An observation on generating functions
with an application to a sum of secant powers

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Suppose that \( P(x), Q(x) \in \mathbb{Z}[x] \) are two relatively prime polynomials, and that \( P(x)/Q(x) = \sum_{n=0}^{\infty} a_n x^n \) has the property that \( a_n \in \mathbb{Z} \) for all \( n \). We show that if \( Q(1/\alpha) = 0 \), then \( \alpha \) is an algebraic integer. Then, we show that this result can be used to provide a solution to Problem 11213(b) of the American Mathematical Monthly (2006).

1. Introduction and statement of results

This paper has two goals. One is to prove this general observation:

**Theorem 1.** Suppose \( P(x), Q(x) \in \mathbb{Z}[x] \) are relatively prime polynomials with integer coefficients and their quotient is the generating function of an integer series:

\[
\frac{P(x)}{Q(x)} = \sum_{n=0}^{\infty} a_n x^n, \quad \text{with } a_n \in \mathbb{Z} \text{ for all } n.
\]

Then the inverse of any root of \( Q \) is an algebraic integer.

The second goal is to apply this result to solve a problem from the American Mathematical Monthly:

**Problem 11213 [AMM 2006].** Proposed by Stanley Rabinowitz, Chelmsford, MA. For positive integers \( n \) and \( m \) with \( n \) odd and greater than 1, let

\[
S(n, m) = \sum_{k=1}^{(n-1)/2} \sec^2 m \left( \frac{k \pi}{n+1} \right).
\]

(a) Show that if \( n \) is one less than a power of 2, then \( S(n, m) \) is a positive integer.

(b*) Show that if \( n \) does not have the form of Part (a), then there exists a positive integer \( m \) such that \( S(n, m) \) is not an integer.

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The * indicates that no solution was known to the *Monthly* editors. (A solution to (a) was provided in [AMM 2008].) We solve part (b) of Problem 11213 by proving the contrapositive:

**Theorem 2.** Let $n > 1$ be an odd integer. If, for every positive integer $m$, the sum

$$S(n, m) = \frac{(n-1)/2}{\sec^2 m \pi} \sum_{k=1}^{(n-1)/2} \frac{k\pi}{n+1}$$

has an integer value, then $n + 1$ is a power of 2.

A similar result to Theorem 1 (but less general) had appeared before in the *Monthly*, as a problem proposed and solved by Michael Larsen:

**Problem E 2993** [AMM 1983; 1986]. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ a complex numbers such that $\sum \alpha_i^m$ is an integer for every positive $m$; then the polynomial $\prod (x - \alpha_i)$ has integer coefficients.

Here is an outline of the paper. After recalling the necessary concepts from algebraic number theory in Section 2, we prove in Section 3 two intermediate results: $S(n, m)$ is always rational, and the generating function of the sequence $\{S(n, m)\}_{m>0}$ (for fixed odd $n > 0$) has integer coefficients. In Section 4 we prove Theorem 1, from which Theorem 2 follows easily given the intermediate results.

**2. Background**

We review some basic algebraic number theory, which is carefully laid out in [Stewart and Tall 2002], for example. (This citation will be abbreviated as [ST].)

An *algebraic number* is any zero of a polynomial with integer coefficients. An *algebraic integer* is any zero of a monic polynomial with integer coefficients. The set of algebraic numbers is a field, and the set of algebraic integers forms a ring [ST, Theorems 2.1 and 2.9].

For example, if $p$ is prime, $\zeta_p = e^{2\pi i / p}$ is an algebraic integer since it is a zero of the polynomial $x^p - 1$.

The *minimal polynomial* of an algebraic number $\alpha$ is the monic polynomial $p(x)$ with rational coefficients and the smallest possible degree such that $p(\alpha) = 0$. All polynomials of which $\alpha$ is a root are divisible by $p$. For example, $r(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 = (x^p - 1)/(x - 1)$ is the minimal polynomial of $\zeta_p$.

**Definition.** If $K$ is a field contained in $L$, we say that $L$ is a field extension of $K$, and we denote this by $L : K$.

If $K$ is a field and $\alpha$ is an algebraic number let $K(\alpha)$ denote the smallest field containing all the elements of $K$ and $\alpha$. One way to think about field extensions is that if $L : K$ is a field extension, then $L$ has a natural structure as a vector space over
$K$. The dimension of this vector space, which is called the degree, is represented with $[L : K]$. If $[L : K]$ is a number the field extension is called finite. If $H$, $K$, and $L$ are fields such that $K$ is a subset of $L$ and $H$ is a subset of $K$, then

$$[L : H] = [L : K][K : H]$$

(1)

[ST, Theorem 1.10].

