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Five-point zero-divisor graphs determined by equivalence classes

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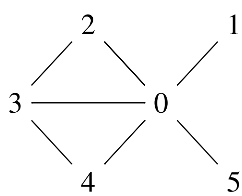
We study condensed zero-divisor graphs (those whose vertices are equivalence classes of zero-divisors of a ring R) having exactly five vertices. In particular, we determine which graphs with exactly five vertices can be realized as the condensed zero-divisor graph of a ring. We provide the rings for the graphs which are possible, and prove that the rest of graphs can not be realized via any commutative ring. There are 34 graphs in total which contain exactly five vertices.

1. Introduction

Beck [1988] introduced, for a commutative ring R, a graph whose vertices are the elements of R and whose edges are given by the rule that two vertices r and s share an edge if and only if $rs = 0$. Thus, for the ring $R = \mathbb{Z}/6\mathbb{Z} = \{0, 1, 2, 3, 4, 5\}$, the associated graph is this:



This is by definition a simple graph (no loops or multiple edges) and it is clearly connected with diameter at most two,[1] since all vertices share an edge with 0.

Anderson and Livingston [1999] later introduced the *zero-divisor graph* $\Gamma(R)$ of a commutative R, by taking the subgraph of Beck's graph consisting of all zero-divisors[2] together with the edges they share — in other words, by discarding from

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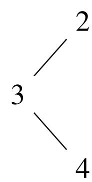
[1]See Definition 2.2 for terms from graph theory.

[2]See Definition 2.1 for terms from ring theory.

Beck's graph the vertex 0 and all vertices that are not zero-divisors. For instance, the zero-divisors of the ring $\mathbb{Z}/6\mathbb{Z}$ are $\{2, 3, 4\}$, so $\Gamma(\mathbb{Z}/6\mathbb{Z})$ is this graph:

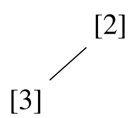


$= \quad \Gamma(\mathbb{Z}/6\mathbb{Z})$

It turns out that the zero-divisor graph, too, is always connected, and its diameter is at most three.

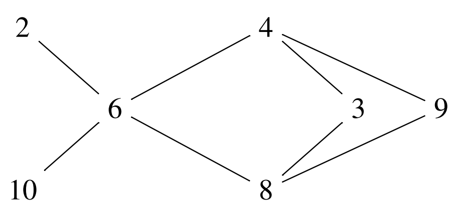
Mulay [2002, (3.5)] demonstrated how a graph $\Gamma_E(R)$ could be constructed from $\Gamma(R)$ by collapsing into equivalence classes zero-divisors that have the same annihilator ideal. Thus, the equivalence class $[r]$ of an element $r \in R$ is the set of zero-divisors s such that $\mathrm{ann}_R(r) = \mathrm{ann}_R(s)$; and such equivalence classes form the vertices of $\Gamma_E(R)$. We call $\Gamma_E(R)$ the *condensed zero-divisor graph* of R. (In [Spiroff and Wickham 2011; Coykendall et al. 2012] the term used was "zero-divisor graph determined by equivalence classes".) Once again, these graphs are simple and connected; the diameter is at most three.

Example 1.1. The equivalence classes of zero-divisors of the ring $R = \mathbb{Z}/6\mathbb{Z}$ are $\{[2], [3]\}$. Note that $\mathrm{ann}_R(2) = \mathrm{ann}_R(4)$, hence $[2] = [4]$.



$= \quad \Gamma_E(\mathbb{Z}/6\mathbb{Z})$

Example 1.2 [Spiroff and Wickham 2011, Example 1.11]. To illustrate the relation between the zero-divisor graph $\Gamma(R)$ and its condensed counterpart $\Gamma_E(R)$, consider $R = \mathbb{Z}/12\mathbb{Z}$:



$= \quad \Gamma(\mathbb{Z}/12\mathbb{Z})$

$[2] — [6] ——— [4] — [3] \qquad = \quad \Gamma_E(\mathbb{Z}/12\mathbb{Z})$

To motivate the study of $\Gamma_E(R)$, we provide an additional example. The ring $(\mathbb{Z}/6\mathbb{Z})[X]$, consisting of polynomials in the variable X with coefficients from $\mathbb{Z}/6\mathbb{Z}$, contains infinitely many elements and zero-divisors. However, there are still just two equivalence classes of zero-divisors, and the graph takes the same form as that in Example 1.1.

