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A numerical investigation on the asymptotic behavior of discrete Volterra equations with two delays

Immacolata Garzilli, Eleonora Messina and Antonia Vecchio

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We describe a numerical approach to the solution of two-delay Volterra integral equations, and we carry out a nonlinear stability analysis on an interesting test equation by means of a parallel investigation both on the continuous and the discrete problem.

1. Introduction

Messina et al. [2008a] present a comparison between the analytical and the numerical solution of the following Volterra integral equation (VIE) with two constant delays:

$$y(t) = \int_{t-\tau_2}^{t-\tau_1} k(t-\tau) g(y(\tau)) d\tau \quad t \in [\tau_2, T], \tag{1}$$

with $y(t) = \varphi(t)$, $t \in [0, \tau_2]$, where $\varphi(t)$ is a known function such that

$$\varphi(\tau_2) = \int_0^{\tau_2-\tau_1} k(\tau_2-\tau) g(\varphi(\tau)) d\tau. \tag{2}$$

The interest of (1) in the applications is mainly in the modeling of age-structured population dynamics, as described in [Messina et al. 2008a] and the references therein. Here, we continue those investigations with the aim of providing a more complete analysis of the dynamics of the solutions. In particular, we add some new results on the global asymptotic behavior of solutions and simplify some already known proofs. In Section 2, the properties of the continuous solution are summarized and a new result on global asymptotic stability of the nontrivial equilibrium is proved. In Section 3, we consider a numerical method of direct quadrature type and look for conditions on the step size h of a direct quadrature method that lead to a numerical solution which mimics the behavior of the continuous one. The

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main novelty of this paper with respect to [Messina et al. 2008a] is the compact form that we use to represent the method: this new form allows us to obtain some new results in the discrete case equivalent to those valid for the continuous case, and so to complete the parallelism between the behaviors of the analytical and the numerical solution. Finally, in Section 4 we report some numerical examples that show the nature of these behaviors.

2. The continuous equation

In this section we provide a summary of the theory related to the stability of equilibria of (1) already developed in [Messina et al. 2008a] and we prove a new result on the global asymptotic stability of the nontrivial equilibrium (Theorem 2.6).

As in that paper, we make certain assumptions on the functions φ, g and k of problem (1):

(a) $\varphi(t) \geq 0$, for all $t \in [0, \tau_2]$;

(b) $k(t)$ not identically zero and $k(t) \geq 0$, for all $t \in [\tau_1, \tau_2]$;

(c) $g \in C^1([0, +\infty))$, $g(x) \geq 0$, for all $x \geq 0$ and $g(0) = 0$, $g'(0) > 0$;

(d) $g(x) - xg'(x) \geq 0$, for all $x \geq 0$;

(e) $1/g'(0) \leq x/g(x)$, for all $x > 0$.

These assumptions include some that are significant from a biological point of view (see [Messina et al. 2008a] and the bibliography therein) and guarantee that the solution $y(t)$ is nonnegative for all $t \geq \tau_2$. Define the positive function

$$a(x) = \begin{cases} x/g(x) & \text{if } x > 0, \\ 1/g'(0) & \text{if } x = 0. \end{cases}$$

By hypotheses (d) and (e), $a(x)$ is an increasing function for all $x \geq 0$. In particular, it is strictly increasing for all $x \geq 0$, if $g(x)$ is a nonlinear function, while it is constant otherwise. From now on, we assume that $g(x)$ is nonlinear, hence (d) and (e) are meant as strict inequalities and, in analogy with [Messina et al. 2008a], we consider the following alternative formulation of (1):

$$y(t) = \rho g(y(t - \xi(t))), \quad \xi(t) \in [\tau_1, \tau_2], \tag{3}$$

where

$$\rho = \int_{\tau_1}^{\tau_2} k(x)\, dx, \tag{4}$$

which is more appropriate for our analysis. Obviously, (1) has at least the trivial solution $y^* = 0$. The following theorem shows that this equilibrium is unique for $\rho_0 < 1/g'(0)$, then the value $\rho_0 = 1/g'(0)$ represents a bifurcation point for the variable ρ; as a matter of fact, when $\rho > \rho_0$, the trivial solution is no longer unique, and another nontrivial equilibrium $y^* = a^{-1}(\rho)$ appears. Let $a^* = \lim_{x\to+\infty} a(x)$.

