A note on moments in finite von Neumann algebras
Jon Bannon, Donald Hadwin and Maureen Jeffery
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By a result of the second author, the Connes embedding conjecture (CEC) is false if and only if there exists a self-adjoint noncommutative polynomial \( p(t_1, t_2) \) in the universal unital \( C^* \)-algebra \( \mathcal{A} = \langle t_1, t_2 : t_j = t_j^*, 0 < t_j \leq 1 \text{ for } 1 \leq j \leq 2 \rangle \) and positive, invertible contractions \( x_1, x_2 \) in a finite von Neumann algebra \( \mathcal{M} \) with trace \( \tau \) such that \( \tau(p(x_1, x_2)) < 0 \) and \( \text{Tr}_k(p(A_1, A_2)) \geq 0 \) for every positive integer \( k \) and all positive definite contractions \( A_1, A_2 \) in \( M_k(\mathbb{C}) \). We prove that if the real parts of all coefficients but the constant coefficient of a self-adjoint polynomial \( p \in \mathcal{A} \) have the same sign, then such a \( p \) cannot disprove CEC if the degree of \( p \) is less than 6, and that if at least two of these signs differ, the degree of \( p \) is 2, the coefficient of one of the \( t_i^2 \) is nonnegative and the real part of the coefficient of \( t_1 t_2 \) is zero then such a \( p \) disproves CEC only if either the coefficient of the corresponding linear term \( t_i \) is nonnegative or both of the coefficients of \( t_1 \) and \( t_2 \) are negative.

1. Introduction

The Connes embedding conjecture (CEC) is true if every separable type II\(_1\) factor \( \mathcal{M} \) embeds in a tracial ultrapower \( \mathcal{R}^\omega \) of the amenable type II\(_1\) factor \( \mathcal{R} \). This question concerns the matricial approximation of the elements of a type II\(_1\) factor \( \mathcal{M} \) with faithful normal trace state \( \tau \) in the sense we now recall. For an \( N \)-tuple \((x_1, \ldots, x_N)\) of self-adjoint elements in \( \mathcal{M} \), \( R > 0 \), \( n, k \in \mathbb{N} \) and \( \varepsilon > 0 \), we let

\[
\Gamma_R(x_1, \ldots, x_N : n, k, \varepsilon)
\]

denote the set of tuples \((A_1, \ldots, A_N)\) of those \( k \times k \) self-adjoint matrices over \( \mathbb{C} \) of operator norm at most \( R \) satisfying

\[
|\tau(x_{i_1} x_{i_2} \ldots x_{i_p}) - \frac{1}{k} \text{Tr}(A_{i_1} A_{i_2} \ldots A_{i_p})| < \varepsilon,
\]

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whenever $1 \leq p \leq n$ and $(i_1, i_2, \ldots, i_p) \in \{1, 2, \ldots, N\}^p$. We call the elements of $\Gamma_R(x_1, \ldots, x_N : n, k, \varepsilon)$ approximating microstates for $(x_1, \ldots, x_N)$ of precision $(n, \varepsilon)$ using $k \times k$ matrices of norm at most $R$. A separable type II$_1$ factor $\mathcal{M}$ embeds in an ultrapower $\mathcal{R}^\omega$ if and only if for all tuples $(x_1, \ldots, x_N)$ of self-adjoint elements in $\mathcal{M}$, all $n \in \mathbb{N}$ and all $\varepsilon > 0$, it is possible to find $k \in \mathbb{N}$ and $R > 0$ such that $\Gamma_R(x_1, \ldots, x_N : n, k, \varepsilon) \neq \emptyset$. In [Rădulescu 1999] it is proved that this statement is true under the restriction that $n \in \{2, 3\}$, and that if the statement were true for $n = 4$, the CEC would follow.