The goal of this project is to examine the five-point condensed zero-divisor graphs and to determine which of them are possible. This work grew out of [Spiroff and Wickham 2011]; we rely on the results there and provide some answers to questions that arose during that initial study. A subsequent paper [Coykendall et al. 2012] generalizes some of the results in this project.

For those graphs that can be constructed from equivalence classes, we provide an associated ring. For those graphs that can not be constructed from equivalence classes, we prove that no ring exists such that $\Gamma_E(R)$ takes the necessary form. The list of all thirty-four graphs with exactly five vertices can be found in [Harary 1969, pages 216–217]. The connected ones are all shown in this paper at the relevant places, and are labeled (1)–(21).

2. Definitions and basic results

Throughout, R will be a commutative ring with identity that satisfies the ascending chain condition on ideals. A good general reference for the ring theory needed here is [Dummit and Foote 1991]. For zero-divisor graphs, see [Anderson and Livingston 1999].

Definition 2.1. Some definitions from ring theory are collected here:

(1) A *zero divisor* of R is a nonzero element r of R for which there is another nonzero element s of R such that $rs = 0$.

(2) The *annihilator ideal of* r in R, denoted by $\mathrm{ann}_R(r)$, is the set of all elements a in R such that $ar = 0$.

(3) A *unit* in R is a nonzero element u that has a multiplicative inverse; that is, $uu^{-1} = 1$ for some u^{-1} in R.

(4) An ideal J of R is *maximal* if, whenever $J \subseteq I$ for any proper ideal I of R, then $J = I$.

(5) An *equivalence relation* on R is a binary relation $\sim$ that is reflexive, symmetric, and transitive.

Definition 2.2. Some definitions from graph theory are collected here:

(1) A *graph* consists of a set of vertices, a set of edges, and an incidence relation, describing which pairs of vertices are joined by an edge. Two vertices joined by an edge are called *adjacent*.

(2) A *path of length* n between two vertices v and w is a finite sequence of vertices $u_0, u_1, \ldots, u_n$ such that $v = u_0$, $w = u_n$, and u_{i-1} and u_i are adjacent for all $1 \le i \le n$.

(3) A graph is said to be *connected* if there is a path between every pair of vertices of the graph.

(4) The *distance* between two vertices v and w in a connected graph is the length of the shortest path between them.

(5) The *diameter* of a connected graph G is the greatest distance between any two vertices.

(6) A graph is said to be *complete* if every vertex in the graph is adjacent to every other vertex in the graph.

Definition 2.3. The condensed zero-divisor graph of a ring R, denoted by $\Gamma_E(R)$, is the graph associated to R whose vertices are the classes of zero-divisors, where a pair of distinct classes $[r], [s]$ is adjacent if and only if $[r] \cdot [s] = 0$, where $[r] \cdot [s] := [rs]$.

Remark 2.4 [Mulay 2002, (3.5)]. Multiplication is well-defined: let $[r_1] = [r_2]$ and $[s_1] = [s_2]$; that is, $\text{ann}_R(r_1) = \text{ann}_R(r_2)$ and $\text{ann}_R(s_1) = \text{ann}_R(s_2)$. Then $r_1 s_1 = 0$ if and only if $s_1 \in \text{ann}_R(r_1) = \text{ann}_R(r_2)$, if and only if $r_2 s_1 = 0$, if and only if $r_2 \in \text{ann}_R(s_1) = \text{ann}_R(s_2)$, if and only if $r_2 s_2 = 0$.

