A note on nonresidually solvable hyperlinear one-relator groups

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We prove that various nonresidually finite, nonresidually solvable groups of the form \( \langle a, b \mid r^w = r^2 \rangle \) are sofic.

This paper concerns the sofic property discussed in the survey [Pestov 2008]. Particularly, we address Question 4.10 in that paper: the problem of Nate Brown asking whether or not every one-relator group is sofic. In [Bannon 2010], it is proved that the example in [Baumslag 1969] of a nonresidually finite nonresidually solvable one-relator group is a sofic group. The purpose of this paper is to exhibit more such examples in the following large class of nonresidually solvable one-relator groups introduced in [Baumslag et al. 2007]. Let \( \mathbb{F}_2 = \langle a, b \rangle \) denote the free group on two generators. Let \( r, w \in \mathbb{F}_2 \) be two elements that do not commute. In [Baumslag et al. 2007], the authors show that the group

\[
\Gamma_{r,w} = \langle a, b \mid r^w = r^2 \rangle = \langle a, b \mid r = [r, (r^{-1})^w] \rangle
\]

has the same finite quotients as the group

\[
\langle a, b \mid r \rangle,
\]

and is therefore not residually finite. We point out that none of the groups \( \Gamma_{r,w} \) are residually solvable, since \( r = [r, (r^{-1})^w] \) lies in every derived subgroup of \( \Gamma_{r,w} \). In [Bannon 2010], it is shown that the group \( \Gamma_{ab,a} \) is sofic. The proof in [Bannon 2010] uses [Dykema 2010, Corollary 3.4], that HNN extensions of sofic groups over amenable subgroups remain sofic. The proof in [Bannon 2010] uses the fact that \( \Gamma_{ab,a} \) is an HNN extension of an amenable one-relator group. We shall extend this result to certain other of the groups \( \Gamma_{r,w} \). If \( r \) and \( w \) generate \( \mathbb{F}_2 \), then \( \Gamma_{r,w} \) embeds naturally as a subgroup of \( \Gamma_{ab,a} \), and since the sofic property passes to subgroups, \( \Gamma_{r,w} \) is sofic. The first result of this short note is that there exist \( r, w \) that do not generate \( \mathbb{F}_2 \), yet the group \( \Gamma_{r,w} \) is sofic. More precisely, we prove:

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Theorem 1. The group $\Gamma_{a,b^{-1}ab}$ is sofic.

Proof. Since $\Gamma_{a,b^{-1}ab} = \langle a, b \mid (bab^{-1})^{-2}a^{-1}(bab^{-1})^{-1}a(bab^{-1})a^{-1}(bab^{-1})a \rangle$, following [McCool and Schupp 1973], we let $a_0 = a$ and $a_{-1} = bab^{-1}$ and realize $\Gamma_{a,b^{-1}ab}$ as the HNN extension

$$\langle a_0, a_{-1}, t \mid (a_{-1})^{-2}a^{-1}_0(a_{-1})^{-1}a_0a_{-1}a^{-1}_0a_{-1}a_0, t^{-1}a_{-1}t = a_0 \rangle$$

of the one-relator group $H_1 = \langle a_0, a_{-1} \mid a_0(a_{-1})^{-2}a^{-1}_0(a_{-1})^{-1}a_0a_{-1}a^{-1}_0a_{-1} \rangle$, where by the Freiheitssatz $\langle a_{-1} \rangle$ and $\langle a_0 \rangle$ are copies of $\mathbb{Z}$ in which the HNN extension we identify by identifying $a_{-1}$ with $a_0$. Letting $b_1 = a_0a_{-1}a^{-1}_0$ and $b_0 = a_{-1}$ we may identify $H_1$ as the HNN extension

$$\langle b_0, b_1, s \mid b_1^{-2}b_0^{-1}b_1b_0, s^{-1}b_1s = b_0 \rangle$$

of the one-relator group $H_2 = \langle b_0, b_1, s \mid b_1^{-2}b_0^{-1}b_1b_0 \rangle$, where we identify the two copies $\langle b_0 \rangle$ and $\langle b_1 \rangle$ of $\mathbb{Z}$ as above. By [Ceccherini-Silberstein and Grigorchuk 1997], the group $H_2$ is amenable, and hence by the argument in [Bannon 2010], the group $H_1$ is sofic. Since $\Gamma_{a,b^{-1}ab}$ is an HNN extension of a sofic group with respect to identified copies of the amenable group $\mathbb{Z}$, it follows that $\Gamma_{a,b^{-1}ab}$ is sofic.

In this proof we used in an essential way that the identified subgroups are amenable and therefore invoke the full hypotheses of Corollary 3.4 of [Dykema 2010], whereas in [Bannon 2010], the group $\Gamma_{ab,a}$ is an HNN extension of an amenable group and so any pair of identified subgroups would work. We next illustrate that there are groups of the form $\Gamma_{r,w}$ that do not in an obvious way fall to the method of [Bannon 2010].

Theorem 2. The group $\Gamma_{a,b^2} = \langle a, b \mid a = [a, (a^{-1})b^2] \rangle$ is isomorphic to

$$(G \ast \mathbb{Z})_{\ast \mathbb{F}_2},$$

where $G$ is a one-relator amenable group.

Proof. Since $\Gamma_{a,b^2} = \langle a, b \mid a^{-2}(b^2ab^{-2})a(b^2ab^{-2})^{-1} \rangle$, then letting $a_0 = a$ and $a_{-2} = b^2ab^{-2}$ we have that $\Gamma_{a,b^2}$ is isomorphic to the HNN extension

$$\langle a_0, a_{-1}, a_{-2}, t \mid a^{-2}_0a^{-2}_0a_0(a_{-2})^{-1}, t^{-1}a_{-2}t = a_{-1}, t^{-1}a_{-1}t = a_0 \rangle$$

of the one-relator group $\langle a_0, a_{-1}, a_{-2} \mid a^{-2}_0a^{-2}_0a_0(a_{-2})^{-1} \rangle$, with the isomorphism from the free subgroup $\langle a_{-2}, a_{-1} \rangle$ with $\langle a_{-1}, a_0 \rangle$ extending the set map that sends $a_{-2}$ to $a_{-1}$ and $a_{-1}$ to $a_0$. But the relator $a^{-2}_0a^{-2}_0a_0(a_{-2})^{-1}$ does not involve $a_{-1}$, hence $\langle a_0, a_{-1}, a_{-2} \mid a^{-2}_0a^{-2}_0a_0(a_{-2})^{-1} \rangle = \langle a_{-1} \rangle \ast \langle a_0, a_{-2} \mid a^{-2}_0a^{-2}_0a_0(a_{-2})^{-1} \rangle$. 

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