In algebraic number theory field extensions of the form $\mathbb{Q}(\alpha)$ are of interest. If $\alpha$ is an algebraic number, then $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ equals the degree of the minimal polynomial of $\alpha$ [ST, Theorem 1.1]. A field $K$ is called an algebraic number field if $[K : \mathbb{Q}]$ is finite. If $K = \mathbb{Q}(\alpha)$ and $\alpha$ is an algebraic number, then the ring of algebraic integers in $K$ is finitely generated as an abelian group [ST, Theorem 2.16].

**Definition.** If $K = \mathbb{Q}(\alpha)$ is an algebraic number field of degree $n$, then there are $n$ distinct monomorphisms $\sigma_1, \ldots, \sigma_n$ from $\mathbb{K}$ to $\mathbb{C}$. The conjugates of an element $\beta \in K$ are the numbers $\sigma_i(\beta)$ for all $i$ between 1 and $n$.

The conjugates of an algebraic number $\alpha$ are the zeros of the minimal polynomial of $\alpha$. For example, if $\alpha = \zeta_n = e^{2\pi i/n}$, where $n > 0$ is an integer, then $\alpha$ has $\phi(n)$ conjugates in $\mathbb{Q}(\alpha)$, where $\phi$ is the Möbius function. The conjugates of $\zeta_n$ are all the elements in the set

$$\{e^{2\pi ik/n} : (k, n) = 1\}.$$  

This information can be found in [Milne 2009, page 93].

**Definition.** Let $K = \mathbb{Q}(\alpha)$ be an algebraic number field, and consider $\beta \in K$. The *trace* of $\beta$ in $K$, denoted by $\text{Tr}_K \beta$, is the sum of all the conjugates of $\beta$. The *norm* of $\beta$ in $K$, denoted by $N_K(\beta)$, is the product of all of the conjugates of $\beta$.

Thus $\text{Tr}_K \zeta_p = -1$ and $N_K(\zeta_p) = (-1)^{p-1}$ for $p$ prime, where $K = \mathbb{Q}(\zeta_p)$. If one notes that

$$\frac{\zeta_n + \zeta_n^{-1}}{2} = \cos \frac{2\pi}{n}$$

and applies (1) one can see that the conjugates of $\alpha = \cos \frac{2\pi}{n}$ in $\mathbb{Q}(\alpha)$ are all the elements in the set

$$\left\{ \cos \frac{2\pi k}{n} : (k, n) = 1, \ 0 < k < n/2 \right\}.$$  

(2)

A formal proof of this can be found in [Milne 2009, pages 95–96]. Also, as a consequence of Theorem 2.6(a), Lemma 2.13, and Lemma 1.7 of [ST], if $\alpha$ is an algebraic number its trace is rational; and as a consequence of Lemma 2.14 of the same reference, if $\alpha$ is an algebraic integer its norm is an integer.
3. Intermediate results

Lemma 3. If \( n > 1 \) is odd and \( m \geq 1 \), the sum \( S(n, m) \) of Theorem 2 is a rational number.

Proof. We make use of the trigonometric identity \( \sec^2 x = \frac{2}{\cos(2x) + 1} \) to write \( \sec^{2m} x = f(\cos 2x) \), where

\[
f(x) := \left( \frac{2}{x+1} \right)^m.
\]

Then, dropping \( m \) from the notation and introducing \( N = n + 1 \) for convenience, we can rewrite our sum as

\[
\sum_{0 < k < N/2} s(k), \quad \text{where} \quad s(k) := f\left( \cos \frac{2\pi k}{N} \right).
\]

We assume at first that \( N/2 \) is an odd prime. All the \( s(k) \) then lie in the extension \( K = \mathbb{Q}(\cos 2\pi/N) \), as follows from the characterization (2) (with \( n \) in that formula equal to \( N \) here). More precisely, if \( k \) is odd, \( \cos 2\pi k/N \) is a conjugate of \( \cos 2\pi/N \) in \( K \). If \( k \) is even, \( \cos 2\pi k/N \) equals \( -\cos 2\pi k'/N \), for \( k' = N/2 - k \) odd; therefore it is a conjugate of \( -\cos 2\pi/N \). Either way, \( \cos 2\pi k/N \) lies in \( K \), and therefore so does \( s(k) \), since \( f \) is a rational function.