Proposition 2.5 [Mulay 2002, (3.5); Spiroff and Wickham 2011, Propositions 1.4, 1.5, 1.8]. *For any ring* R, $\Gamma_E(R)$:

(a) *is connected*;

(b) *has diameter at most three*;

(c) *is not a cycle graph*; *that is*, *does not take the form of an* n*-gon*, *for any* n;

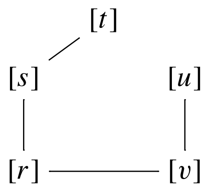
(d) *is not complete if it has at least three vertices.*

Lemma 2.6. *If* u *is a unit in* R *and* r *is a zero-divisor in* R, *then* $\text{ann}_R(ur) = \text{ann}_R(r)$.

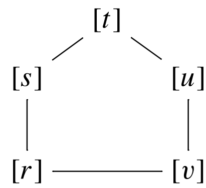
Proof. If $s \in \text{ann}_R(r)$, then $s(ur) = u(sr) = 0$, hence $s \in \text{ann}_R(ur)$. Conversely, if $s \in \text{ann}_R(ur)$, then $0 = s(ur) = u(sr)$ implies $u^{-1} \cdot 0 = u^{-1} \cdot u(sr)$, and hence $0 = sr$. Thus, $s \in \text{ann}_R(r)$. □

3. Negative results

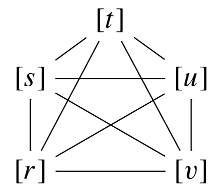
In this section, we prove that all but four of the five-point graphs can not be realized as the condensed zero-divisor graph of a ring. (Recall that we are assuming that all rings are commutative with identity and satisfy the ascending chain condition on ideals.) By part (a) of Proposition 2.5, only connected graphs need to be considered. By parts (b)–(d) of the same proposition, graphs of types (1)–(3) are not possible:



(1)



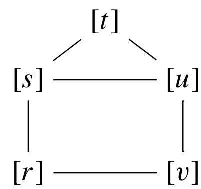
(2)



(3)

The rest of the arguments proceed by contradiction. Namely, we assume that there exists R such that $\Gamma_E(R)$ has exactly the graph in question, which means, in particular, that R has exactly five distinct equivalence classes as represented by the graph. Then from the classes and relations, we show that there must be, in fact, a distinct sixth class, and hence arrive at a contradiction.

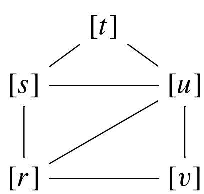
Consider this graph:



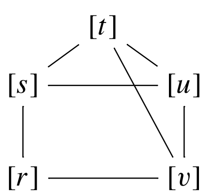
(4)

We show that the element $t+v$ determines a sixth class. First, $t+v$ is annihilated by u, but not by s: indeed, $s(t+v) = 0 + sv \neq 0$, as there is no edge between s and v. Likewise, r does not annihilate $t+v$. However, based on the graph, every class is annihilated by $[r]$ or $[s]$. Thus, $[t+v]$ is not represented by any vertex, and hence must determine a new class.

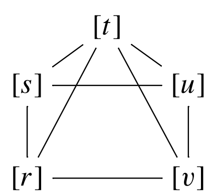
The proofs for graphs (5)–(8) below proceed along the same lines: in (5) and (6), the element $t+v$ determines a new class, and in (7) and (8), the elements $u+v$ and rt determine a new class, respectively.



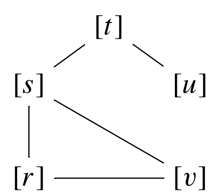
(5)



(6)



(7)

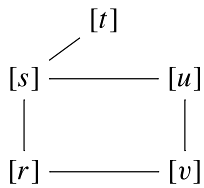


(8)

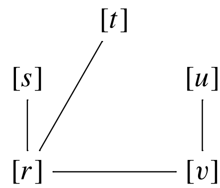
The remaining proofs rely on two key strategies.