Our paper concerns the following reformulation of the CEC:

**Theorem 1.1** [Hadwin 2001, Corollary 2.3]. Let $\mathcal{H}$ be a separable Hilbert space. The Connes embedding conjecture is false if and only if there is a positive integer $n$, a noncommutative polynomial $p(t_1, t_2, \ldots, t_n)$ in the universal unital $C^*$-algebra $\mathcal{A}_n = \langle t_1, t_2, \ldots, t_n : t_j = t_j^*, -1 < t_j \leq 1 \text{ for } 1 \leq j \leq n \rangle$ and an $n$-tuple $(x_1, \ldots, x_n)$ of self-adjoint contractions in $B(H)$ such that

(i) $\text{Tr}_k(p(A_1, A_2, \ldots, A_n)) \geq 0$ for every positive integer $k$ and every $n$-tuple $(A_1, \ldots, A_n)$ of self-adjoint contractions $A_1, A_2, \ldots, A_n$ in $M_k(\mathbb{C})$, and

(ii) $W^*(x_1, x_2, \ldots, x_n)$ has a faithful tracial state $\tau$ and $\tau(p(x_1, x_2, \ldots, x_n)) < 0$.

It is well known that a separable type II$_1$ factor $\mathcal{M}$ embeds in an $\mathcal{R}^\omega$ if and only if $\mathcal{M} \otimes M_k(\mathbb{C})$ does for all $k \in \mathbb{N}$. If $\mathcal{M}$ is generated by $k$ self-adjoint elements then $\mathcal{M} \otimes M_k(\mathbb{C})$ is generated by two self-adjoint elements [Sinclair and Smith 2008, Proposition 16.1.1]. Whenever $x \in B(H)$ is a self-adjoint contraction and $\varepsilon > 0$, it follows (e.g., by the continuous functional calculus for $x$) that

$$\frac{(1 + \varepsilon) + x}{2 + \varepsilon}$$

is a positive invertible contraction. Therefore, if we replace $\mathcal{A}_n$ by

$$\mathcal{A} = \{t_1, t_2 : t_j = t_j^*, 0 < t_j \leq 1 \text{ for } 1 \leq j \leq 2\},$$

and repeat the argument in [Hadwin 2001, Section 2], we obtain the following.

**Theorem 1.2.** Let $\mathcal{H}$ be a separable Hilbert space. The Connes embedding conjecture is false if and only if there is a noncommutative polynomial $p(t_1, t_2)$ in the universal unital $C^*$-algebra $\mathcal{A} = \langle t_1, t_2 : t_j = t_j^*, \text{ with } 0 < t_j \leq 1 \text{ for } 1 \leq j \leq 2\rangle$, and positive, invertible contractions $x_1$ and $x_2$ in $B(H)$ such that

(i) $\text{Tr}_k(p(A_1, A_2)) \geq 0$ for every positive integer $k$ and all positive definite contractions $A_1$ and $A_2$ in $M_k(\mathbb{C})$, and

(ii) $W^*(x_1, x_2)$ has a faithful tracial state $\tau$ and $\tau(p(x_1, x_2)) < 0$. 
Also note that if a polynomial $p \in \mathcal{A}$ satisfies (i) and (ii) in the theorem, then so does the polynomial $p + p^*$. We may therefore assume that the polynomial appearing in the theorem is self-adjoint.

Note that, even if we restrict our attention in Theorem 1.1 (or Theorem 1.2) to the case where the degree of $p$ is less than or equal to 3, we cannot use [Rădulescu 1999] to rule out the possibility of finding such a $p$ that will disprove the CEC, because existing methods only allow us to use, when $R' < R$, the existence of a microstate in $\Gamma_R(x_1, \ldots, x_N : n, k, \varepsilon)$ to guarantee the existence of a microstate in $\Gamma_{R'}(x_1, \ldots, x_N : n', k, \varepsilon')$, where $\varepsilon' < \varepsilon$ and $n' > n$ — that is, decreasing $R$ comes at the expense of increasing $n$. See, for example, Proposition 2.4 of [Voiculescu 1994] or Lemma 4 of [Dostál and Hadwin 2003]. Even if this difficulty were overcome, there is no guarantee that the matrices in any approximating microstates found would be positive definite. It behooves us, therefore, to either look for a noncommutative polynomial that may be used to disprove the CEC as prescribed in Theorem 1.1, or to proceed inductively, by degree, to show that such a polynomial cannot exist.

