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Jeffrey Mudrock

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An observation on generating functions with an application to a sum of secant powers

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Suppose that $P(x),\ Q(x) \in \mathbb{Z}[x]$ are two relatively prime polynomials, and that $P(x)/Q(x) = \sum_{n=0}^{\infty} a_n x^n$ has the property that $a_n \in \mathbb{Z}$ for all n. We show that if $Q(1/\alpha) = 0$, then α is an algebraic integer. Then, we show that this result can be used to provide a solution to Problem 11213(b) of the *American Mathematical Monthly* (2006).

1. Introduction and statement of results

This paper has two goals. One is to prove this general observation:

Theorem 1. *Suppose $P(x),\ Q(x) \in \mathbb{Z}[x]$ are relatively prime polynomials with integer coefficients and their quotient is the generating function of an integer series*:

$$\frac{P(x)}{Q(x)} = \sum_{n=0}^{\infty} a_n x^n, \quad \text{with } a_n \in \mathbb{Z} \text{ for all } n.$$

Then the inverse of any root of Q is an algebraic integer.

The second goal is to apply this result to solve a problem from the *American Mathematical Monthly*:

Problem 11213 [AMM 2006]. *Proposed by Stanley Rabinowitz, Chelmsford, MA.* For positive integers n and m with n odd and greater than 1, let

$$S(n,m) = \sum_{k=1}^{(n-1)/2} \sec^{2m}\left(\frac{k\pi}{n+1}\right).$$

(a) Show that if n is one less than a power of 2, then $S(n,m)$ is a positive integer.

(b*) Show that if n does not have the form of Part (a), then there exists a positive integer m such that $S(n,m)$ is not an integer.

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The * indicates that no solution was known to the *Monthly* editors. (A solution to (a) was provided in [AMM 2008].) We solve part (b) of Problem 11213 by proving the contrapositive:

Theorem 2. *Let $n > 1$ be an odd integer. If, for every positive integer m, the sum*

$$S(n,m) = \sum_{k=1}^{(n-1)/2} \sec^{2m} \frac{k\pi}{n+1}$$

has an integer value, then $n+1$ is a power of 2.

A similar result to Theorem 1 (but less general) had appeared before in the *Monthly*, as a problem proposed and solved by Michael Larsen:

Problem E 2993 [AMM 1983; 1986]. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ a complex numbers such that $\sum_1^n \alpha_i^m$ is an integer for every positive m; then the polynomial $\prod_1^n (x - \alpha_i)$ has integer coefficients.

Here is an outline of the paper. After recalling the necessary concepts from algebraic number theory in Section 2, we prove in Section 3 two intermediate results: $S(n,m)$ is always rational, and the generating function of the sequence $\{S(n,m)\}_{m>0}$ (for fixed odd $n > 0$) has integer coefficients. In Section 4 we prove Theorem 1, from which Theorem 2 follows easily given the intermediate results.

2. Background

We review some basic algebraic number theory, which is carefully laid out in [Stewart and Tall 2002], for example. (This citation will be abbreviated as [ST].)

An *algebraic number* is any zero of a polynomial with integer coefficients. An *algebraic integer* is any zero of a monic polynomial with integer coefficients. The set of algebraic numbers is a field, and the set of algebraic integers forms a ring [ST, Theorems 2.1 and 2.9].

For example, if p is prime, $\zeta_p = e^{2\pi i/p}$ is an algebraic integer since it is a zero of the polynomial $x^p - 1$.

The *minimal polynomial* of an algebraic number α is the monic polynomial $p(x)$ with rational coefficients and the smallest possible degree such that $p(\alpha) = 0$. All polynomials of which α is a root are divisible by p. For example, $r(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 = (x^p - 1)/(x-1)$ is the minimal polynomial of ζ_p.

Definition. If K is a field contained in L, we say that L is a field extension of K, and we denote this by $L : K$.

If K is a field and α is an algebraic number let $K(\alpha)$ denote the smallest field containing all the elements of K and α. One way to think about field extensions is that if $L : K$ is a field extension, then L has a natural structure as a vector space over

K. The dimension of this vector space, which is called the *degree*, is represented with $[L : K]$. If $[L : K]$ is a number the field extension is called finite. If H, K, and L are fields such that K is a subset of L and H is a subset of K, then

$$[L : H] = [L : K][K : H] \tag{1}$$

[ST, Theorem 1.10].