Strategy I. *If two points on the condensed zero-divisor graph are adjacent to the same set of vertices, but are not adjacent to one another, then at least one is self-annihilating; otherwise, the two points would represent the same class.*

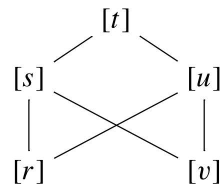
Consider graph (9). One can assume that $r^2 = 0$, else $[r] = [u]$. Then the element $r + v$, which is annihilated by r, but not s or u, determines a new class since, based on the graph, every class is annihilated by $[s]$ or $[u]$. Similarly, for (10), we have $s^2 = 0$, else $[s] = [t]$, hence sv determines a new class. In (11) we have $u^2 = 0$, else $[s] = [u]$, and $v^2 = 0$, else $[r] = [v]$, hence $s + v$ determines a new class.



(9)



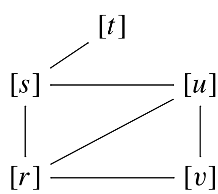
(10)



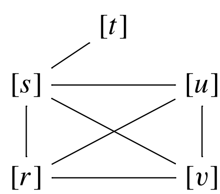
(11)

Strategy II. *If two points on the condensed zero-divisor graph are adjacent to the same set of vertices and are also adjacent to one another, then at least one of the points must not annihilate itself; otherwise, the two points would represent the same class.*

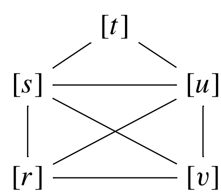
More specifically, in graph (12), one can assume that $r^2 \neq 0$, else $[r] = [u]$. Then the element $r + v$, which is annihilated by u, but not r or s, determines a new class since, based on the graph, every class is annihilated by $[r]$ or $[s]$. Similarly, in (13), $r^2 \neq 0$ and $v^2 \neq 0$, else $[r] = [u] = [v]$; hence $r + v$ determines a new class; and in (14), $r^2 \neq 0$, else $[r] = [v]$ and $s^2 \neq 0$, else $[s] = [u]$; hence $r + s$ determines a new class.



(12)

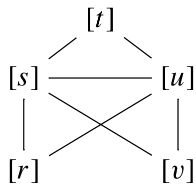


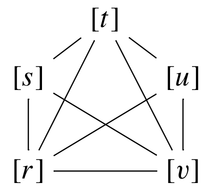
(13)



(14)

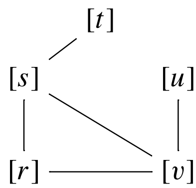
The proofs for graphs (15) and (16), shown on the next page, use both strategies. In (15), one can assume that $v^2 = 0$, by Strategy I, else $[r] = [v]$, and that $s^2 \neq 0$, by Strategy II, else $[s] = [u]$. Then the element $s + v$, which is annihilated by u and v, but not r, s or t, determines a new class since, based on the graph, every class is annihilated by $[r]$, $[s]$ or $[t]$. Similarly, in (16), one can assume that $u^2 = 0$, else $[s] = [u]$, and that $v^2 \neq 0$, else $[r] = [v]$; hence the element $u + v$ determines a new class.

(15) 

 (16)

The last negative case is more complicated.

Proposition 3.1. *The graph in* (17) *can not be realized as* $\Gamma_E(R)$ *for any ring* R.

 (17)

Proof. Suppose that R is a ring such that $\Gamma_E(R)$ takes the form in (17). Note that $su \neq 0$, but $rsu = tsu = vsu = 0$, hence $[su] = [s]$. As a result, $su^2 \neq 0$, and hence $u^2 \neq 0$. By symmetry, $[tv] = [v]$ and $t^2 \neq 0$. Next, consider $s + v$, which is annihilated by r, but not t or u. The only candidate for $[s + v]$ is $[r]$, which means that r is self-annihilating. Moreover, it implies the same of s and v, since $0 = rs = (s + v)s$ and $0 = rv = (s + v)v$.