In Section 2 of this paper we prove, in Corollary 2.5 and Theorem 2.6 that if the real parts of all coefficients but the constant coefficient of a self-adjoint noncommutative polynomial $p \in \mathcal{A}$ share the same sign, then such a $p$ cannot disprove the CEC if the degree of $p$ is less than 6. We prove in Section 3 that if the degree of a self-adjoint noncommutative polynomial $p \in \mathcal{A}$ is 2, the real part of the coefficient of $t_1t_2$ is zero and the coefficient of one of the $t_i^2$ is nonnegative, then such a $p$ disproves the CEC only if either the coefficient of the corresponding linear term $t_i$ is nonnegative or if both of the coefficients of $t_1$ and $t_2$ are negative.

From here on in this paper, the symbols $t_1$ and $t_2$ will denote the standard generators of the universal $C^*$-algebra

$$\mathcal{A} = \langle t_1, t_2 : t_j = t_j^*, 0 < t_j \leq 1 \text{ for } 1 \leq j \leq 2 \rangle.$$ 

We refer the reader to [Kadison and Ringrose 1983; Sinclair and Smith 2008] for the basic theory of finite von Neumann algebras.

2. $\tau$-symmetrizable monomials

We prove that if the real parts of all coefficients but the constant coefficient of self-adjoint $p \in \mathcal{A}$ share the same sign, and the constant coefficient is positive, then $p$ cannot disprove the CEC if its degree is less than six. Let $\mathcal{M}$ be a finite von Neumann algebra with faithful trace state $\tau$, and $0 < x_1, x_2 \leq 1$ self-adjoint contractions in $\mathcal{M}$.

**Definition 2.1.** A symmetric expression in $x_1, x_2$ is a finite sequence

$$(w_0, w_1, \ldots, w_{N-1}, w_N)$$
of elements in $\mathcal{M}$, where $N \in \mathbb{N}$, $w_k = x_i^s$ with $i \in \{1, 2\}$, $s \in \{1, 1/2\}$ and $w_k = w_{N-k}$ for all $k \in \{0, 1, \ldots, N\}$. A monic monomial $m(x_1, x_2) = x_{i_1}x_{i_2} \ldots x_{i_l} \in \mathcal{M}$ with $i_j \in \{1, 2\}$ for $j \in \{1, 2, \ldots, l\}$ is $\tau$-symmetrizable if there exists a symmetric expression $(w_0, w_1, \ldots, w_{N-1}, w_N)$ in $x_1, x_2$ such that

$$\tau(x_{i_1}x_{i_2} \ldots x_{i_l}) = \tau(w_0w_1 \ldots w_{N-1}w_N).$$

The element $w_0w_1 \ldots w_{N-1}w_N \in \mathcal{M}$ is called the element associated to the symmetric expression $(w_0, w_1, \ldots, w_{N-1}, w_N)$.

**Lemma 2.2.** If $(w_0, w_1, \ldots, w_{N-1}, w_N)$ is a symmetric expression in $x_1, x_2$, then the associated element $w_0w_1 \ldots w_{N-1}w_N$ in $\mathcal{M}$ is a nonnegative contraction.

**Proof.** We prove this by induction on $N + 1$. If $N + 1 = 1$, then $N = 0$ and the result is clear from the assumptions on the $x_i$.

Assume now that the result holds for $N + 1 \leq l$, that is, for all symmetric expressions $(w_0, w_1, \ldots, w_{j-1}, w_j)$ in $x_1, x_2$ with $j < l$. Let $(w_0, w_1, \ldots, w_{l-1}, w_l)$ be a symmetric expression in $x_1, x_2$. Then so is $(w_1, \ldots, w_{l-1})$. By the induction hypothesis, $w_1 \ldots w_{l-1} \in \mathcal{M}$ is a nonnegative contraction. Since $w_0 = w_l = x_i^s$ for some $i \in \{1, 2\}$ and $s \in \{1, 1/2\}$, we have

$$0 \leq w_0w_1 \ldots w_{l-1}w_l = x_i^s w_1 \ldots w_{l-1}x_i^s \leq x_i^{2s} \leq x_i \leq 1.$$ 

** Remark 2.3.** It is a straightforward exercise to verify that every monic noncommutative monomial $m(x_1, x_2)$ of degree less than six is $\tau$-symmetrizable in any finite von Neumann algebra $\mathcal{M}$ with faithful trace state $\tau$. (Here, of course, it is essential that $0 < x_1, x_2 \leq 1$)

**Corollary 2.4.** If $m(x_1, x_2) = x_{i_1}x_{i_2} \ldots x_{i_l} \in \mathcal{M}$ is a $\tau$-symmetrizable monic monomial, then $1 - \tau(m(x_1, x_2)) \geq 0$.