Theorem 2.1 [Iannelli 1994; Messina et al. 2008a]. *Let ρ be defined as in* (4).

(i) *Equation* (1) *has one and only one nontrivial equilibrium $y^* = a^{-1}(\rho)$ if and only if $1/g'(0) < \rho < a^*$.*

(ii) *Equation* (1) *has only the trivial equilibrium if $\rho \le 1/g'(0)$.*

To analyze the nature of these equilibria we recall the following definitions.

Definition 2.1. Let y^* be an equilibrium point for (1). Then y^* is said to be:

- stable if, for all $\epsilon > 0$, there exists $\delta > 0$ such that
$$|\varphi(t) - y^*| < \delta,\ \forall t \ge \tau_2 \implies |y(t) - y^*| < \epsilon,\ \forall t \in [\tau_2, T];$$
- locally attractive if there exists $\delta > 0$ such that
$$|\varphi(t) - y^*| < \delta,\ \forall t \in [0, \tau_2] \implies \lim_{t\to+\infty} |y(t) - y^*| = 0;$$
- globally attractive if, for all $\varphi(t) > 0$,
$$\lim_{t\to+\infty} |y(t) - y^*| = 0;$$
- locally asymptotically stable if it is stable and locally attractive;
- globally asymptotically stable if it is stable and globally attractive.

We now quote some propositions proved in earlier papers, and we prove Theorem 2.6, which assures the global asymptotic stability of the solution $y(t)$ of (1).

Theorem 2.2 [Iannelli 1994; Messina et al. 2010]. *Let y^* be an equilibrium point for* (1).

(i) *If $\rho|g'(y^*)| < 1$, then y^* is locally asymptotically stable*;

(ii) *If $\rho|g'(y^*)| > 1$, then y^* is unstable.*

Theorem 2.3 [Messina et al. 2008a]. *If $g(x)$ is nondecreasing, then the nontrivial equilibrium y^* is locally asymptotically stable.*

Theorem 2.4 [Iannelli 1994]. *If $\rho g'(0) < 1$, then the trivial equilibrium is globally asymptotically stable.*

We recall that, from the biological point of view, the threshold value $\rho g'(0)$ plays the role of the *basic reproduction number.*[1] Furthermore, while it is known

[1] In population dynamics, the basic reproduction number represents the average number of offspring produced over the lifetime of an individual under ideal conditions. In epidemiological models, it represents the mean number of secondary cases that a single infected case causes in a population with no immunity.

in [Messina et al. 2008a] that the global attractivity of $y^* = 0$ implies $\rho \le 1/g'(0)$, a result on the behavior of $y^* = 0$, when $\rho = 1/g'(0)$ is still missing.

Since in many examples of applications the form of the nonlinearity in (1) is of unimodal type (e.g., $g(x) = xe^{-x}$; see for instance [Breda et al. 2007, Section 6; Iannelli 1994, page 81 (5.19)], where, as we explain in the introduction of [Messina et al. 2008a], $\Phi(x) = (g(x))/x$), we assume, from now on, that $g(x)$ is an unimodal function with mode $\bar{y}$.

Theorem 2.5 [Messina et al. 2008a]. *Let $g(x)$ in* (1) *be unimodal and let*

$$\frac{1}{g'(0)} \le \rho \le \frac{\bar{y}}{g(\bar{y})}.$$

Then

$$\lim_{t\to+\infty} y(t) = a^{-1}(\rho), \quad \textit{for all } \varphi(t) \ge 0.$$

Thanks to these results we can prove the following theorem on the global asymptotic stability of the nontrivial equilibrium.