The operation of taking conjugates commutes with applying \( f \) (monomorphisms preserve sums, products and inverses, and fix the numbers 1 and 2). Putting this together with the previous paragraph, we conclude that half of the \( s(k) \) (those where \( k \) is odd) make up the conjugates in \( K \) of \( s(1) \), while the other half make up the conjugates of \( s(2) \) (taking \( k = 2 \) as a representative of the even \( k \)'s). It follows that

\[
\sum_{k=1}^{N/2-1} s(k) = \text{Tr}_K s(1) + \text{Tr}_K s(2) = \text{Tr}_K f\left( \cos \frac{2\pi k}{N} \right) + \text{Tr}_K f\left( \cos \frac{2 \times 2\pi k}{N} \right).
\]

Thus \( S(n, m) \) is the sum of two traces of algebraic numbers, and so rational.

Now let \( N/2 \) be arbitrary. Our strategy is the same: we partition the values of \( k \) according to their \( \gcd \) with \( N \). Let \( d_1, \ldots, d_l \) be all the divisors of \( N \) apart from \( N \) and \( N/2 \), and define

\[
D_i := \{ k : \gcd(k, N) = d_i, \ 0 < k < N/2 \} = \{ d_i j : \gcd(j, N/d_i) = 1, \ 0 < j < N/(2d_i) \}.
\]

The \( D_i \) are disjoint, and together they account for all the \( k \) in the sum (3). Moreover,

\[
\sum_{k \in D_i} s(k) = \sum_{j : \gcd(j, N/d_i) = 1} \sum_{0 < j < N/(2d_i)} f\left( \cos \frac{2\pi j}{N/d_i} \right) = \text{Tr}_\mathbb{Q}(\cos \frac{2\pi}{N/d_i}) f\left( \cos \frac{2\pi}{N/d_i} \right),
\]

where the last equality follows from the same reasoning used earlier for \( k \) odd (with \( N \) replaced by \( N/d_i \)). We have expressed \( S(n, m) \) as a sum of traces of algebraic numbers, which means it is rational. 

This result allows us to prove that the generating function for the sequence \( \{S(n, m)\}_{m \geq 0} \) (for fixed odd \( n > 0 \)) is a rational function.

**Lemma 4.** If \( n > 1 \) is odd, \( m \geq 1 \), and

\[
F_n(x) = \sum_{m=0}^{\infty} S(n, m) x^m,
\]

then there exist \( P(x), Q(x) \in \mathbb{Z}[x] \) such that \( F_n(x) = P(x)/Q(x) \).

**Proof.** Using the formula for the sum of a geometric series, we write

\[
F_n(x) = \sum_{m=0}^{\infty} \left( \sum_{k=1}^{(n-1)/2} \sec^2 \frac{k\pi}{n+1} \right) x^m = \sum_{k=1}^{(n-1)/2} \frac{1}{1 - x \sec^2 \frac{k\pi}{n+1}},
\]

so that

\[
Q(x) = \prod_{k=1}^{(n-1)/2} \left( 1 - x \sec^2 \frac{k\pi}{n+1} \right).
\]

We will show that \( Q(x) \) is a polynomial with rational coefficients. Set

\[
b_k := \sec^2 \frac{k\pi}{n+1},
\]

where \( 1 \leq k \leq (n-1)/2 \). Let \( s_i \) be the sum of the products of each \( i \)-element subset of the set \( \{b_1, b_2, \ldots, b_{(n-1)/2}\} \) (in other words, \( s_i \) is the \( i \)-th elementary symmetric polynomial applied to the \( b_j \)). The coefficient of \( x^i \) in \( Q(x) \) is \((-1)^i s_i \). Also, let

\[
p_i := \sum_{k=1}^{n} b_k^i.
\]

The Newton–Girard formulas tell us that

\[
p_i = s_1 p_{i-1} + s_2 p_{i-2} + \cdots + (-1)^{i-1} s_{i-1} p_1 + (-1)^i i s_i = 0,
\]

for all \( 1 \leq i \leq (n-1)/2 \). It is clear that \( p_i \) is rational for all \( i \) by Lemma 4. An easy induction argument implies that \( s_i \) is rational for all \( i \). Since the coefficients of \( Q(x) \) can be expressed in terms of the \( s_i \), we see that \( Q(x) \) has rational coefficients. Thus \( P(x) = F_n(x)/Q(x) \) has rational coefficients. The desired result follows. \( \square \)