Consider tu, which is annihilated by s and v. We will show that $[tu]$ must represent a new class. The candidates for $[tu]$ are $[r]$, $[s]$, and $[v]$. By symmetry, we need only consider $[tu] = [r]$ and $[tu] = [s]$.

Case I: $[tu] = [r]$. This means that $t^2u \neq 0$, $tu^2 \neq 0$, but $t^2u^2 = 0$ since r is self-annihilating. Here we are using the fact from [Mulay 2002, (3.5), page 3552] that if $y \in [x]$ and $x^n = 0$, then $y^n = 0$ as well. Now $[t^2] \neq [v]$ since t^2 is not annihilated by u; likewise, $[u^2] \neq [s]$. Thus, $[t^2] \neq [t]$, else $t^2u^2 = 0$ implies that $[u^2] = [s]$. Next, if $[t^2] = [r]$, then $t^2v = 0$, which contradicts $[tv] = [v]$, and for the same reason, $[t^2] \neq [s]$. Finally, t^2 is annihilated by s, hence $[t^2] \neq [u]$. Thus, $[tu]$ determines a new class; contradiction.

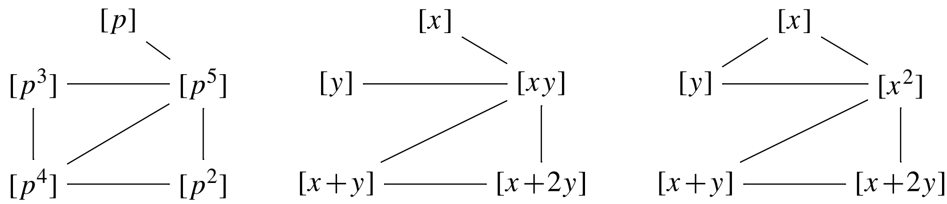
Case II: $[tu] = [s]$. This means that $t^2u = 0$. Thus $[t^2] = [v]$, and hence $t^2v = 0$ since v is self-annihilating. But this contradicts the fact that $[tv] = [v]$. □

4. Positive results

The graphs in this section, labeled (18)–(21), can be realized as condensed zero-divisor graphs. In Proposition 4.1 we prove that when $R = \mathbb{Z}/p^6\mathbb{Z}$, for any prime number p, we get (18) for $\Gamma_E(R)$. In Proposition 4.2 we show that the ring

$$\frac{(\mathbb{Z}/3\mathbb{Z})[[X, Y]]}{(X^2, Y^2)} \tag{$*$}$$

has a graph of the form (19), where lowercase letters match the corresponding uppercase letters in the quotient rings; that is, $x = X + (X^2, Y^2)$ in the ring $(*)$.



(18) (19) (20)

The graph (20) is the condensed zero-divisor graph of the ring

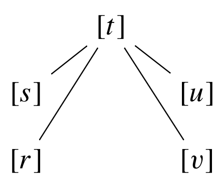
$$\frac{(\mathbb{Z}/3\mathbb{Z})[[X, Y]]}{(X^3, Y^3, XY, X^2+2Y^2)}.$$

This was first reported in [Spiroff and Wickham 2011, Example 3.9], but without details; we supply the details in Proposition 4.3.

Finally, the graded ring

$$R = \frac{A[T]}{(T^3, T^2x, T^2y, Txy)}, \quad \text{where} \quad A = \frac{(\mathbb{Z}/2\mathbb{Z})[[X, Y]]}{(X^2, Y^2)}$$

and x and y represent the cosets of X and Y in A, has the graph shown in (21); a summary of the proof is given in Proposition 4.4. This is an example of a *star graph* or *fan graph*; such graphs are studied in our context in [Coykendall et al. 2012, Section 2], and we refer the interested reader to that paper for a full proof that this ring has the graph shown.



(21)

Proposition 4.1. *If $R = \mathbb{Z}/p^6\mathbb{Z}$, then $\Gamma_E(R)$ has the graph* (18).