Theorem 2.6. *Let $g(x)$ in* (1) *be an unimodal function with mode $\bar{y}$. If*

$$\frac{1}{g'(0)} \le \rho \le \frac{\bar{y}}{g(\bar{y})},$$

then y^* is globally asymptotically stable.

Proof. If $y(t)$ is a solution of (1), then

$$y(t) = \rho g(y(t - \xi(t))) \le \rho g(\bar{y}) \le \bar{y}.$$

This means that each $y(t)$ which is a solution of (1) falls in the interval $[0, \bar{y}]$ where $g(y)$ is increasing; in particular $g'(y^*) > 0$. Since also ρ is positive, then $\rho|g'(y^*)| = \rho g'(y^*)$. What is more, thanks to hypothesis (d), $g(y^*) - y^* g'(y^*) > 0$ and thus $\rho g'(y^*) < 1$ (this last inequality holds since $\rho = y^*/g(y^*)$). Hence, we are in the hypotheses of Theorem 2.2 and so y^* is locally asymptotically stable. Since $g(x)$ is an unimodal function, we are in the hypotheses of Theorem 2.5. Hence, $y^* = a^{-1}(\rho)$ is a globally attractive equilibrium. □

The hypothesis $\rho \le (\bar{y})/g(\bar{y})$ plays a crucial role in the proof because it implies that each $y(t)$ which is a solution of (1) falls in the interval where $g(x)$ in increasing. As a consequence, the previous results on unimodal functions can be extended to increasing functions $g(x)$. In particular, the following theorem holds.

Theorem 2.7. *Let $g(x)$ in* (1) *be an increasing function. If $\rho \ge 1/g'(0)$, then y^* is globally asymptotically stable.*

Theorem 2.8 [Messina et al. 2008a]. *Let $g(x)$ in* (1) *be unimodal with mode $\bar{y}$. Assume $\rho > \bar{y}/g(\bar{y})$ and let $k'(x)$ be constant in sign for all $x \in [\tau_1, \tau_2]$. Then the nonequilibrium solutions of* (1) *cannot be definitively monotone.*

3. The discrete equation

Let a partition of the interval $[0, T]$ be given by

$$\Pi_N = \{t_n : 0 = t_0 < t_1 < \cdots < t_N = T\},$$

where $t_{n+1} - t_n = h$, $n = 0, \ldots, N$, for some fixed h, called the step size. Assume

$$h = \frac{\tau_1}{r_1} = \frac{\tau_2}{r_2}, \tag{5}$$

with r_1, r_2 positive integers. In [Messina et al. 2009] the following direct quadrature method [Brunner and van der Houwen 1986; Linz 1985], adapted to the form of (1), is proposed:

$$y_n = h \sum_{j=r_1}^{r_2} w_j k(jh) g(y_{n-j}), \qquad n > r_2, \tag{6}$$

where $y_n \simeq y(t_n)$ and $y_l = \varphi(lh)$, $l = 0, 1, \ldots, r_2$, for $\varphi(t)$ is a known function satisfying condition (2). In [Messina et al. 2008a; 2008b; 2009] some conditions on the step size h were derived for which the numerical solution mimics the behavior of the continuous one. Now, with the help of a new reformulation of (6) we are able to complete such analysis by deriving the discrete version of Theorems 2.2 and 2.3 (Theorems 3.3 and 3.4 respectively) and a new result on the global asymptotic stability of the nontrivial equilibrium (Theorem 3.8).

In order to write (6) as the discrete analogous of (3), we will make use of the discrete mean value theorem that we report and prove here for the sake of completeness.