**Lemma 5.** Suppose that \( a \) and \( b \) are algebraic numbers, and

\[
F(x) = \frac{a}{1 - bx} = \sum_{n=0}^{\infty} a_n x^n.
\]

If \( a_n \) is an algebraic integer for all \( n \), then \( b \) is an algebraic integer.
Proof. The assumption implies that \( a_n = ab^n \). We know that \( ab^n \) is an algebraic integer for all \( n \), and so lies in the ring of algebraic integers of the field \( K = \mathbb{Q}(b) \). This ring is finitely generated as an abelian group. Suppose that it is generated by \( \{v_1, v_2, \ldots, v_l\} \). Then \( b^n \) must be in the finitely generated abelian group generated by \( \{v_1/a, \ldots, v_l/a\} \) for all \( n \). Lemma 2.8 of [ST] states that a complex number \( \theta \) is an algebraic integer if and only if the additive group generated by all powers \( 1, \theta, \theta^2, \ldots \) is finitely generated. Thus, \( b \) is an algebraic integer. \( \square \)

Now, we wish to expand upon the ideas presented in Lemma 5.

Definition. A sequence \( \{a_n\} \) of algebraic numbers has a bounded denominator if there exists a positive integer \( m \) such that \( ma_n \) is an algebraic integer for all \( n \).

Lemma 6. Let
\[
F(x) = \sum_{n=0}^{\infty} a_n x^n,
\]
where \( \{a_n\} \) is a sequence with bounded denominator. Suppose \( p(x) \) is a polynomial whose coefficients are algebraic numbers and let
\[
F(x) p(x) = \sum_{n=0}^{\infty} b_n x^n.
\]
Then, the sequence \( \{b_n\} \) has bounded denominator.

Proof. This follows from the fact that the algebraic numbers form a subfield of the complex numbers and the fact that given an algebraic number \( a \) there exists a positive integer \( n \) such that \( na \) is an algebraic integer. \( \square \)

Lemma 7. Let \( \zeta_{4p} = e^{2\pi i / 4p} \), where \( p \) is an odd prime. Then
\[
N_{\mathbb{Q}(\zeta_{4p})}(\zeta_{4p} + \zeta_{4p}^{-1}) = p^2.
\]

Proof. First note that
\[
\zeta_{4p} + \zeta_{4p}^{-1} = 2 \cos \frac{\pi}{2p},
\]
and recall the characterization of the conjugates of \( \cos 2\pi / n \) given in (2). We have
\[
N_{\mathbb{Q}(\zeta_{4p})}(\zeta_{4p} + \zeta_{4p}^{-1}) = \prod_{(k,4p) = 1}^{\frac{2\pi ik}{4p}} (e^{\frac{2\pi ik}{4p}} + e^{-\frac{2\pi ik}{4p}}) = \zeta_{4p}^{-\phi(4p)2p} \prod_{(k,4p) = 1}^{\frac{4\pi ik}{4p}} (e^{\frac{4\pi ik}{4p}} + 1).
\]

Now, we know that
\[
N_{\mathbb{Q}(\zeta_{2p})}(\zeta_{2p} + 1) = \prod_{(k,2p) = 1}^{\frac{2\pi ik}{2p}} (e^{\frac{2\pi ik}{2p}} + 1).
\]
This implies
\[ \zeta_{4p}^{-\phi(4p)2p} \prod_{(k,4p)=1}^{4\pi i k} (e^{4\pi i k p} + 1) = N_{Q(\zeta_{2p})}(\zeta_{2p} + 1)^2(e^{-2\pi i}). \]

Now, the minimal polynomial of \( \zeta_{2p} \) is the same as that of \( -\zeta_p \). Furthermore,
\[ r(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 = \prod_{k=1}^{p-1} (x - \zeta_p^k). \]
So, \( N_{Q(\zeta_p)}(1 - \zeta_p) = r(1) = p \) since the minimal polynomial of \( \zeta_p \) is \( r(x) \). Thus,
\[ N_{Q(\zeta_{2p})}(\zeta_{2p} + 1)^2(e^{-2\pi i}) = N_{Q(\zeta_p)}(1 - \zeta_p)^2 = p^2, \]
as desired. \( \square \)

**Lemma 8.** If, for all \( k \) satisfying \( 1 \leq k \leq (n-1)/2 \), the value of \( \sec^2 \frac{k\pi}{n+1} \) is an algebraic integer, then \( n+1 \) is a power of two.