Proof. Every nonzero element $\bar r = r + p^6\mathbb{Z}$ in R is either a unit, in which case $\gcd(r, p) = 1$, or a zero-divisor, in which case $\bar r = \overline{up^k}$, where $\bar u$ is a unit, and $k \in \{1, 2, 3, 4, 5\}$. By Lemma 2.6, $\operatorname{ann}_R(\bar u\overline{p^k}) = \operatorname{ann}_R(\overline{p^k})$, therefore the elements $\bar p, \overline{p^2}, \overline{p^3}, \overline{p^4}$, and $\overline{p^5}$ represent the classes. They are all distinct since $\overline{p^i} \in \operatorname{ann}_R(\overline{p^{6-i}})$, but $\overline{p^i} \notin \operatorname{ann}_R(\overline{p^{6-j}})$, for $j > i$. From this the relations follow. □

Proposition 4.2. *If* $R = \dfrac{(\mathbb{Z}/3\mathbb{Z})[[X, Y]]}{(X^2, Y^2)}$, *then $\Gamma_E(R)$ has the graph shown in* (19).

Proof. The ring has a unique maximal ideal $\mathfrak{m} = (x, y)$. Note that $\mathfrak{m}^2 = (x^2, xy, y^2)$ and hence $\mathfrak{m}^2 = (xy)$ since xy is the only nonzero generator. Moreover, $\mathfrak{m}^3 = 0$ in R; that is, both x and y, annihilate xy. Therefore, a general element of R looks like $a + bx + cy + dxy$, where the coefficients a, b, c, d lie in {0, 1, 2}. However, whenever $a \neq 0$, this element is a unit since the other terms all lie in $\mathfrak{m}$; see, for instance, [Matsumura 1989, page 3]. We have shown this:

The only possible zero-divisors live in $\mathfrak{m}$ *and have the form* $bx + cy + dxy$.

We now proceed to describe each class.

***First class:* $[xy]$.** $\mathrm{Ann}_R(dxy) = \mathfrak{m}$, for any $d \neq 0$. To see this, note by $\mathrm{ann}_R(xy) \subseteq \mathfrak{m}$, by the statement proved immediately above. On the other hand, since both generators of $\mathfrak{m}$ annihilate xy, $\mathfrak{m} \subseteq \mathrm{ann}_R(xy)$. Thus, $\mathrm{ann}_R(xy) = \mathfrak{m}$. Also, since 2 is a unit in R, Lemma 2.6 implies that $[2xy] = [xy]$.

***Second class:* $[x]$.** $\mathrm{Ann}_R(bx + dxy) = (x)$, for $b \neq 0$.

Let $b'x + c'y + d'xy \in \mathrm{ann}_R(bx + dxy)$. Then

$$0 = (bx + dxy)(b'x + c'y + d'xy) = bc'xy,$$

which is zero if and only if $bc' = 0$. Since b, c' are elements of a field and $b \neq 0$, we must have $c' = 0$. Therefore, the annihilators of $bx + dxy$ have the form

$$b'x + d'xy = x(b' + d'y) = x(b' + b''x + d'y + d''xy),$$

for any $b', b'', d', d'' \in \mathbb{Z}/3\mathbb{Z}$; that is, $\mathrm{ann}_R(bx + dxy) = (x)$.

***Third class:* $[y]$.** An analogous argument shows that $\mathrm{ann}_R(cy + dxy) = (y)$, for $c \neq 0$.

***Fourth class:* $[x + y]$.** $\mathrm{Ann}_R(bx + by + dxy) = (x + 2y)$, for $b \neq 0$.

Let $b'x + c'y + d'xy \in \mathrm{ann}_R(bx + by + dxy)$. Then

$$0 = (bx + by + dxy)(b'x + c'y + d'xy) = bc'xy + bb'xy = b(b' + c')xy,$$

which is zero if and only if $b(b' + c') = 0$. Since $b \neq 0$, we must have $b' + c' \equiv 0$ in $\mathbb{Z}/3\mathbb{Z}$. Therefore, the elements that annihilate $bx+by+dxy$ are $d'xy$, $x+2y+d'xy$ and $2x + y + d'xy$. However, these last two differ by a unit, for example, $2x + y = 2(x + 2y)$, and $d'xy = d'y(x + 2y)$, hence only $x + 2y$ is necessary as a generator. Thus, $\mathrm{ann}_R(bx + by + dxy) = (x + 2y)$.

