWITT, JILLIAN HAMILTON, ALYS RODRIGUEZ AND JENNIFER DANIEL	