involve
a journal of mathematics

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mathematical sciences publishers

2010 vol. 3, no. 4

INVOLVE 3:4(2010)

A topological generalization of partition regularity

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(Communicated by Chi-Kwong Li)

In 1939, Richard Rado showed that any complex matrix is partition regular over $\mathbb{C}$ if and only if it satisfies the columns condition. Recently, Hogben and McLeod explored the linear algebraic properties of matrices satisfying partition regularity. We further the discourse by generalizing the notion of partition regularity beyond systems of linear equations to topological surfaces and graphs. We begin by defining, for an arbitrary matrix Φ, the metric space (M_Φ, δ). Here, M_Φ is the set of all matrices equivalent to Φ that are (not) partition regular if Φ is (not) partition regular; and for elementary matrices, E_i and F_j, we let $\delta(A, B) = \min\{m = l + k : B = E_1 \ldots E_l A F_1 \ldots F_k\}$. Subsequently, we illustrate that partition regularity is in fact a local property in the topological sense, and uncover some of the properties of partition regularity from this perspective. We then use these properties to establish that all compact topological surfaces are partition regular.

1. Introduction

Let $\mathbb{C}$ be the set of complex numbers, and let $\mathbb{M}_{u,v}(\mathbb{C})$ be the set of all $u \times v$ matrices with complex entries. Let $A = [a_{i,j}] \in \mathbb{M}_{u,v}(\mathbb{C})$ be given, and let $\vec{a}_j$ denote the column j of A. Then A satisfies the *columns condition* if and only if there exists an $m \in \{1, \ldots, v\}$ and a partition $\{I_1, \ldots, I_m\}$ of $\{1, \ldots, v\}$ into nonempty sets such that

(i) $\sum_{j \in I_1} \vec{a}_j = \bar{\mathbf{0}}$, and

(ii) for each $t \in \{2, 3, \ldots, m\}$ (if any), $\sum_{i \in I_t} \vec{a}_i$ is in the span of $\{\vec{a}_i : i \in \bigcup_{j=1}^{t-1} I_j\}$.

A is said to be *partition regular* if it satisfies the *columns condition* [Hindman 2007; Rado 1943]. The study of partition regularity has long been a combinatorial endeavor, which mostly uses the columns condition to check if a given matrix is partition regular. However, Hogben and McLeod [2010] recently showed that the columns condition is interesting in its own right, and provided a more linear

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MSC2000: primary 05C99, 05E99, 15A06, 54H10, 57N05; secondary 15A99, 54E35.

Keywords: partition regularity, columns condition, graphs, metric space, discrete topology, topological surface, triangulation.

algebraic perspective on partition regularity. We employ this new perspective to extend the notion of partition regularity into geometrical and topological settings.

For an arbitrary complex matrix Φ, we construct a metric space characterized by the partition regularity of Φ (Section 2). We then use this metric space about Φ to generate a topological space that recasts partition regularity as a local property. We show a few topological properties of these spaces, and then demonstrate how their systems of neighborhoods can describe the "degree" of partition regularity as applied to a given matrix (Section 2). Finally, using some well known connections between graph theory and linear algebra, we construct topological spaces that allow us to define partition regularity as a property of topological surfaces and graphs. In Section 3, we show that all compact topological surfaces are partition regular. We then demonstrate that not all graphs are partition regular.

We take a *topological surface* to be a two-dimensional real manifold that is Hausdorff. A *graph* $G = (V, E)$ is a nonempty set V of vertices, along with a set E of edges, where an edge is a two-element subset of vertices. A *walk* is an alternating sequence $(v_0, e_1, v_1, e_2, \dots, e_m, v_m)$ of vertices and edges. A graph G is *connected* if there exists a walk between any two distinct vertices of G. A *component* is a connected subgraph of G, and a set S of edges of G is a *disconnecting set* if $G \setminus S$ has more than one component. The *edge connectivity* of G is the minimum size of a disconnecting set of G. An *orientation* Γ of G is obtained by assigning a direction to each edge of G, and thus replacing the edge $\{i, j\}$ with the arc (i, j). An orientation Γ of G is *strongly connected* if there exists an alternating sequence $(v_0, e_1, v_1, e_2, \dots, e_m, v_m)$ of vertices and arcs between any two vertices of Γ. The *oriented incidence matrix* of Γ is the rational matrix denoted $D_\Gamma = [d_{i,e}]$, where if $e = (i, j)$, then $d_{i,e} = -1$, $d_{j,e} = 1$, and $d_{k,e} = 0$ for $k \neq i$ and $k \neq j$.