***Fifth class:* $[x + 2y]$.** A similar analysis shows that $\mathrm{ann}_R(bx + 2by + dxy) = (x + y)$, where $b \neq 0$. □

Proposition 4.3 [Spiroff and Wickham 2011, Example 3.9]. *If*

$$R = \frac{(\mathbb{Z}/3\mathbb{Z})[[X, Y]]}{(X^3, Y^3, XY, (X+Y)(X+2Y))},$$

then $\Gamma_E(R)$ *has the graph shown in* (20).

Proof. The ring has unique maximal ideal $\mathfrak{m} = (x, y)$. The nonzero generators of $\mathfrak{m}^2$ are (x^2, y^2) and $\mathfrak{m}^3 = 0$ in R; that is, both x and y, annihilate every element in $\mathfrak{m}^2$. Therefore, a general element of R looks like $a + bx + cy + dx^2 + ey^2$, where the coefficients a, b, c, d, and e, are all either 0, 1 or 2. However, whenever $a \neq 0$, this polynomial is a unit since the other terms all lie in $\mathfrak{m}$; see [Matsumura 1989, page 3]. Moreover, the relation $(x+y)(x+2y) = 0$ simplifies to $x^2 = y^2$. Therefore, *the only possible zero-divisors live in* $\mathfrak{m}$ *and have the form* $bx + cy + dx^2$.

***First class:* $[x^2]$.** $\mathrm{Ann}_R(dx^2) = \mathfrak{m}$, $d \neq 0$.

To see this, we first note that $\mathrm{ann}_R(x^2) \subseteq \mathfrak{m}$. On the other hand, since both generators of $\mathfrak{m}$ annihilate x^2, $\mathfrak{m} \subseteq \mathrm{ann}_R(x^2)$. Thus, $\mathrm{ann}_R(x^2) = \mathfrak{m}$. Moreover, since 2 is a unit in R, Lemma 2.6 implies that $[2x^2] = [x^2]$.

***Second class:* $[x]$.** $\mathrm{Ann}_R(bx + dx^2) = (y)$, for $b \neq 0$.

Let $b'x + c'y + d'x^2 \in \mathrm{ann}_R(bx + dx^2)$. Then $0 = (bx + dx^2)(b'x + c'y + d'x^2) = bb'x^2$, which is zero if and only if $bb' = 0$. Since b, b' are elements of a field and $b \neq 0$, we must have $b' = 0$. Therefore, the annihilators of $bx + dx^2$ have the form $c'y + d'x^2$, or $c'y + d'y^2$, since $x^2 = y^2$, and $c'y + d'y^2 = y(c' + d'y) = y(c' + b''x + d'y + d''x^2)$, for any $c', b'', d', d'' \in \mathbb{Z}/3\mathbb{Z}$; that is, $\mathrm{ann}_R(bx + dx^2) = (y)$.