**Proof.** Since $m$ is $\tau$-symmetrizable, there exists a symmetric expression

$$(w_0, w_1, \ldots, w_{N-1}, w_N)$$

in $x_1, x_2$ such that

$$\tau(x_{i_1}x_{i_2} \ldots x_{i_l}) = \tau(w_0w_1 \ldots w_{N-1}w_N).$$

By **Lemma 2.2** and the fact that $\tau$ is a state, $\tau(w_0w_1 \ldots w_{N-1}w_N) \leq 1$.

In the following two results, $J = J \setminus \{0\}$ denotes a finite index set, and for all $j \in J$, $c_j \in \mathbb{C}$, and $m_j(t_1, t_2) \neq 1$ denotes a monic monomial in $\mathcal{A}$.

**Corollary 2.5.** If $0 < x_1, x_2 \leq 1$ in $\mathcal{M}$ and $p(t_1, t_2) = c_0 1 + \sum_{j \in J} c_j m_j(t_1, t_2)$ is a self-adjoint noncommutative polynomial in $\mathcal{A}$ such that, such that $c_0 > 0$, $\text{Re}(c_j) \geq 0$ for all $j \in J$, $p(1, 1) \geq 0$ and $m_j(x_1, x_2)$ is $\tau$-symmetrizable for every $j \in J$, then $\tau(p(x_1, x_2)) \geq 0$. 
Proof. This is trivial application of Corollary 2.4. □

Theorem 2.6. If \( 0 < x_1, x_2 \leq 1 \) in \( \mathcal{M} \) and \( p(t_1, t_2) = c_0 l + \sum_{j \in J} c_j m_j(t_1, t_2) \) is a self-adjoint noncommutative polynomial in \( \mathcal{A} \) such that \( c_0 > 0 \), \( \Re(c_j) < 0 \) for all \( j \in J \), \( p(1, 1) \geq 0 \) and \( m_j(x_1, x_2) \) is \( \tau \)-symmetrizable for every \( j \in J \), then \( \tau(p(x_1, x_2)) \geq 0 \).

Proof. Suppose \( p(t_1, t_2) \) satisfies the hypotheses. We have
\[
p(1, 1) = c_0 l + \sum_{j \in J} c_j \geq 0,
\]
and therefore
\[
\tau(p(x_1, x_2)) \geq \sum_{j \in J} c_j (m_j(x_1, x_2) - 1) \geq 0. \quad \square
\]

3. Degrees 1 and 2

In degree 1 it is convenient to consider the statement of Theorem 1.1 above. The next result rules out the possibility of finding a polynomial \( p \) of degree 1 that will disprove the CEC via Theorem 1.1. Observe that if \( p(s, t) = c_0 + c_1 s + c_2 t = \tilde{c}_0 + \tilde{c}_1 s + \tilde{c}_2 t \) for any real numbers \(-1 \leq s, t \leq 1\) and that \( p(s, t) \geq 0 \) for any such \( s \) and \( t \), then \( c_0 \geq |c_1 + c_2| \).

Theorem 3.1. Let \( \mathcal{H} \) be a separable Hilbert space. Let \( x_1 \) and \( x_2 \) be self-adjoint contraction operators in \( B(H) \) such that \( W^*(x_1, x_2) \) has a faithful trace state \( \tau \). If \( p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 = \tilde{c}_0 + \tilde{c}_1 t_1 + \tilde{c}_2 t_2 \) is a self-adjoint polynomial in \( \mathcal{A} \) with \( c_0 \geq |c_1 + c_2| \) then \( \tau(p(x_1, x_2)) \geq 0 \).