For any matrix A in $\mathbb{M}_{m,n}(\mathbb{C})$, we let a *type-1* elementary operation be a row (column) permutation, a *type-2* elementary operation be multiplication of a given row (column) of A by a scalar $\beta \in \mathbb{C}$, and a *type-3* elementary operation be the addition of a scalar multiple of one row (column) of A to another. We call the associated matrices of each elementary operation T1, T2, and T3 matrices, respectively.

2. Topologically rich spaces associated with partition regularity

Let $A, B \in \mathbb{M}_{m,n}(\mathbb{C})$. We say that B is *equivalent* to A if there exist invertible matrices P and Q for which $B = PAQ$. This is an equivalence relation on $\mathbb{M}_{m,n}(\mathbb{C})$, and we let $[A]$ denote the equivalence class of A. Since P and Q are each the product of a finite number of elementary matrices we can identify P and Q, with the sequence of nonidentity elementary matrices $\langle x \rangle_{i=1}^{l}$ that when applied to matrix A produces matrix B. Since A and B are both in $[A]$, there must exist a minimal sequence of elementary operations. Let $l_{A,B}$ be the nonnegative integer denoting the length of this minimal sequence. Then we can define the function

$$\delta : [A] \times [A] \longrightarrow \mathbb{R}$$

such that

$$\delta(A, B) = l_{A,B}.$$

Theorem 2.1. *Let A be in $\mathbb{M}_{m,n}(\mathbb{C})$. Then $([A], \delta)$ is a metric space.*

Proof. The nonnegativity of δ follows trivially from the definition of $l_{A,B}$ for any pair of matrices A, B in $[A]$.

To see that δ is symmetric, notice that if $\delta(A, B) = l_{A,B}$, then there exist invertible matrices P and Q associated with the minimal sequence $\langle x\rangle_{i=1}^{l_{A,B}}$ such that $B = PAQ$. It follows that $A = P^{-1}BQ^{-1}$, and thus P^{-1} and Q^{-1} may be associated with the sequence $\langle x\rangle_{i=1}^{m}$, where $m = l_{A,B}$ is equal to the number of elementary matrices in the product $P^{-1}Q^{-1}$. Assume that $\langle x\rangle_{i=1}^{m}$ is not minimal, then there must exist a sequence $\langle x\rangle_{i=1}^{l_{B,A}}$ such that $l_{B,A} < m$. Find its associated invertible matrices U and V such that $A = UBV$. So, $B = U^{-1}AV^{-1}$ and there is an associated sequence $\langle x\rangle_{i=1}^{n}$. But $n = l_{B,A}$, and since $m = l_{A,B}$, this contradicts the minimality of $\langle x\rangle_{i=1}^{l_{A,B}}$. Thus,

$$\delta(A, B) = l_{A,B} = m = l_{B,A} = \delta(B, A),$$

and δ is a symmetric function.

To see that δ satisfies the triangle inequality, let A, B, and C be in $[A]$, and pick G, P, U, V, N, and Q such that $B = GAP$, $C = UAV$, and $B = NCQ$. Then

$$\delta(A, B) = l_{A,B}, \quad \delta(A, C) = l_{A,C}, \quad \delta(C, B) = l_{C,B}.$$

Now let $m = \delta(A, C) + \delta(C, B)$. Then there exists a sequence $\langle x\rangle_{i=1}^{m}$ associated with the invertible matrices NU and VQ such that

$$B = (NU)A(VQ).$$

Thus, since A can be changed to B using $\delta(A, C)+\delta(C, B)$ elementary operations, it follows from the minimality of $\delta(A, B)$ that

$$\delta(A, B) \leq \delta(A, C) + \delta(C, B).$$

So, δ satisfies the triangle inequality, and $([A], \delta)$ is indeed a metric space. $\square$

Let $\Phi \in \mathbb{M}_{m,n}(\mathbb{C})$ be a partition regular matrix, and let M_Φ denote the set of all matrices that are partition regular and equivalent to Φ. Notice that (M_Φ, δ) is a metric space. Furthermore, the range of our metric δ is a subset of the nonnegative integers.

Theorem 2.2. *Let $\mathcal{T}$ be the metric topology induced by δ on M_Φ. Then $(M_\Phi, \mathcal{T})$ is a discrete topological space.*

Proof. Let A, $B \in \mathbb{M}_{m,n}(\mathbb{C})$. We need only prove that $\delta(A, B) = 0$ if and only if $A = B$. To see this, notice that for any A, B in M_Φ, $\delta(A, B) = l_{A,B}$ is a nonnegative integer. So, if $\delta(A, B) = 0$, then the minimal number of nonidentity elementary matrices that must be applied to A to produce B is zero. So it must be that $A = B$. Conversely, if $A = B$, then the minimal number of elementary operations that must be applied to A to reach B is equal to 0. Thus, $\delta(A, B) = 0$.