Theorem 3.1. *Assume $f \in C([a, b])$, with $-\infty < a < b < \infty$ and let $x_1, \ldots, x_n \in [a, b]$. If $\alpha_1, \ldots, \alpha_n$ are n real numbers, all of the same sign, there exists $\xi \in (a, b)$ such that*

$$\sum_{i=1}^{n} \alpha_i f(x_i) = f(\xi) \sum_{i=1}^{n} \alpha_i.$$

Proof. Let $m = \min_{x \in [a,b]} f(x)$ and $M = \max_{x \in [a,b]} f(x)$ and assume $\alpha_j \geq 0$, for all $j = 1, \ldots, n$. Then,

$$m \sum_{j=1}^{n} \alpha_j \leq \sum_{j=1}^{n} \alpha_j f(x_j) \leq M \sum_{j=1}^{n} \alpha_j$$

and hence,

$$m \leq \frac{\sum_{j=1}^{n} \alpha_j f(x_j)}{\sum_{j=1}^{n} \alpha_j} \leq M.$$

Since $f(x)$ takes on all values between m and M (*intermediate value theorem*), there exists a point $\xi \in (a, b)$ such that

$$f(\xi) = \frac{\sum_{j=1}^{n} \alpha_j f(x_j)}{\sum_{j=1}^{n} \alpha_j}. \qquad \square$$

Now, define the quantity

$$\rho_h = h \sum_{j=r_1}^{r_2} w_j k(jh). \tag{7}$$

Observe that $k(jh)w_j$ is constant in sign for all $j = r_1, \ldots, r_2$. By Theorem 3.1, then, there exists $\xi_n \in \big[\min_{n-r_2\le j\le n-r_1} y_j, \max_{n-r_2\le j\le n-r_1} y_j\big]$ such that

$$y_n = hg(\xi_n) \sum_{j=r_1}^{r_2} w_j k(jh).$$

Thus, (6) can be formulated, in analogy with the continuous case, in the form

$$y_n = \rho_h g(\xi_n), \text{ with } \xi_n \in \Big[\min_{n-r_2\le j\le n-r_1} y_j, \max_{n-r_2\le j\le n-r_1} y_j\Big]. \tag{8}$$

As for the continuous case, hypotheses (a), (b) and (c) and the positiveness of weights w_j guarantee that the discrete solution y_n is nonnegative for all $n \geq 0$. With regard to the existence of equilibrium solutions, we have:

Theorem 3.2 [Messina et al. 2008a]. *Let ρ_h be defined by* (7).

(i) *Equation* (6) *has one and only one nontrivial equilibrium $y^*(h) = a^{-1}(\rho_h)$ if and only if* $1/g'(0) < \rho_h < a^*$.

(ii) *Equation* (6) *has only the trivial equilibrium if $\rho_h \leq 1/g'(0)$.*

Now we can prove the following results.

Theorem 3.3. *Let $y^*(h)$ be an equilibrium point for* (6).

(i) *If $\rho_h|g'(y^*(h))| < 1$, then $y^*(h)$ is locally asymptotically stable.*

(ii) *If $\rho_h|g'(y^*(h))| > 1$, then $y^*(h)$ is unstable.*

Proof. (1) Suppose $\rho_h|g'(y^*(h))| < 1$. To show that $y^*(h)$ is stable, we fix $\epsilon > 0$ and consider φ such that $|\varphi_j - y^*(h)| < \delta_\epsilon$, $j = 0, \ldots, r_2$, for some $\delta_\epsilon > 0$. Let n take values in $\{r_2, \ldots, r_2 + r_1\}$. From (8) we have $y_n = \rho_h g(\xi_n)$, with

$$\xi_n \in \Big[\min_{j=0\ldots,r_1} \varphi_j, \max_{j=0\ldots,r_1} \varphi_j\Big];$$

hence, $|\xi_n - y^*(h)| < \delta_\epsilon$. For the difference $y_n - y^*(h)$, we have

$$y_n - y^*(h) = \rho_h(g(\xi_n) - g(y^*)) = \rho_h g'(\theta)(\xi_n - y^*(h)),$$

where $\theta \in [\min\{\xi_n, y^*(h)\}, \max\{\xi_n, y^*(h)\}]$; for this reason, $|\theta - y^*(h)| < \delta_\epsilon$. Moreover, since $g'(x)$ is a continuous function such that $|\rho_h g'(y^*(h))| < 1$, there exists $\tilde{y}$ such that

$$|\rho_h g'(y)| < 1 \quad \text{for } y \in [y^*(h) - \tilde{y}, y^*(h) + \tilde{y}].$$

Thus, if we choose $\delta_\epsilon = \min\{\epsilon, \tilde{y}\}$, then $|y_n - y^*(h)| < \epsilon$, $n = r_2, \ldots, r_2 + r_1$. Using this, we easily prove that $|y_n - y^*(h)| < \epsilon$ also for $n = r_2 + r_1, \ldots, r_2 + 2r_1$, and, in general for all $n \geq r_2$. Thus, stability is proved.