Proof. Observe that \( \tau(c_0 + c_1 x_1 + c_2 x_2) = c_0 + c_1 \tau(x_1) + c_2 \tau(x_2) \geq c_0 - |c_1 + c_2| \), since \(-1 \leq \tau(x_i) \leq 1 \) for \( i \in \{1, 2\} \). □

We now turn to degree 2. We first prove in Theorem 3.4 that if
\[
p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_4 t_1 t_2 + c_5 t_2^2
\]
is a quadratic, self-adjoint noncommutative polynomial such that either \( c_4 \) is the only nonzero degree 2 term with \( 2 \Re(c_4) \neq 0 \) or one of \( c_3 \) or \( c_5 \) is positive, then whenever \( p(s, t) \) is nonnegative for all real numbers \( 0 < s, t \leq 1 \), it follows that \( \text{Tr}_k(p(A, B)) \geq 0 \) for all positive definite contractions \( A \) and \( B \) in \( M_k(\mathbb{C}) \), for any \( k \in \mathbb{N} \).

To prove the result above, we shall need the fact that any positive definite square matrix has strictly positive entries on its main diagonal. This is a direct consequence of Sylvester’s minorant criterion for positive definiteness.

Lemma 3.2. Let \( A = (A_{ij})_{i=1}^k \in M_k(\mathbb{C}) \) be positive definite. Then \( A_{ii} > 0 \) for all \( i \in \{1, 2, \ldots, k\} \).
Proof. We prove this by induction on \( k \). Recognize that the case \( k = 1 \) is clear. Assume the claim holds for \( k = l \), and that \( A = (A_{ij})_{i=1}^{l+1} \) is a positive definite matrix. By Sylvester’s criterion, \( A = (A_{ij})_{i=1}^{l+1} \) is also positive definite, and therefore, by the induction hypothesis, \( A_{ii} > 0 \) if \( i \in \{1, 2, \ldots, l\} \). We need only show \( A_{(l+1)(l+1)} > 0 \). Let \( v \in \mathbb{C}^{l+1} \) be the vector with 1 in its \((l+1)\)-st row and zero elsewhere. Then \( \langle Av, v \rangle = A_{(l+1)(l+1)} > 0 \) by the positive definiteness of \( A \). \( \square \)

We now observe that if a polynomial is nonnegative on \((0, 1] \times (0, 1]\), then its constant term must be nonnegative.

**Lemma 3.3.** If \( p(s, t) = c_0 + c_1 s + c_2 t + c_3 s^2 + 2 \Re(c_4)st + c_5 t^2 \geq 0 \) for all real numbers \( 0 < s, t \leq 1 \), then \( c_0 \geq 0 \).

**Proof.** For any \( \varepsilon > 0 \) we have

\[
0 < p(\varepsilon, \varepsilon) = c_0 + (c_1 + c_2 + (c_3 + 2 \Re(c_4) + c_5)\varepsilon)\varepsilon;
\]

hence \( c_0 \geq 0 \). \( \square \)

**Theorem 3.4.** Let \( p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_4 t_1 t_2 + c_5 t_2^2 \) be a self-adjoint noncommutative polynomial in \( \mathcal{A} \). Suppose

\[ p(s, t) = c_0 + c_1 s + c_2 t + c_3 s^2 + 2 \Re(c_4)st + c_5 t^2 \geq 0 \]

for all real numbers \( 0 < s, t \leq 1 \), and either \( c_3 = 0 \), \( c_5 = 0 \) and \( 2 \Re(c_4) \neq 0 \) or \( c_3 > 0 \) or \( c_5 > 0 \). Then \( \Tr_k(p(A, B)) \geq 0 \) for any positive definite contractions \( A, B \) in \( M_k(\mathbb{C}) \).

**Proof.** For simplicity, let us assume \( c_5 \geq 0 \). Let \( A, B \) be positive definite contractions in \( M_k(\mathbb{C}) \). By the spectral theorem, we may assume \( A = \text{diag}(A_i)_{i=1}^{k} \) is diagonal. A simple computation establishes that, for all \( i \in \{1, 2, \ldots, k\} \),

\[
(p(A, B))_{ii} = p(A_i, B_{ii}) + \sum_{j \in \{1, 2, \ldots, k\} \setminus \{i\}} c_5 |B_{ij}|^2.
\]