For A_0 in M_Φ consider the open ball of radius $\frac{1}{2}$ about A_0:

$$\mathcal{B}_{A_0,1/2} = \left\{A \in M_\Phi : \delta(A_0, A) < \tfrac{1}{2}\right\} = \{A \in M_\Phi : \delta(A_0, A) = 0\} = \{A_0\}.$$

Therefore, the singletons of M_Φ are all open sets, and so $\mathcal{T}$ is equal to the power set of M_Φ. Thus, the pair $(M_\Phi, \mathcal{T})$ is a discrete topological space. □

Notice that if Φ is not partition regular, then there is a corresponding discrete space consisting of all matrices that are not partition regular and equivalent to Φ. This allows us to establish a "degree" of partition regularity for any arbitrary matrix.

Definition 2.3. Let $A \in \mathbb{M}_{u,v}(\mathbb{C})$ be given.

(a) The *progress* of A is the minimum number, l, of elementary operations that must be performed on A to produce a partition regular matrix. We say that A *has progress* l, and write $\mathrm{pr}(A) = l$, moreover, we write $pr(A) = \infty$ if A cannot be changed into a partition regular matrix via elementary operations.

(b) The *antiprogress* of A is the minimum number, l, of elementary operations that must be performed on A to produce a matrix that is not partition regular. We say that A *has antiprogress* l, and write $\mathrm{apr}(A) = l$, moreover, we write $apr(A) = \infty$ if A cannot be changed into a matrix that is not partition regular via elementary operations.

Any $A \in \mathbb{M}_{u,v}(\mathbb{C})$ has both a progress and an antiprogress. Moreover, A has progress 0 if and only if A is partition regular, and A has antiprogress 0 if and only if A is not partition regular.

We are interested in the collection of matrices that proceed from a given matrix A in M_Φ. The following definitions describe such collections.

Definition 2.4. (a) A *filament* is a sequence of equivalent matrices in M_Φ satisfying the following conditions:

(i) The sequence begins with Φ.

(ii) No matrix in the sequence is repeated.

(iii) The sequence is finite if and only if the last matrix in the sequence has antiprogress 1.

(iv) Each matrix of the sequence is obtained by performing a single elementary operation on the preceding matrix in the sequence.

A filament is called *finite* if the sequence is finite. Otherwise, it is called *infinite*.

(b) A *subfilament associated with* A is a sequence of equivalent matrices in M_Φ starting with matrix A, that satisfies (ii), (iii), and (iv). A subfilament is called *finite* if the sequence is finite. Otherwise, it is called *infinite*.

Example 2.5. Let $\Phi = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then

$$\left(\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\right) \quad \text{and} \quad \left(\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}\right)$$

are finite filaments in M_Φ since

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

are both not partition regular. If $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$, then

$$\left(\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}\right)$$

is a finite subfilament associated with A.

The following theorem provides a better description of exactly how the size of M_Φ relates to the degree of partition regularity of Φ.

Theorem 2.6. *For any partition regular matrix Φ in $\mathbb{M}_{m,n}(\mathbb{C})$, M_Φ contains an infinite number of infinite filaments.*

Proof. Since row equivalent matrices share the same nullspace, we have, as an immediate consequence of [Hogben and McLeod 2010, Theorem 2.3], that partition regularity is invariant under elementary row operations. It follows from the definition of the columns condition that partition regularity is also invariant under type-1 column operations. Thus, there exist exactly four types of elementary operations that cannot produce a matrix that has antiprogress 0 from a partition regular matrix. For any $A \in M_\Phi$ we may apply a type-1 row operation for every possible pair of rows in A, and similarly for any type-1 column operation. This gives $\binom{m}{2}$ possible type 1 row operations, and $\binom{n}{2}$ possible type-1 column operations that can be applied to A and produce a matrix in M_Φ. type-2 row operations also cannot produce a matrix that has antiprogress 0 from A. Since any scalar in $\mathbb{C}-\{0\}$ may be applied to a single row of A, there exist $m\,|\mathbb{C}-\{0\}|$ possible type-2 row operations that can be applied to A. Finally, there are $|\mathbb{C}-\{0\}|$ ways to scale a given row of A, and $m-1$ rows to which this scaled row may be added. Since this may be done for any row of A, there exist $m\,(m-1)\,|\mathbb{C}-\{0\}|$ type-3 row operations that can be applied to A and can produce a matrix in M_Φ. Thus, for any $A \in M_\Phi$, there are exactly

$$m\,(m-1)\,|\mathbb{C}-\{0\}|+m\,|\mathbb{C}-\{0\}|+\binom{m}{2}+\binom{n}{2}$$

elementary operations that will not produce a matrix that has antiprogress 0 when applied to A. Thus, for each matrix produced by applying exactly these operations to Φ, we can produce another $m\,(m-1)\,|\mathbb{C}-\{0\}|+m\,|\mathbb{C}-\{0\}|+\binom{m}{2}+\binom{n}{2}$ that are still partition regular. Continuing in this fashion, we produce

$$\left[m\,(m-1)\,|\mathbb{C}-\{0\}|+m\,|\mathbb{C}-\{0\}|+\binom{m}{2}+\binom{n}{2}\right]^{\aleph_0}$$

infinite sequences of matrices contained in M_Φ.