**Proof.** Assume that \( n+1 \) is not a power of two. Let \( p \) be an odd prime factor of \( 2(n+1) \). Since \( n \) is odd, \( 2(n+1) \) is a multiple of 4 and so \( 4p \) divides \( 2(n+1) \).
Let \( k = 2(n+1)/(4p) \), so \( 2(n+1)/k = 4p \). Then
\[ \sec^2 \frac{k\pi}{n+1} = \left( \frac{2}{\zeta_{2(n+1)} + \zeta_{2(n+1)}^{-1}} \right)^2 = \left( \frac{2}{\zeta_{4p} + \zeta_{4p}^{-1}} \right)^2. \]
Now, from the previous lemma, \( N_{Q(\zeta_{4p})}(\zeta_{4p} + \zeta_{4p}^{-1}) = p^2 \). This implies
\[ N_{Q(\zeta_{4p})} \left( \frac{2}{\zeta_{4p} + \zeta_{4p}^{-1}} \right)^2 = \frac{2^{\phi(4p)}}{p^4}. \]
Then, since \( p \) is an odd prime we know that \( 2^{\phi(4p)}/p^4 \) is not an integer. This means that with the chosen \( k \), \( \sec^2 k\pi/(n+1) \) is not an algebraic integer. This proves the desired result. \( \square \)

### 4. Proof of the theorems

**Proof of Theorem 2.** This is a more general version of Lemma 5. Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) be all the numbers whose reciprocals are zeros of \( Q(x) \). Then \( F(x) \) has a partial fraction expansion whose terms are of the form
\[ \frac{A_{i,l}}{(1 - \alpha_i x)^l}, \]
plus a polynomial part. Write

\[ Q(x) = \prod_{i=1}^{n} (1 - \alpha_i x)^{k_i}. \]

Let \( j \) be the largest positive integer such that in the partial fraction decomposition of \( F(x) \) the term \( A_{i,j}/(1 - \alpha_i x)^j \) is nonzero. Clearly \( j > 0 \), since \( P(x) \) and \( Q(x) \) are relatively prime. Now, let

\[ Q_i(x) = \frac{Q(x)}{(1 - \alpha_i x)^{k_i-j+1}}, \]

The highest power of \((1 - \alpha_i x)\) that divides \( Q_i(x) \) is clearly \( j - 1 \).

We have

\[ F(x)Q_i(x) = \sum_{n=0}^{\infty} b_n x^n. \]

Then, by Lemma 6, \( \{b_n\} \) has a bounded denominator. Now, we will consider the effect of multiplying \( F(x) \) and \( Q_i(x) \) by considering what happens to each term in the partial fraction expansion of \( F(x) \). With the exception of the term

\[ \frac{A_{i,j}}{(1 - \alpha_i x)^j}, \]

\( Q_i(x) \) times a term in the partial fraction expansion of \( F(x) \) is a polynomial of finite degree. Now, one can see that

\[ Q_i(x) \frac{A_{i,j}}{(1 - \alpha_i x)^j} = \frac{Q_i(x)}{(1 - \alpha_i x)^{j-1}} \frac{A_{i,j}}{(1 - \alpha_i x)^{j-1}}. \]

It is clear that \( Q_i(x)/(1 - \alpha_i x)^{j-1} \) is a polynomial. Thus,

\[ F(x)Q_i(x) = q(x) + \frac{D_i}{1 - \alpha_i x}, \]

where \( q(x) \) is a polynomial and \( D_i \) is some algebraic number. So, we can say that for sufficiently large \( n \), \( b_n = D_i \alpha_i^n \) where \( D_i \) and \( b_n \) are algebraic numbers. Then, by Lemma 5, \( \alpha_i \) is an algebraic integer. \( \square \)

**Proof of Theorem 1.** Suppose \( S(n, m) \) is an integer for all \( m > 0 \). By Lemma 4,

\[ F_n(x) = \sum_{m=0}^{\infty} \left( \sum_{k=1}^{(n-1)/2} \frac{\sec^{2m} k \pi}{n+1} \right) x^m = \sum_{k=1}^{(n-1)/2} \left( \frac{1}{1 - x \sec \left( \frac{k \pi}{n+1} \right)} \right) \]

is a rational function. Hence, \( F_n(x) = P(x)/Q(x) \) where \( P(x), Q(x) \in \mathbb{Q}[x] \). Theorem 1 now implies that \( \sec^2(k \pi/(n+1)) \) is an algebraic integer for all \( k \) with \( 1 \leq k \leq (n-1)/2 \). According to Lemma 8, this means \( n+1 \) is a power of two. \( \square \)
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