In algebraic number theory field extensions of the form $\mathbb{Q}(\alpha)$ are of interest. If α is an algebraic number, then $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ equals the degree of the minimal polynomial of α [ST, Theorem 1.1]. A field K is called an *algebraic number field* if $[K : \mathbb{Q}]$ is finite. If $K = \mathbb{Q}(\alpha)$ and α is an algebraic number, then the ring of algebraic integers in K is finitely generated as an abelian group [ST, Theorem 2.16].

Definition. If $K = \mathbb{Q}(\alpha)$ is an algebraic number field of degree n, then there are n distinct monomorphisms $\sigma_1, \ldots, \sigma_n$ from K to $\mathbb{C}$. The *conjugates* of an element $\beta \in K$ are the numbers $\sigma_i(\beta)$ for all i between 1 and n.

The conjugates of an algebraic number α are the zeros of the minimal polynomial of α. For example, if $\alpha = \zeta_n = e^{2\pi i/n}$, where $n > 0$ is an integer, then α has $\phi(n)$ conjugates in $\mathbb{Q}(\alpha)$, where ϕ is the Möbius function. The conjugates of ζ_n are all the elements in the set

$$\{e^{2\pi ik/n} : (k, n) = 1\}.$$

This information can be found in [Milne 2009, page 93].

Definition. Let $K = \mathbb{Q}(\alpha)$ be an algebraic number field, and consider $\beta \in K$. The *trace* of β in K, denoted by $\mathrm{Tr}_K\beta$, is the sum of all the conjugates of β. The *norm* of β in K, denoted by $N_K(\beta)$, is the product of all of the conjugates of β.

Thus $\mathrm{Tr}_K\zeta_p = -1$ and $N_K(\zeta_p) = (-1)^{p-1}$ for p prime, where $K = \mathbb{Q}(\zeta_p)$. If one notes that

$$\frac{\zeta_n + \zeta_n^{-1}}{2} = \cos\frac{2\pi}{n}$$

and applies (1) one can see that the conjugates of $\alpha = \cos\frac{2\pi}{n}$ in $\mathbb{Q}(\alpha)$ are all the elements in the set

$$\left\{\cos\frac{2\pi k}{n} : (k, n) = 1,\ 0 < k < n/2\right\}. \tag{2}$$

A formal proof of this can be found in [Milne 2009, pages 95–96]. Also, as a consequence of Theorem 2.6(a), Lemma 2.13, and Lemma 1.7 of [ST], if α is an algebraic number its trace is rational; and as a consequence of Lemma 2.14 of the same reference, if α is an algebraic integer its norm is an integer.

3. Intermediate results

Lemma 3. *If $n > 1$ is odd and $m \geq 1$, the sum $S(n,m)$ of Theorem 2 is a rational number.*

Proof. We make use of the trigonometric identity $\sec^2 x = \dfrac{2}{\cos(2x)+1}$ to write $\sec^{2m} x = f(\cos 2x)$, where

$$f(x) := \left(\frac{2}{x+1}\right)^m.$$

Then, dropping m from the notation and introducing $N = n+1$ for convenience, we can rewrite our sum as

$$\sum_{0<k<N/2} s(k), \quad \text{where } s(k) := f\left(\cos\frac{2\pi k}{N}\right). \tag{3}$$

We assume at first that $N/2$ is an odd prime. All the $s(k)$ then lie in the extension $K = \mathbb{Q}(\cos 2\pi/N)$, as follows from the characterization (2) (with n in that formula equal to N here). More precisely, if k is odd, $\cos 2\pi k/N$ is a conjugate of $\cos 2\pi/N$ in K. If k is even, $\cos 2\pi k/N$ equals $-\cos 2\pi k'/N$, for $k' = N/2 - k$ odd; therefore it is a conjugate of $-\cos 2\pi/N$. Either way, $\cos 2\pi k/N$ lies in K, and therefore so does $s(k)$, since f is a rational function.