Now, if

$$(\Phi, A_1, A_2, A_3, \ldots)$$

is an infinite sequence of matrices in M_Φ, where each A_i is produced by applying exactly one of the infinitely many operations described above to A_{i-1}, then there exist only a finite subset of these operations such that $A_i = A_j$ for some

$$A_j \in \{\Phi, A_1, \ldots, A_{i-1}\}.$$

Notice that only the identity operation may be applied to A_{i-1} to produce A_{i-1}. Then to see the result, assume there exists some elementary operation E_0 such that E_0 applied to A_{i-1} produces $A_i \in M_\Phi$ and

$$A_i \in \{\Phi, A_1, \ldots, A_{i-2}\}.$$

If E_0 is a type-1 row operation, there exists some

$$A_j \in \{\Phi, A_1, \ldots, A_{i-2}\}$$

such that switching exactly two rows of A_j produces A_{i-1}. Since there exist for each

$$A_j \in \{\Phi, A_1, \ldots, A_{i-2}\}$$

exactly two rows that may be switched to produce A_{i-1}, there exist at most $(i-1)$ possibilities for E_0 to be a type-1 row operation such that

$$A_i = E_0 A_{i-1} \in \{\Phi, A_1, \ldots, A_{i-2}\}.$$

Similarly, there exist at most $(i-1)$ possibilities for E_0 to be a type-1 column operation such that

$$A_i = A_{i-1} E_0 \in \{\Phi, A_1, \ldots, A_{i-2}\}.$$

In the case that E_0 is a type-2 row operation, and

$$E_0 A_{i-1} \in \{\Phi, A_1, \ldots, A_{i-2}\},$$

then we may produce some $A_j \in \{\Phi, A_1, \ldots, A_{i-2}\}$ by scaling exactly one row of A_{i-1} by a single $\alpha \in \mathbb{C}$. Since $\mathbb{C}$ is a unique factorization domain, it follows that if $E_0 A_{i-1} = A_j$, E_0 is unique. Therefore, at most, we may find one E_0 such that $E_0 A_{i-1} = A_j$ for each $A_j \in \{\Phi, A_1, \ldots, A_{i-2}\}$. Thus, there are at most $(i-1)$ type-2 column operations such that $A_i \in \{\Phi, A_1, \ldots, A_{i-2}\}$.

Finally, let E_0 be a type-3 column operation such that

$$E_0 A_{i-1} = A_j \in \{\Phi, A_1, \ldots, A_{i-2}\}.$$

Then exactly one row of A_j, call it row $(A_j)_k$, is not equal to row $(A_{i-1})_k$, and

$$(A_j)_k = \alpha (A_{i-1})_{k'} + (A_{i-1})_k,$$

where $\alpha \in \mathbb{C}$, and $(A_{i-1})_{k'}$ is one of the $(m-1)$ rows of A_{i-1} that is not row $(A_{i-1})_k$. It follows that

$$(A_j)_k - (A_{i-1})_k = \alpha (A_{i-1})_{k'},$$

and again $\alpha \in \mathbb{C}$ must be unique. Thus, for a given

$$A_j \in \{\Phi, A_1, \ldots, A_{i-2}\},$$

there are at most $(m-1)$ possibilities for E_0 such that $E_0 A_{i-1} = A_j$. Therefore, there are at most $(m-1)(i-1)$ possibilities for E_0 to be a type-3 column operation such that $A_i \in \{\Phi, A_1, \ldots, A_{i-2}\}$.

We may now conclude from these various cases that there are at most

$$3(i-1) + (m-1)(i-1) + 1$$

elementary operations that when applied to A_{i-1} will yield for A_i an element of $\{\Phi, A_1, \ldots, A_{i-1}\}$. Thus, for each A_i in the sequence there still exist an infinite number of elementary operations such that $A_{i+1} \in M_\Phi$ and $A_{i+1} \notin \{\Phi, A_1, \ldots, A_i\}$. It follows that there exist an infinite number of infinite filaments in M_Φ, for any partition regular matrix Φ. $\square$

For any partition regular matrix, Φ, the space M_Φ is large. Moreover, for any two partition regular matrices, Φ and Ψ, the cardinality of the collection of infinite filaments in M_Φ is the same as the cardinality of the collection of infinite filaments in M_Ψ. This makes it difficult to use the size of these spaces to say that one matrix is “more” partition regular than another.