Since \( A \) is a positive definite contraction, each \( A_i \) satisfies \( 0 < A_i \leq 1 \). If we could establish that the matrix \( B_0 := \text{diag}(B_{ii})_{i=1}^{k} \) is a positive definite contraction, then each \( p(A_i, B_{ii}) \) would follow nonnegative by assumption and therefore \( \Tr_k(p(A, B)) = \sum_{i=1}^{k} (p(A, B))_{ii} \geq 0 \). Positivity of \( B_0 \) is a simple consequence of the positive definiteness of \( B \), since every diagonal entry of a positive definite matrix is strictly positive by **Lemma 3.2**. It remains to show that \( B_0 \) is a contraction, which is equivalent to proving that \( I - B_0 \) is positive semidefinite. We know, however, that \( I - B \) is positive semidefinite, and hence that for all \( \varepsilon > 0 \) that \((I + \varepsilon) - B \) is positive definite. Again as a consequence of Sylvester’s criterion, \(((I + \varepsilon) - B)_{ii} > 0 \) for all \( i \in \{1, 2, \ldots, n\} \), therefore for all such \( i \) it follows that \( 1 + \varepsilon > B_{ii} \), and hence \( 1 \geq B_{ii} \). It follows that \( I - B_0 \) is positive semidefinite, hence \( B_0 \) is a contraction. \( \square \)
Let $\mathcal{M}$ be a von Neumann algebra with faithful trace state $\tau$. Below,

$$\langle x, y \rangle_2 = \tau(y^*x) \quad \text{and} \quad \|x\|_2^2 = \tau(x^*x)^{1/2}, \quad \text{for } x, y \in \mathcal{M}. $$

Let $n \in \mathbb{N}$ and $x_1, x_2$ be positive invertible contractions in $\mathcal{M}$. For every $k \in \mathbb{N}$, there are spectral projections $\{P_i^{(k)}\}_{i=1}^k$ in $\{1, x_1\}''$ such that $\tau(P_i^{(k)}) = 1/k$ for each $i$ and

$$\left\| x_1 - \sum_{i=1}^k \frac{i - 1}{k} P_i^{(k)} \right\| < \frac{1}{k}. $$

If $i = j$, let $V_{ij}^{(k)} = P_i^{(k)}$, and if $i \neq j$, let $V_{ij}^{(k)}$ be a partial isometry in $\mathcal{M}$ with initial projection $P^{(k)}_i$ (meaning that $V_{ij}^{(k)}(V_{ij}^{(k)})^* = P_j$) and final projection $P_i^{(k)}$ (meaning that $(V_{ij}^{(k)})^* V_{ij}^{(k)} = P_i^{(k)}$). We now prove that if $x_2$ is sufficiently close (in $\|\cdot\|_2$) to a positive definite element in the type $I$ subfactor of $\mathcal{M}$ generated by $\{V_{ij}^{(k)}\}_{i,j=1}^k$, then $\tau(p(x_1, x_2)) \geq 0$ when $p$ satisfies the hypotheses of Theorem 3.4.

In the statement of the theorem, we regard $x_2$ as an operator matrix and compare it entry-wise to the element $(b_{ij} V_{ij}^{(k)})_{i,j=1}^k$.

**Theorem 3.5.** Let $\mathcal{M}$ be a finite von Neumann algebra with faithful trace state $\tau$, let $x_1, x_2$ be positive, invertible elements in $\mathcal{M}$, and adopt the notation in the previous paragraph. Let $p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_4 t_1 t_2 + \tilde{c}_4 t_2 t_1 + c_5 t_2^2$ be a self-adjoint noncommutative polynomial in $\mathcal{A}$. Suppose that

$$p(s, t) = c_0 + c_1 s + c_2 t + c_3 s^2 + 2 \text{Re}(c_4)st + c_5 t^2 \geq 0$$

for all real numbers $0 < s, t \leq 1$, that either $c_3 = 0$, $c_5 = 0$ and $2 \text{Re}(c_4) \neq 0$ or $c_3 > 0$ or $c_5 > 0$, and that for all $k \in \mathbb{N}$ there exists a type $I$ subfactor of $\mathcal{M}$ generated by $\{V_{ij}^{(k)}\}_{i,j=1}^k$ as in the previous paragraph, and a positive definite contraction $(b_{ij} V_{ij}^{(k)})_{i,j=1}^k \in M_k(\mathbb{C})$ such that

$$\left\| P_i^{(k)} x_2 P_j^{(k)} - b_{ij} V_{ij}^{(k)} \right\|_2 < \frac{1}{k^{100}}, \quad \text{for all } i, j \in \{1, 2, \ldots, k\}.$$

Then $\tau(p(x_1, x_2)) \geq 0$.