Local attractivity follows straightforwardly, by observing that there exists $\delta > 0$ such that $\rho_h |g'(y)| \leq p < 1$, for all $y \in [y^*(h) - \delta, y^*(h) + \delta]$. Thus, by choosing

$$\varphi_1, \ldots, \varphi_{r_2} \in [y^*(h) - \delta, y^*(h) + \delta],$$

and proceeding step by step as n grows, we see that in the k-th interval

$$|y_n - y^*(h)| \leq p^k \delta, \tag{9}$$

where $k \to +\infty$ for $n \to +\infty$. Therefore, $\lim_{n \to +\infty} y_n = y_h^*$.

(2) Consider $\rho_h |g'(y^*(h))| > 1$. To prove the instability of $y^*(h)$ we must find ϵ_0 such that

$$\forall \delta > 0,\ \exists n \in \{0, \ldots, r_2\} \ :\ |y_n - y^*| < \delta,$$

and

$$\exists \bar{n} > r_2 \ :\ |y_{\bar{n}} - y^*(h)| > \epsilon_0.$$

By the continuity of the function g', there exists $d > 0$ such that

$$|g'(y)| \geq r > 1,\ \forall y \in [y^*(h) - d, y^*(h) + d].$$

Take $n \in \{r_2, \ldots, r_2 + r_1\}$. In view of (8) there results

$$|y_n - y^*(h)| = \rho_h |g(\xi_n) - g(y^*(h))| = \rho_h |g'(z)||\xi_n - y^*(h)|, \tag{10}$$

with $|z - y^*(h)| \leq |\xi_n - y^*(h)|$. Then, for all $\delta > 0$, it is possible to choose the starting values φ_l different from $y^*(h)$, for all $l = 0, \ldots, r_2$ and such that $|\varphi_l - y^*(h)| < \min\{d, \delta\}$, for all $n = 0, \ldots, r_2$. Thus,

$$|\xi_n - y^*(h)| < d, \quad n = r_2, \ldots, r_2 + r_1$$

and, form (10), $|\bar{y} - y^*(h)| < d$. This implies that $\rho_h |g'(z)| > 1$. Hence, choosing $\epsilon_0 = \min_{n \in [0, r_2]} |y_n - y^*(h)|$, we have

$$|y_n - y^*(h)| > |\xi_n - y^*(h)|,\ \forall n \in \{r_2, \ldots, r_2 + r_1\}. \qquad \square$$

Now we prove the discrete counterpart of Theorem 2.3.

Theorem 3.4. *If $g(x)$ is a nondecreasing function, then the nontrivial equilibrium $y^*(h)$ is locally asymptotically stable.*

Proof. Let $y^*(h) \neq 0$, for hypothesis (d), $g(x) - xg'(x) > 0$, for all $x > 0$, then $g(y^*(h)) - y^*(h)g'(y^*(h)) > 0$; since $y^*(h) = \rho_h g(y^*(h))$, we have

$$\rho_h g'(y^*(h)) < 1,$$

that is, $\rho_h |g'(y^*(h))| < 1$, since $g'(x) \geq 0$. The result follows from Theorem 3.3. □

The following theorem was proved in [Messina et al. 2008a], but here the proof has been simplified by the new formulation (8) of (6).