Lemma 2.7. *Let $A' \in M_\Phi$, and let $\kappa(A')$ be an infinite subfilament associated with A', such that A' is the only matrix on $\kappa(A')$ that may also be on a finite filament. Consider the map*

$$p : M_\Phi \longrightarrow M_\Phi$$

such that

$$p(A) = \begin{cases} A & \text{if } A \text{ is on a finite filament,} \\ \Phi & \text{if } A \text{ is on some } \kappa(A') \text{ for some } A' \in M_\Phi, \text{ and } A \neq A'. \end{cases}$$

Let $P_\Phi = Im(p)$, the image of p. Then the space P_Φ has the quotient topology induced by p.

Proof. It is clear p is a surjection. Furthermore, since M_Φ has the discrete topology, then if U is open in P_Φ, it must be that U is open in M_Φ. □

This new topological space consists of exactly those finite sequences of matrices that will allow Φ to escape the condition of partition regularity. Therefore, the sizes of these spaces offer a better characterization of the degree of partition regularity of a given matrix. Now consider the following corollary to Theorem 2.1.

Corollary 2.8. *Let $\mathscr{D}_\Phi = \{A \in P_\Phi : \operatorname{apr}(A) = 1\}$. Then $(\mathscr{D}_\Phi, \delta)$ is a metric space, and $(\mathscr{D}_\Phi, \mathscr{T})$ is a discrete topological space.*

Proof. By Theorem 2.1 we know that

$$\delta : [\Phi] \text{ X } [\Phi] \longrightarrow \mathbb{R}$$

is a metric on $[\Phi]$. Since $\mathscr{D}_\Phi$ is a subset of $[\Phi]$, we know that

$$\delta : \mathscr{D}_\Phi \text{ X } \mathscr{D}_\Phi \longrightarrow \mathbb{R}$$

is a metric on $\mathscr{D}_\Phi$. Thus, $(\mathscr{D}_\Phi, \delta)$ is a metric space. Since every subset of $\mathscr{D}_\Phi$ is contained in M_Φ, then we know for every A_0 in $\mathscr{D}_\Phi$, there exists an open ball

$$\mathscr{B}_{A_0,1/2} = \{A_0\}.$$

Therefore, every singleton in $\mathscr{D}_\Phi$ is open, and we conclude that $(\mathscr{D}_\Phi, \mathscr{T})$ is a discrete topological space. □

Lemma 2.9. *If P_Φ is compact, then it is finite. Similarly, if $\mathscr{D}_\Phi$ is compact, then it is finite.*

Proof. We will demonstrate the result for P_Φ. The proof works analogously for $\mathscr{D}_\Phi$. Let P_Φ be compact. Since the collection $\mathscr{F}$ consisting of all singletons in P_Φ forms an open cover of P_Φ, there exists a finite subcover $\mathscr{F}'$ contained in $\mathscr{F}$. Assume that $\mathscr{F}'$ is a proper subcollection of $\mathscr{F}$. Then the set $\mathscr{Q} = \{A \in P_\Phi : \{A\} \in \mathscr{F}'\}$ has power set $\mathscr{P}(\mathscr{Q})$ equal to the set of all sets that may be formed by taking the union of elements of $\mathscr{F}'$. Similarly, $\mathscr{R} = \{A \in P_\Phi : \{A\} \in \mathscr{F}\}$ has power set $\mathscr{P}(\mathscr{R})$

equal to the set of all sets that may be formed by taking the union of elements of $\mathcal{F}$. Since $\mathcal{F}'$ is a proper subcollection of $\mathcal{F}$, it must be that $\mathcal{P}(\mathcal{Q})$ is a proper subcollection of $\mathcal{P}(\mathcal{R})$. Thus, we can choose $\mathcal{U} \in \mathcal{P}(\mathcal{R}) \setminus \mathcal{P}(\mathcal{Q})$ that is not equal to the empty set. Then

$$\mathcal{U} = \bigcup_{B \in \mathcal{U}} \{B\},$$

and so any subset $\{B\}$ is not in $\mathcal{P}(\mathcal{Q})$. Thus, the matrix B cannot be in any open set contained in $\mathcal{F}'$, a contradiction. Therefore, $\mathcal{F}' = \mathcal{F}$, and consequently, $\mathcal{F}$ is a finite open cover of P_Φ. Since $\mathcal{F}$ is the collection of all singletons in P_Φ, we conclude that P_Φ must be finite. □

Notice that for any Φ in $\mathbb{M}_{m,n}(\mathbb{C})$, the topological spaces M_Φ, P_Φ, and $\mathcal{D}_\Phi$ are all *Hausdorff*. We can think of $\mathcal{D}_\Phi$ as the boundary set of P_Φ, since no matrix in P_Φ can be "closer" to leaving P_Φ than those matrices with antiprogress 1. This relationship between P_Φ and $\mathcal{D}_\Phi$ grants us the following theorem.