**Proof.** Let $D_k = \sum_{i=1}^k \frac{i - 1}{k} P_i^{(k)}$ and $B_k = \sum_{i,j=1}^k b_{ij} V_{ij}^{(k)}$. Writing $x_1 = D_k + (x_1 - D_k)$ and $x_2 = B_k + (x_2 - B_k)$, we have

$$\tau(p(x_1), p(x_2))$$

$$= \tau(p(D_k) + (x_1 - D_k)) + \tau(p(B_k) + (x_2 - B_k)) + c_3 \tau((D_k + (x_1 - D_k))^2) + 2 \text{Re}(c_4) \tau((D_k + (x_1 - D_k))(B_k + (x_2 - B_k))) + c_5 \tau((B_k + (x_2 - B_k))^2)$$

$$= p(\tau(D_k), \tau(B_k)) + c_1 \tau(x_1 - D_k) + c_2 \tau(x_2 - B_k) + 2c_3 \tau(D_k(x_1 - D_k)) + c_3 \tau(x_1 - D_k)^2 + 2 \text{Re}(c_4) \tau(D_k(x_2 - B_k)) + 2 \text{Re}(c_4) \tau((x_1 - D_k)(x_2 - B_k)) + 2c_5 \tau(B_k(x_2 - B_k)) + c_5 \tau((x_2 - B_k)^2).$$
Therefore, by the triangle and Cauchy–Schwartz inequalities and the fact that the operator norm dominates the \( \| \cdot \|_2 \)-norm,

\[
|\tau(p(x_1), p(x_2)) - p(\tau(D_k), \tau(B_k))| \leq (|c_1| + |c_2| + 3|c_3| + 6\Re(c_4) + 3c_5) \frac{1}{k}.
\]

Since \( W^*(D_k, B_k) \cong W^*(\text{diag}((i - 1)/k, i \in \{1, \ldots, k\}), (b_{ij})_{i,j=1}^k) \subseteq M_k(\mathbb{C}) \) via the obvious trace-preserving *-isomorphism, it follows that

\[
\tau(p(x_1), p(x_2)) \geq 0.
\]

**Proposition 3.6.** Let \( p(t_1, t_2) = c_0 + c_1t_1 + c_2t_2 + c_3t_1^2 + c_4t_1t_2 + c_5t_2^2 \) be a self-adjoint noncommutative polynomial in \( \mathcal{A} \) satisfying the hypotheses of Theorem 3.4, and let \( \mathcal{M} \) be a finite von Neumann algebra with faithful trace state \( \tau \). If \( 0 < x_1, x_2 \leq 1 \) in \( \mathcal{M} \) then \( \tau(p(x_1, x_2)) < 0 \) if and only if

\[
c_5\|x_2 - \tau(x_2)\|^2_2 + c_3\|x_1 - \tau(x_1)\|^2_2 + 2\Re(c_4)\langle x_1 - \tau(x_1), x_2 - \tau(x_2)\rangle_2
\]

\[
< -p(\tau(x_1), \tau(y_1)).
\]

**Proof.** Writing each \( \tau(x_i x_j) \) as \( \tau((x_i - \tau(x_i)1)(x_j - \tau(x_j)1)) + \tau(x_i)\tau(x_j) \), we see that

\[
\tau(p(x_1, x_2)) = p(\tau(x_1), \tau(y_1)) + c_5\|x_2 - \tau(x_2)\|^2_2 + c_3\|x_1 - \tau(x_1)\|^2_2
\]

\[
+ 2\Re(c_4)\langle x_1 - \tau(x_1), x_2 - \tau(x_2)\rangle_2.
\]

The result follows. \( \square \)

In the rest of this section, we narrow down the possibilities for disproving the CEC using polynomials satisfying the hypotheses of Theorem 3.4 in the nonrotated case, where \( \Re(c_4) = 0 \). We point out that if \( p(t_1, t_2) = c_0 + c_1t_1 + c_2t_2 + c_3t_1^2 + c_5t_2^2 \) is a self-adjoint noncommutative polynomial in \( \mathcal{A} \) satisfying the hypotheses of Theorem 3.4 with both \( c_5 \geq 0 \) and \( c_3 \geq 0 \), then \( \tau(p(x_1, x_2)) \geq 0 \) by the proof of Proposition 3.6.