The operation of taking conjugates commutes with applying f (monomorphisms preserve sums, products and inverses, and fix the numbers 1 and 2). Putting this together with the previous paragraph, we conclude that half of the $s(k)$ (those where k is odd) make up the conjugates in K of $s(1)$, while the other half make up the conjugates of $s(2)$ (taking $k=2$ as a representative of the even k's). It follows that

$$\sum_{k=1}^{N/2-1} s(k) = \mathrm{Tr}_K s(1) + \mathrm{Tr}_K s(2) = \mathrm{Tr}_K f\left(\cos\frac{2\pi k}{N}\right) + \mathrm{Tr}_K f\left(\cos\frac{2\times 2\pi k}{N}\right).$$

Thus $S(n,m)$ is the sum of two traces of algebraic numbers, and so rational.

Now let $N/2$ be arbitrary. Our strategy is the same: we partition the values of k according to their gcd with N. Let $d_1, \ldots, d_l$ be all the divisors of N apart from N and $N/2$, and define

$$D_i := \{k : \gcd(k,N) = d_i,\ 0 < k < N/2\} = \{d_i j : \gcd(j, N/d_i) = 1,\ 0 < j < N/(2d_i)\}.$$

The D_i are disjoint, and together they account for all the k in the sum (3). Moreover,

$$\sum_{k\in D_i} s(k) = \sum_{\substack{j:\gcd(j,N/d_i)=1\\ 0<j<N/(2d_i)}} f\left(\cos\frac{2\pi j}{N/d_i}\right) = \mathrm{Tr}_{\mathbb{Q}(\cos\frac{2\pi}{N/d_i})} f\left(\cos\frac{2\pi}{N/d_i}\right),$$

where the last equality follows from the same reasoning used earlier for k odd (with N replaced by N/d_i). We have expressed $S(n,m)$ as a sum of traces of algebraic numbers, which means it is rational. □

This result allows us to prove that the generating function for the sequence $\{S(n,m)\}_{m>0}$ (for fixed odd $n>0$) is a rational function.

Lemma 4. *If $n>1$ is odd, $m\geq 1$, and*

$$F_n(x)=\sum_{m=0}^{\infty} S(n,m)x^m,$$

then there exist $P(x)$, $Q(x)\in\mathbb{Z}[x]$ such that $F_n(x)=P(x)/Q(x)$.

Proof. Using the formula for the sum of a geometric series, we write

$$F_n(x)=\sum_{m=0}^{\infty}\left(\sum_{k=1}^{(n-1)/2}\sec^{2m}\frac{k\pi}{n+1}\right)x^m=\sum_{k=1}^{(n-1)/2}\frac{1}{1-x\sec^2\frac{k\pi}{n+1}},$$

so that

$$Q(x)=\prod_{k=1}^{(n-1)/2}\left(1-x\sec^2\frac{k\pi}{n+1}\right).$$

We will show that $Q(x)$ is a polynomial with rational coefficients. Set

$$b_k:=\sec^2\frac{k\pi}{n+1},$$

where $1\leq k\leq (n-1)/2$. Let s_i be the sum of the products of each i-element subset of the set $\{b_1,b_2,\ldots,b_{(n-1)/2}\}$ (in other words, s_i is the i-th elementary symmetric polynomial applied to the b_i). The coefficient of x^i in $Q(x)$ is $(-1)^i s_i$. Also, let

$$p_r:=\sum_{k=1}^{n} b_k^r.$$

The Newton–Girard formulas tell us that

$$p_i-s_1p_{i-1}+s_2p_{i-2}+\cdots+(-1)^{i-1}s_{i-1}p_1+(-1)^i i s_i=0,$$

for all $1\leq i\leq (n-1)/2$. It is clear that p_i is rational for all i by Lemma 4. An easy induction argument implies that s_i is rational for all i. Since the coefficients of $Q(x)$ can be expressed in terms of the s_i, we see that $Q(x)$ has rational coefficients. Thus $P(x)=F_n(x)Q(x)$ has rational coefficients. The desired result follows. □

Lemma 5. *Suppose that a and b are algebraic numbers, and*

$$F(x)=\frac{a}{1-bx}=\sum_{n=0}^{\infty}a_nx^n.$$

If a_n is an algebraic integer for all n, then b is an algebraic integer.