Theorem 2.10. *For any partition regular matrix* Φ, *the quotient space* P_Φ *is compact if and only if* $\mathcal{D}_\Phi$ *is finite.*

Proof. Necessity of the statement follows from Lemma 2.9. To demonstrate sufficiency, recall that P_Φ consists of only finite filaments. Every A in $\mathcal{D}_\Phi$ is the last matrix of a finite filament. For such an A, pick P and Q such that $A = P\Phi Q$, and represent P and Q with the finite sequence of elementary operations $\langle x \rangle_{i=1}^{l}$. Then we can think of the finite filament ending in A as the finite, ordered set of matrices

$$\big(\Phi, \langle x \rangle_{i=1}^{1}(\Phi), \langle x \rangle_{i=1}^{2}(\Phi), \dots, \langle x \rangle_{i=1}^{l-1}(\Phi), A\big),$$

where $\langle x \rangle_{i=1}^{n}(\Phi)$ represents the matrix produced by applying the first n operations of $\langle x \rangle_{i=1}^{l}$ to Φ. If we let

$$\langle x \rangle_A(\Phi) = \big(\Phi, \langle x \rangle_{i=1}^{1}(\Phi), \langle x \rangle_{i=1}^{2}(\Phi), \dots, \langle x \rangle_{i=1}^{l-1}(\Phi), A\big),$$

then

$$P_\Phi = \bigcup_{A \in \mathcal{D}_\Phi} \langle x \rangle_A(\Phi).$$

It follows that P_Φ contains a finite number of matrices if and only if $\mathcal{D}_\Phi$ is finite. Now let $\mathbb{X}$ denote the set of all open sets in P_Φ. Since every singleton is open in P_Φ, we know that $\mathbb{X}$ is finite if and only if $\mathcal{D}_\Phi$ is finite.

Assume that $\mathcal{D}_\Phi$ is finite. Let $\mathcal{F}$ be an open cover of P_Φ (without duplicates), and assume that $\mathcal{F}$ is not finite. Then $|\mathbb{X}| < |\mathcal{F}|$. Thus, $\mathcal{F}$ must contain more open sets of P_Φ than are in the set $\mathbb{X}$, a contradiction. We conclude that $\mathcal{F}$ must be finite, and since any open cover $\mathcal{F}$ is a subcover of itself, then it must be that every open cover of P_Φ contains a finite subcover. Thus, P_Φ is compact. □

Corollary 2.11. *$\mathcal{D}_\Phi$ is compact if and only if $\mathcal{D}_\Phi$ is finite.*

Proof. Necessity of the statement again follows from Lemma 2.9. So assume that $\mathcal{D}_\Phi$ is finite. Then, P_Φ is compact. Since $P_\Phi \setminus \mathcal{D}_\Phi$ is open in P_Φ, then $\mathcal{D}_\Phi$ is closed. Therefore, $\mathcal{D}_\Phi$ is a closed subset of a compact space, and we conclude that $\mathcal{D}_\Phi$ is compact. □

Theorem 2.12. *Let Φ be a partition regular matrix, and let the set $\mathcal{D}_\Phi$ be infinite. Then P_Φ is the union of an infinite number of disjoint, compact subspaces.*

Proof. Let $\mathcal{G} = \{G_i : i \in \mathbb{N}\}$ be a partition of $\mathcal{D}_\Phi$ into nonempty, disjoint and finite subsets. Since each subset is finite and $\mathcal{D}_\Phi$ contains an infinite number of elements, there must exist an infinite number of subsets in $\mathcal{G}$. Now let $B_1 \subset P_\Phi$ be the set of all matrices on a filament that terminates with a matrix contained in G_1. Then for $i > 1$, let $B_i \subset P_\Phi$ be the set of all matrices on a filament that terminates with a matrix contained in G_i, but are not contained in $\bigcup_{j=1}^{i-1} B_j$.