**Theorem 3.7.** Let \( p(t_1, t_2) = c_0 + c_1t_1 + c_2t_2 + c_3t_1^2 + c_5t_2^2 \) be a self-adjoint noncommutative polynomial in \( \mathcal{A} \) satisfying the hypotheses of Theorem 3.4 with \( c_3 > 0, c_5 < 0 \) and such that \( c_1 \geq 0 \) and \( c_2 \leq 0 \). Then, for any finite von Neumann algebra \( \mathcal{M} \) with faithful trace state \( \tau \), we have

\[
\tau(p(x_1, x_2)) \geq 0,
\]

for any positive definite contractions \( x_1 \) and \( x_2 \) in \( \mathcal{M} \).

**Proof.** Assume that \( p(t_1, t_2) \) satisfies the hypotheses. Suppose that there exists a finite von Neumann algebra \( \mathcal{M} \) with faithful trace state \( \tau \) and positive definite
contractions $x_1$ and $x_2$ such that $\tau(p(x_1, x_2)) < 0$. If $c_1 \geq 0$ and $c_2 \leq 0$, then

$$p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2,$$

so $c_0 + (c_1 + c_3 \varepsilon) \varepsilon + c_2 + c_5 \geq 0$ for every $\varepsilon > 0$, and hence $c_0 \geq -c_5 - c_2$. Thus

$$0 > c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2$$

$$\geq -c_5 - c_2 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2$$

$$= -c_5(1 - t_2^2) + c_3 t_1^2 + c_1 t_1 - c_2(1 - t_2),$$

and

$$0 > -c_5 \tau(1 - x_2^2) + c_3 \tau(x_1^2) + c_1 \tau(x_1) - c_2 \tau(1 - x_2) \geq 0.$$ 

This is a contradiction. \hfill \qed

**Theorem 3.8.** Let $p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2$ be a self-adjoint non-commutative polynomial in $\mathcal{A}$ satisfying the hypotheses of Theorem 3.4 with $c_3 > 0$, $c_5 < 0$ and such that $c_1 < 0$ and $c_2 = 0$. Then for any finite von Neumann algebra $\mathcal{M}$ with faithful trace state $\tau$,

$$\tau(p(x_1, x_2)) \geq 0,$$

for any positive definite contractions $x_1$ and $x_2$ in $\mathcal{M}$.

**Proof.** Assume that $p(t_1, t_2)$ satisfies the hypotheses. Let $\mathcal{M}$ be a finite von Neumann algebra with faithful trace state $\tau$ and let $x_1$ and $x_2$ be positive definite contractions. If $c_1 < 0$ and $c_2 = 0$, then for every $\varepsilon > 0$ letting $t_1 = \varepsilon - c_1/(2c_3)$,

$$c_0 + c_3 \varepsilon^2 - \frac{c_1^2}{4c_3} + c_5 \geq 0,$$

and therefore $c_0 \geq \frac{c_1^2}{4c_3} - c_5$. Then

$$p(t_1, t_2) = c_0 + c_3 \left(t_1 + \frac{c_1}{2c_3}\right)^2 - \frac{c_1^2}{4c_3} + c_5 t_2^2$$

$$\geq \frac{c_1^2}{4c_3} - c_5 + c_3 \left(t_1 + \frac{c_1}{2c_3}\right)^2 - \frac{c_1^2}{4c_3} + c_5 t_2^2 = -c_5(1 - t_2^2) + c_3 \left(t_1 + \frac{c_1}{2c_3}\right)^2.$$ 

Therefore

$$\tau(p(x_1, x_2)) = -c_5 \tau(1 - x_2^2) + c_3 \tau \left(\left(x_1 + \frac{c_1}{2c_3}\right)^2\right) \geq 0. \quad \square$$

The previous two theorems establish that any polynomial $p(t_1, t_2) = c_0 + c_1 t_1 + c_2 t_2 + c_3 t_1^2 + c_5 t_2^2$ in $\mathcal{A}$ that has a chance to disprove the CEC must satisfy either $c_2 > 0$ or both $c_1 < 0$ and $c_2 < 0$. 
References


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