Proof. The assumption implies that $a_n = ab^n$. We know that ab^n is an algebraic integer for all n, and so lies in the ring of algebraic integers of the field $K = \mathbb{Q}(b)$. This ring is finitely generated as an abelian group. Suppose that it is generated by $\{v_1, v_2, \ldots, v_l\}$. Then b^n must be in the finitely generated abelian group generated by $\{v_1/a, \ldots, v_l/a\}$ for all n. Lemma 2.8 of [ST] states that a complex number θ is an algebraic integer if and only if the additive group generated by all powers $1, \theta, \theta^2, \ldots$ is finitely generated. Thus, b is an algebraic integer. □

Now, we wish to expand upon the ideas presented in Lemma 5.

Definition. A sequence $\{a_n\}$ of algebraic numbers has a *bounded denominator* if there exists a positive integer m such that ma_n is an algebraic integer for all n.

Lemma 6. *Let*

$$F(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where $\{a_n\}$ is a sequence with bounded denominator. Suppose $p(x)$ is a polynomial whose coefficients are algebraic numbers and let

$$F(x)p(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Then, the sequence $\{b_n\}$ has bounded denominator.

Proof. This follows from the fact that the algebraic numbers form a subfield of the complex numbers and the fact that given an algebraic number a there exists a positive integer n such that na is an algebraic integer. □

Lemma 7. *Let $\zeta_{4p} = e^{2\pi i/4p}$, where p is an odd prime. Then*

$$N_{\mathbb{Q}(\zeta_{4p})}(\zeta_{4p} + \zeta_{4p}^{-1}) = p^2.$$

Proof. First note that

$$\zeta_{4p} + \zeta_{4p}^{-1} = 2\cos\frac{\pi}{2p},$$

and recall the characterization of the conjugates of $\cos 2\pi/n$ given in (2). We have

$$N_{\mathbb{Q}(\zeta_{4p})}(\zeta_{4p} + \zeta_{4p}^{-1}) = \prod_{\substack{(k,4p)=1 \\ 1\le k\le 4p}} \left(e^{\frac{2\pi ik}{4p}} + e^{\frac{-2\pi ik}{4p}}\right) = \zeta_{4p}^{-\phi(4p)2p} \prod_{\substack{(k,4p)=1 \\ 1\le k\le 4p}} \left(e^{\frac{4\pi ik}{4p}} + 1\right).$$

Now, we know that

$$N_{\mathbb{Q}(\zeta_{2p})}(\zeta_{2p} + 1) = \prod_{\substack{(k,2p)=1 \\ 1\le k\le 2p}} (e^{\frac{2\pi ik}{2p}} + 1).$$

This implies

$$\zeta_{4p}^{-\phi(4p)2p} \prod_{\substack{(k,4p)=1\\1\le k\le 4p}} \left(e^{\frac{4\pi ik}{4p}}+1\right) = N_{\mathbb{Q}(\zeta_{2p})}(\zeta_{2p}+1)^2(e^{-2\pi i}).$$

Now, the minimal polynomial of ζ_{2p} is the same as that of $-\zeta_p$. Furthermore,

$$r(x) = x^{p-1}+x^{p-2}+\cdots+x+1 = \prod_{k=1}^{p-1}(x-\zeta_p^k).$$

So, $N_{\mathbb{Q}(\zeta_p)}(1-\zeta_p) = r(1) = p$ since the minimal polynomial of ζ_p is $r(x)$. Thus,

$$N_{\mathbb{Q}(\zeta_{2p})}(\zeta_{2p}+1)^2(e^{-2\pi i}) = N_{\mathbb{Q}(\zeta_p)}(1-\zeta_p)^2 = p^2,$$

as desired. □

Lemma 8. *If, for all k satisfying $1 \le k \le (n-1)/2$, the value of* $\sec^2 \dfrac{k\pi}{n+1}$ *is an algebraic integer, then $n+1$ is a power of two.*