If a filament ends in a matrix $A \in G_i$, then there exists a pair of invertible matrices P and Q, such that $A = P\Phi Q$, that can be represented by a finite sequence of elementary operations $\langle x\rangle_{i=1}^{l}$. Thus, the filament terminating with A may be written as the finite, ordered set of matrices

$$\langle x\rangle_A(\Phi) = \big(\Phi, \langle x\rangle_{i=1}^{1}(\Phi), \langle x\rangle_{i=1}^{2}(\Phi), \ldots, \langle x\rangle_{i=1}^{l-1}(\Phi), A\big),$$

where $\langle x\rangle_{i=1}^{n}(\Phi)$ represents the matrix produced by applying the first n operations of $\langle x\rangle_{i=1}^{l}$ to Φ. Then, for all $i \in \{1, 2, 3, \ldots\}$, the set

$$\bigcup_{A \in G_i} \langle x\rangle_A(\Phi)$$

is the union of a finite number of finite sets, and therefore is also finite. Consequently, each B_i is finite for all i since,

$$B_i \subseteq \bigcup_{A \in G_i} \langle x\rangle_A(\Phi).$$

Now let $\mathbb{X}$ be the set of all open sets in B_i. Since B_i is finite and has the discrete topology, $\mathbb{X} = \mathcal{P}(B_i)$, and is also finite. Assume that $\mathcal{F}$ is an open cover of B_i that contains an infinite number of open sets. Since $\mathcal{F}$ is a collection of open sets of B_i, it must be that $\mathcal{F} \subseteq \mathbb{X}$. Thus, $|\mathcal{F}| \leq |\mathbb{X}|$, which contradicts the finite size of $\mathbb{X}$. So $\mathcal{F}$ must be finite, and every open cover of B_i is finite. Thus, every open cover of B_i contains a finite subcover, namely itself. Therefore, B_i is a compact subspace of P_Φ. Since $\{B_i : i \in \mathbb{N}\}$ is an infinite set of disjoint subspaces of P_Φ and $P_\Phi = \bigcup_{i \in \mathbb{N}} B_i$, it follows that P_Φ is the union of an infinite number of disjoint, compact subspaces. □

3. Partition regular topological surfaces and graphs

In this section, we will create a topological space that has geometry describing the degree of partition regularity for an associated topological surface, and then we will show how these spaces may also be created for an arbitrary, finite graph.

Every topological surface $\mathcal{S}$ has a triangulation, and $\mathcal{S}$ is compact if and only if it has a triangulation consisting of a finite number of triangles. So let $\mathcal{S}$ be a topological surface, and let $T(\mathcal{S})$ be a triangulation of $\mathcal{S}$. Then, the set of all vertices and edges in $T(\mathcal{S})$ form a connected graph G. Let $\{\Gamma_i : i \in \mathbb{N}\}$ be the collection of all orientations of G.

Proposition 3.1. *The collection* $\{\Gamma_i : i \in \mathbb{N}\}$ *is finite if and only if the surface* $\mathcal{S}$ *is compact.*

Proof. For necessity, let $\{\Gamma_i : i \in \mathbb{N}\}$ be finite and assume $\mathcal{S}$ is not compact. Then $T(\mathcal{S})$ does not consist of a finite number of triangles. Thus, there exists an infinite number of edges in G, each of which may be assigned one of two directions. Let Γ_0 be in $\{\Gamma_i : i \in \mathbb{N}\}$ with $\mathcal{K}$ as the edge index set. Now, let Γ_n be the orientation of G obtained by reversing only the direction of edge n in Γ_0. Then the set $\{\Gamma_n : i \in \mathcal{K}\}$ is infinite. However,

$$\{\Gamma_n : i \in \mathcal{K}\} \subset \{\Gamma_i : i \in \mathbb{N}\},$$

which gives a contradiction.

For sufficiency, notice that if $\mathcal{S}$ is compact, then $T(\mathcal{S})$ contains a finite number of triangles. Therefore, E, the set of all edges in G, is a finite set. Since each edge may have one of two directions, then $|\{\Gamma_i : i \in \mathbb{N}\}| = 2^{|E|}$. □

Let G be a finite graph and Γ_i an orientation of G. We know that D_{Γ_i}, the oriented incidence matrix of Γ_i, is partition regular if and only if Γ_i is strongly connected [Hogben and McLeod 2010, Theorem 2.4]. Here we consider the subcollection $\mathcal{C}$ of $\{\Gamma_i : i \in \mathbb{N}\}$ consisting of all strongly connected orientations of G.

Theorem 3.2. *The collection* $\mathcal{C}$, *of all strongly connected orientations of* G, *is a nonempty and proper subcollection of* $\{\Gamma_i : i \in \mathbb{N}\}$, *for any triangulation of any topological surface* $\mathcal{S}$.