Theorem 3.5. *If $\rho_h g'(0) < 1$, then the trivial equilibrium is globally asymptotically stable.*

Proof. We already know from Theorem 3.3 that $y^*(h) = 0$ is locally asymptotically stable. Now we prove the global attractivity. Let $r_2 \leq n \leq r_2 + r_1$ then, from (8),

$$y_n = \rho_h[g(\xi_n) - g(0)] = \rho_h g'(\xi_{n_0})\xi_n, \tag{11}$$

with $0 \leq \xi_{n_0} \leq \xi_n$ and $\xi_n \in [0, \max_{0 \leq j \leq r_2} \varphi_j]$. Thanks to hypotheses (d) and (e),

$$g'(\xi_{n_0}) < \frac{g(\xi_{n_0})}{\xi_{n_0}} < g'(0). \tag{12}$$

From (11) and (12) we obtain $y_n < \rho_h g'(0)\xi_n \leq \rho_h g'(0)\phi$, where $\phi = \max_{0 \leq j \leq r_2} \varphi_j$. Let $\alpha = \rho_h g'(0)$. Then $y_n \leq \alpha\phi$, with $\alpha < 1$.

By similar arguments, for $n = r_2 + r_1 \ldots r_2 + 2r_1$, we get $y_n < \alpha^2\phi$, with $\alpha < 1$. The conclusion follows by iterating the same procedure in all the next intervals. □

In analogy with the continuous case we report the following result:

Theorem 3.6 [Messina et al. 2008a]. *If $y^*(h) = 0$ is globally attractive, then*

$$\rho_h \leq \frac{1}{g'(0)}.$$

Next we consider the special case where the function $g(x)$ is unimodal.

Theorem 3.7 [Messina et al. 2008a]. *Assume that $g(x)$ in (1) is unimodal with mode $\bar{y}$. It $1/g'(0) \leq \rho_h \leq \bar{y}/g(\bar{y})$, then*

$$\lim_{n \to +\infty} y_n = a^{-1}(\rho_h).$$

Now, we can prove the discrete version of Theorem 2.6.

Theorem 3.8. *Assume that $g(x)$ in (1) is an unimodal function with mode $\bar{y}$. If $1/g'(0) \leq \rho_h \leq \bar{y}/g(\bar{y})$, then $y^*(h)$ is globally asymptotically stable.*

Proof. If y_n is a solution of (6), then

$$y_n = \rho_h g(\xi_n) \le \rho_h g(\bar{y}) \le \bar{y}.$$

This means that each y_n which is a solution of (6) falls in the interval $[0, \bar{y}]$, where $g(y)$ is increasing; in particular $g'(y^*(h)) > 0$. Since ρ_h is positive as well, we have $\rho_h|g'(y^*(h))| = \rho_h g'(y^*(h))$. What is more, thanks to hypothesis (d), $g(y^*(h)) - y^*(h)g'(y^*(h)) > 0$ and thus $\rho_h g'(y^*(h)) < 1$ (this last inequality holds since $\rho_h = (y^*(h))/(g(y^*(h))))$. Hence, for Theorem 3.3, $y^*(h)$ is locally asymptotically stable. Furthermore, $y^* = a^{-1}(\rho)$ is globally attractive, because, since $g(x)$ is unimodal, we can apply the result in Theorem 3.7. So it is a globally asymptotically stable equilibrium. □

In analogy with the continuous case the following result holds.

Theorem 3.9. *Assume that $g(x)$ is an increasing function. If*

$$\rho_h \ge \frac{1}{g'(0)},$$

then $y^*(h)$ is globally asymptotically stable.

Now, we report a known result that characterize the behavior of the solutions of (6) when the parameter ρ_h is greater than the threshold value $\bar{y}/g(\bar{y})$.