Proof. Assume that $n+1$ is not a power of two. Let p be an odd prime factor of $2(n+1)$. Since n is odd, $2(n+1)$ is a multiple of 4 and so $4p$ divides $2(n+1)$. Let $k = 2(n+1)/(4p)$, so $2(n+1)/k = 4p$. Then

$$\sec^2\frac{k\pi}{n+1} = \left(\frac{2}{\zeta_{2(n+1)}^{k}+\zeta_{2(n+1)}^{-k}}\right)^2 = \left(\frac{2}{\zeta_{4p}+\zeta_{4p}^{-1}}\right)^2.$$

Now, from the previous lemma, $N_{\mathbb{Q}(\zeta_{4p})}(\zeta_{4p}+\zeta_{4p}^{-1}) = p^2$. This implies

$$N_{\mathbb{Q}(\zeta_{4p})}\left(\frac{2}{\zeta_{4p}+\zeta_{4p}^{-1}}\right)^2 = \frac{2^{2\phi(4p)}}{p^4}.$$

Then, since p is an odd prime we know that $2^{2\phi(4p)}/p^4$ is not an integer. This means that with the chosen k, $\sec^2 k\pi/(n+1)$ is not an algebraic integer. This proves the desired result. □

4. Proof of the theorems

Proof of Theorem 2. This is a more general version of Lemma 5. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be all the numbers whose reciprocals are zeros of $Q(x)$. Then $F(x)$ has a partial fraction expansion whose terms are of the form

$$\frac{A_{i,l}}{(1-\alpha_i x)^l},$$

plus a polynomial part. Write

$$Q(x) = \prod_{i=1}^{n} (1 - \alpha_i x)^{k_i}.$$

Let j be the largest positive integer such that in the partial fraction decomposition of $F(x)$ the term $A_{i,j}/(1-\alpha_i x)^j$ is nonzero. Clearly $j > 0$, since $P(x)$ and $Q(x)$ are relatively prime. Now, let

$$Q_i(x) = \frac{Q(x)}{(1-\alpha_i x)^{k_i - j + 1}}.$$

The highest power of $(1-\alpha_i x)$ that divides $Q_i(x)$ is clearly $j-1$.

We have

$$F(x)Q_i(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Then, by Lemma 6, $\{b_n\}$ has a bounded denominator. Now, we will consider the effect of multiplying $F(x)$ and $Q_i(x)$ by considering what happens to each term in the partial fraction expansion of $F(x)$. With the exception of the term

$$\frac{A_{i,j}}{(1-\alpha_i x)^j},$$

$Q_i(x)$ times a term in the partial fraction expansion of $F(x)$ is a polynomial of finite degree. Now, one can see that

$$Q_i(x)\frac{A_{i,j}}{(1-\alpha_i x)^j} = \frac{Q_i(x)}{(1-\alpha_i x)^{j-1}} \frac{A_{i,j}}{(1-\alpha_i x)}.$$

It is clear that $Q_i(x)/(1-\alpha_i x)^{j-1}$ is a polynomial. Thus,

$$F(x)Q_i(x) = q(x) + \frac{D_i}{1-\alpha_i x},$$

where $q(x)$ is a polynomial and D_i is some algebraic number. So, we can say that for sufficiently large n, $b_n = D_i \alpha_i^n$ where D_i and b_n are algebraic numbers. Then, by Lemma 5, α_i is an algebraic integer. □

Proof of Theorem 1. Suppose $S(n,m)$ is an integer for all $m > 0$. By Lemma 4,

$$F_n(x) = \sum_{m=0}^{\infty}\left(\sum_{k=1}^{(n-1)/2} \sec^{2m}\frac{k\pi}{n+1}\right)x^m = \sum_{k=1}^{(n-1)/2}\left(\frac{1}{1 - x\sec(\frac{k\pi}{n+1})}\right)$$

is a rational function. Hence, $F_n(x) = P(x)/Q(x)$ where $P(x),\ Q(x) \in \mathbb{Q}[x]$. Theorem 1 now implies that $\sec^2(k\pi/(n+1))$ is an algebraic integer for all k with $1 \le k \le (n-1)/2$. According to Lemma 8, this means $n+1$ is a power of two. □

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mudrock2@illinois.edu

Mathematics Department,
University of Illinois at Urbana-Champaign,
1409 West Green Street, Urbana, IL 61801, United States

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