Proof. Let $T(\mathcal{S})$ be a triangulation of some topological surface $\mathcal{S}$. Then the graph G consisting of all vertices and edges in $T(\mathcal{S})$ is a connected graph. It is well known that a graph G has a strongly connected orientation if and only if the edge connectivity of G is greater than or equal to 2. Therefore, G associated with $T(\mathcal{S})$ will have a strongly connected orientation if and only if it does not have edge connectivity equal to 1. We know that an edge $\{i, j\}$ of G must be an edge of at least one triangle. Now let

$$w = \big\{v_1, \{1, 2\}, v_2, \dots, v_i, \{i, j\}, v_j, \dots, \{n-1, n\}, v_n\big\}$$

be a walk on G that uses edge $\{i, j\}$. If we remove edge $\{i, j\}$, then we can define the walk

$$w' = \Big\{v_1, \{1,2\}, v_2, \dots, v_i, \{i,k\}, v_k, \{k, j\}, v_j, \dots, \{n-1, n\}, v_n\Big\},$$

where v_k is the third vertex in some triangle containing edge $\{i, j\}$. Thus, $G\backslash\{\{i, j\}\}$ is still connected, and consequently, G cannot have edge connectivity equal to 1. Therefore, there exists a strongly connected orientation of G, and $\mathscr{C}$ is nonempty.

To see that $\mathscr{C} \neq \{\Gamma_i : i \in \mathbb{N}\}$, notice that for any vertex v_i of G, there exists an orientation Γ_i in $\{\Gamma_i : i \in \mathbb{N}\}$ such that any edge connected to v_i has v_i as its head. Thus, Γ_i cannot be strongly connected, and $\mathscr{C}$ must be a proper subcollection of $\{\Gamma_i : i \in \mathbb{N}\}$. □

Therefore, if $\mathscr{S}$ is compact, then for each Γ_i in $\mathscr{C}$, D_{Γ_i} is partition regular. Furthermore, for each D_{Γ_i} we can create the associated quotient space $P_{D_{\Gamma_i}} = P_{\Gamma_i}$.

Definition 3.3. Given a compact topological surface $\mathscr{S}$, let $\mathfrak{T}$ be the set of all triangulations of $\mathscr{S}$. For $t \in \mathfrak{T}$, let $\mathscr{C}_t$ be the set of all strongly connected orientations of the graph associated with t. Then $\mathscr{S}$ is a *partition regular surface* if the product space

$$\prod_{t\in\mathfrak{T}}\Big(\prod_{\Gamma_i\in\mathscr{C}_t} P_{\Gamma_i}\Big)$$

is nonempty.

Theorem 3.4. *Let $\mathscr{S}$ be a compact topological surface. Then $\mathscr{S}$ is partition regular.*

Proof. Recall that the set $\mathscr{C}_t$ is nonempty for any triangulation t, of any surface $\mathscr{S}$. Thus, there is at least one quotient space for each distinct triangulation of $\mathscr{S}$ in the product topology associated with $\mathscr{S}$. Consequently, the product topology associated with $\mathscr{S}$ is not an empty space, and thus $\mathscr{S}$ is partition regular. □

Definition 3.5. For any finite graph G, let $\mathscr{C}$ be the set of all strongly connected orientations of G. Then G is a *partition regular graph* if the product space

$$\prod_{\Gamma_i\in\mathscr{C}_t} P_{\Gamma_i}$$

is nonempty.

In contrast to Theorem 3.4, the following theorem shows that not all finite graphs are partition regular.

Theorem 3.6. *Let G be a finite tree. Then G is not a partition regular graph.*

Proof. Since no orientation of a tree graph is strongly connected, then every quotient space in the product topology associated with a tree graph is empty. Consequently, any such product topology is empty, and no tree graph is a partition regular graph. □

4. Conclusion

As is captured in the contrasting scenarios presented in Theorems 3.4 and 3.6, partition regularity may be thought of as a property with varying degree that is dependent on the object being studied. For instance, we began with a compact topological surface $\mathcal{S}$, traced the notion of order in the context of topological surfaces, through the graph theoretical context, and finally the matrix theoretical context. Consequently, we were able to construct topological spaces characterizing the degree of order for the surface $\mathcal{S}$. We have also seen that every compact topological surface is a partition regular surface, and thus exhibits, as should be expected, some level of order. We may now explore the concept of partition regularity for compact topological surfaces and finite graphs. Moreover, we now possess structures that allow us to no longer think of an object as simply being partition regular, but instead, as having some degree of partition regularity. Subsequently, we may begin to relate matrices, graphs, and topological surfaces based on their relative degrees of partition regularity.

Acknowledgements

The author thanks the 2010 NSF REU Program at Mount Holyoke College, and Professor Jillian McLeod of Mount Holyoke College for her critical review of the material in this paper.

The author also thanks the faculty and staff of the Oberlin College Department of Mathematics.

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Received: 2010-08-02 Revised: 2010-12-21 Accepted: 2010-12-22

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

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Typeset in LaTeX

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2010 vol. 3 no. 4