Theorem 3.10 [Messina et al. 2008a]. *Assume $g(x)$ in* (6) *is unimodal with mode $\bar{y}$ and let $\rho_h > \bar{y}/g(\bar{y})$. Then the nonequilibrium solutions of* (6) *cannot be definitively monotone.*

4. A case study

All the previous analysis is well illustrated by means of the following problem of the type (1):

$$y(t) = 8R_0 \int_{t-\tau_2}^{t-\tau_1} \Big(1 - \frac{1}{\tau_2}(t-\tau)\Big) e^{-y(\tau)} y(\tau)\, d\tau, \quad t \in [\tau_2, T]. \tag{13}$$

This equation was considered in [Messina et al. 2008a], where the analytical properties of the solutions were listed and some plots of the numerical solution with respect to time were reported. Here, we summarize the results in that paper and show new ones using a different approach — namely, a comparison between the bifurcation diagrams of the continuous and numerical solutions and plots of the orbits of the numerical solution. Experiments of this kind are quite common for describing the dynamics of population problems.

In (13), $y(t)$ represents the number of adults in the population at time t, while τ_1 is the maturation age, τ_2 the maximum age, and R_0 the basic reproduction number. This equation represents an interesting case study because it includes the major

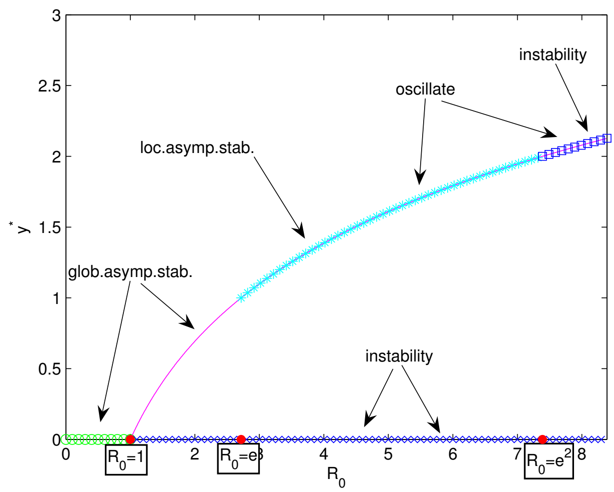

Figure 1. Bifurcation diagram for the parameter R_0 of (13).

features of more complicated models. In particular, $k(t-s)$ is positive and $g(x) = xe^{-x}$ is unimodal with mode $\bar{y} = 1$. If we choose $\tau_1 = 1/2$ and $\tau_2 = 1$, then the parameter $\rho = R_0$ and the nontrivial equilibrium is $y^* = \ln R_0$.

We underline that (13) corresponds to the partial derivative equation described in [Breda et al. 2007, Section 6], modeling a juveniles-adult dynamic.

What makes this equation simple with respect to other problems is that the two classes in which the population is divided (adult $y(t)$ and juveniles $x(t)$) are described by uncoupled equations. More precisely, the number of juveniles $x(t)$ is described by the following integral

$$x(t) = 8R_0 \int_{t-\tau_1}^{t} \Big(1 - \frac{1}{\tau_2}(t-\tau)\Big) e^{-y(\tau)} y(\tau)\, d\tau, \qquad t \in [\tau_2, T]. \tag{14}$$

Hence, the complete problem is represented by Equations (13)+(14), where $y(t)$ depends only on itself and $x(t)$ is a function of $y(t)$.

From the diagram in Figure 1, it is clear that, if $R_0 < 1$ only the trivial equilibrium exists and it is globally asymptotically stable, after this threshold value it becomes unstable; as usual we don't know what happens when $R_0 = 1$. At $R_0 = 1$ the solution bifurcates giving rise to a new nontrivial equilibrium which is globally asymptotically stable for all values of $R_0 \le e = \bar{y}/g(\bar{y})$. When $e < R_0 \le e^2 = 1/g'(\rho)$ the solution oscillates and then converges to $y^* = \ln R_0$, while for $R_0 > e^2$ the equilibrium becomes unstable.