***Third class:* $[y]$.** An analogous argument shows that $\mathrm{ann}_R(cy + dx^2) = (x)$, for $c \neq 0$.

***Fourth class:* $[x + y]$.** $\mathrm{Ann}_R(bx + by + dx^2) = (x + 2y)$, for $b \neq 0$.

Let $b'x + c'y + d'x^2 \in \mathrm{ann}_R(bx + by + dx^2)$. Then

$$0 = (bx + by + dx^2)(b'x + c'y + d'x^2) = bb'x^2 + bc'y^2 = b(b' + c')x^2,$$

which is zero if and only if $b(b' + c') = 0$. Since $b \neq 0$, we must have $b' + c' \equiv 0$ in $\mathbb{Z}/3\mathbb{Z}$. Therefore, the elements that annihilate $bx + by + dx^2$ are $d'x^2$, $x + 2y + d'x^2$ and $2x + y + d'x^2$. However, these last two differ by a unit, for example, $2x + y = 2(x + 2y)$, and $d'x^2 = d'x(x + 2y)$, hence only $x + 2y$ is necessary as a generator. Thus, $\mathrm{ann}_R(bx + by + dx^2) = (x + 2y)$.

***Fifth class:* $[x + 2y]$.** A similar analysis shows that $\mathrm{ann}_R(bx + 2by + dxy) = (x + y)$, where $b \neq 0$. □

Proposition 4.4. *If*

$$R = \frac{A[T]}{(T^3, T^2x, T^2y, Txy)}, \quad \textit{where } A = \frac{(\mathbb{Z}/2\mathbb{Z})[[X,Y]]}{(X^2, Y^2)},$$

then $\Gamma_E(R)$ *has the graph shown in* (21).

Outline of proof. (See [Coykendall et al. 2012] for details.) The ring A is similar to the ring in Proposition 4.2, but with a smaller coefficient ring, and an analogous argument to the one there shows that zero-divisors in A take the form $bx+cy+dxy$, where $b, c, d \in \mathbb{Z}/2\mathbb{Z}$, and there are four distinct classes, given by $\operatorname{ann}_A(x) = (x)$, $\operatorname{ann}_A(y) = (y)$, $\operatorname{ann}_A(xy) = (x, y)$, and $\operatorname{ann}_A(x + y) = (x + y)$. In fact, these determine four distinct classes in R. Note that R has the direct sum decomposition

$$A \oplus \frac{A}{(xy)} \cdot t \oplus \frac{A}{(x, y)} \cdot t^2$$

as an abelian group. We describe the first four classes in R and the last class, determined by t.

***First class:* $[\boldsymbol{xy}]$.** $\operatorname{Ann}_R(xy + \overline{\gamma}t^2) = (x, y)A \oplus \dfrac{A}{(xy)} \cdot t \oplus \dfrac{A}{(x, y)} \cdot t^2$, for $\overline{\gamma}$ in $A/(x, y)$.

***Second class:* $[\boldsymbol{x}]$.** $\operatorname{Ann}_R(x + \overline{\gamma}t^2) = (x)A \oplus \dfrac{(x, y)}{(xy)} \cdot t \oplus \dfrac{A}{(x, y)} \cdot t^2$.

***Third class:* $[\boldsymbol{y}]$.** $\operatorname{Ann}_R(y + \overline{\gamma}t^2) = (y)A \oplus \dfrac{(x, y)}{(xy)} \cdot t \oplus \dfrac{A}{(x, y)} \cdot t^2$.

***Fourth class:* $[\boldsymbol{x+y}]$.** $\operatorname{Ann}_R(x + y + \overline{\gamma}t^2) = (x + y)A \oplus \dfrac{(x, y)}{(xy)} \cdot t \oplus \dfrac{A}{(x, y)} \cdot t^2$.

***Fifth class:* $\boldsymbol{t}$.** $\operatorname{Ann}_R(t + \overline{\gamma}t^2) = (xy)A \oplus \dfrac{(x, y)}{(xy)} \cdot t \oplus \dfrac{A}{(x, y)} \cdot t^2$.

Remark 4.5. The (nonzero) elements $\alpha + \overline{\beta}t$, where $\alpha \in (x, y)A$ and $\overline{\beta} \in A/(xy)$, fall into the above categories. If $\alpha = 0$ and $\beta \in (x, y)A$, then the element is in the first class; if $\alpha \neq 0$ and $\beta \in (x, y)A$, then the element is in $[\alpha]$; finally, if $\beta \notin (x, y)A$, then the element is in $[t]$. □

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