In Figure 2 we report the bifurcation diagram related to the numerical solution of the problem described in (13). From the figure it is clear that the dynamics of the continuous and discrete solutions coincide. In particular, the threshold values $1,\ e,\ e^2$ are the same. What makes the difference is that the dynamic of $y(t)$ is

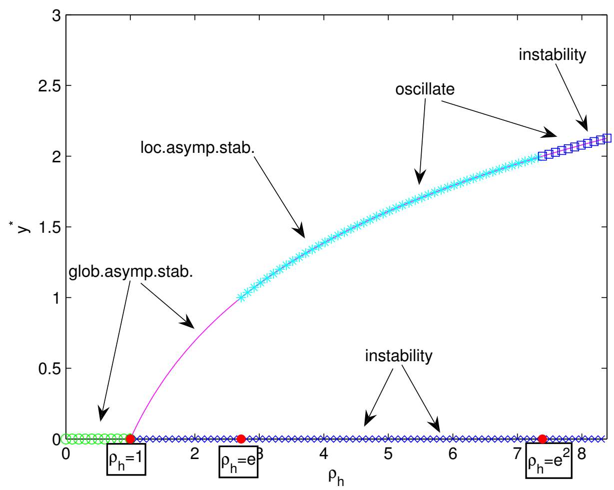

Figure 2. Bifurcation diagram for the parameter ρ_h.

described by the parameter ρ given in (4) (that in our case corresponds to R_0), while the one of y_n by the parameter ρ_h given in (7). However, since $\rho_h \to \rho$ it is always possible to find a sufficiently small step size h such that the two solutions show the same asymptotic behavior.

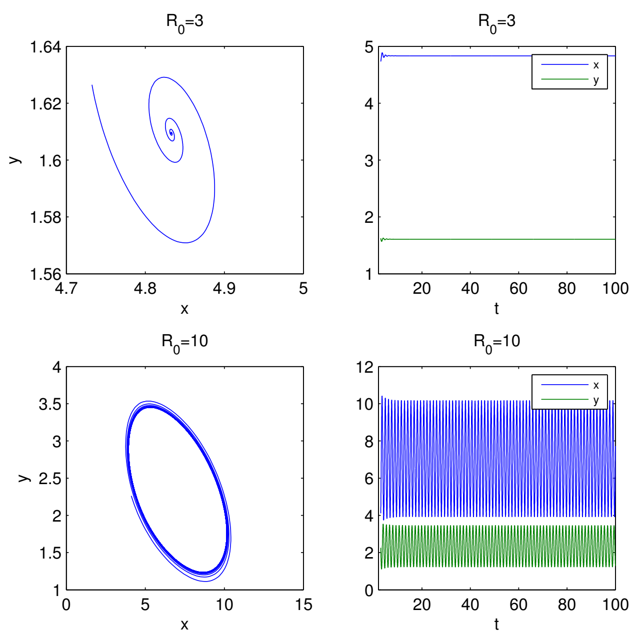

Figure 3. Orbits obtained by numerical computation of the solution of (13)+(14) and corresponding time-dependent plots.

To complete our analysis of problem (13)+(14), we report, in Figure 3, some numerical simulations that show the dynamics of the complete system (13)+(14) for $\rho > 1/g'(0)$ (that is, $R_0 > 1$). In this case, there exists a unique nontrivial equilibrium $P^* = (x^*, y^*) = (\ln R_0, 3 \ln R_0)$. In the first column of the figure, the orbits of the numerical solution clearly show that, in accordance with our investigations, for $1 < R_0 < e^2$ ($R_0 = 5$ in the plot), P^* is a stable equilibrium, while for $R_0 > e^2$, P^* becomes unstable. The two plots reported in the second column show the corresponding time-dependent behaviors, where it is evident that the solution tends to the equilibrium for $R_0 < e^2$ and presents nonstable oscillations after that.

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immacolata.garzilli@gmail.com *Università degli Studi di Napoli "Federico II", Dipartimento di Matematica e Applicazioni R.Caccioppoli, via Cintia, I-80126 Napoli, Italy*

eleonora.messina@unina.it *Università degli Studi di Napoli "Federico II", Dipartimento di Matematica e Applicazioni R.Caccioppoli, via Cintia, I-80126 Napoli, Italy*

antonia.vecchio@cnr.it *Sede di Napoli – CNR, Istituto per le Applicazioni del Calcolo "Mauro Picone", Via P. Castellino, 111, I-80131 Napoli, Italy*

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