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Clique-relaxed graph coloring

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We define a generalization of the chromatic number of a graph G called the k-clique-relaxed chromatic number, denoted $\chi^{(k)}(G)$. We prove bounds on $\chi^{(k)}(G)$ for all graphs G, including corollaries for outerplanar and planar graphs. We also define the k-clique-relaxed game chromatic number, $\chi_g^{(k)}(G)$, of a graph G. We prove $\chi_g^{(2)}(G) \leq 4$ for all outerplanar graphs G, and give an example of an outerplanar graph H with $\chi_g^{(2)}(H) \geq 3$. Finally, we prove that if H is a member of a particular subclass of outerplanar graphs, then $\chi_g^{(2)}(H) \leq 3$.

1. Introduction

The *chromatic number* of a graph G, denoted $\chi(G)$, is the least number of colors required to color the vertices of G such that adjacent vertices receive different colors. The study of this characteristic of graphs is interesting in itself, and several extensions have also been explored. For example, the *k-relaxed chromatic number* of a graph G, denoted $\chi^k(G)$, is the least number of colors necessary to color the vertices of G such that each vertex is adjacent to at most k vertices of the same color. Note that $\chi^0(G) = \chi(G)$. This parameter has been studied in many papers, including [Cowen et al. 1986; 1997; Eaton and Hull 1999]. In this paper we introduce a relaxation to vertex coloring which forbids monochromatic $(k+1)$-cliques, where a *k-clique* is a set of k pairwise-adjacent vertices.

Another area of research branching from graph coloring is competitive graph coloring. Two players, Alice and Bob, take turns (with Alice going first) coloring uncolored vertices of a graph G with legal colors from a set X of m colors, where the definition of a legal color for a vertex varies depending on the version of the game. In the standard game [Bodlaender 1992], a color $\alpha \in X$ is *legal* for an uncolored vertex u if u has no neighbors already colored α. Alice wins this game if all vertices of G are eventually colored. Bob wins when there is an uncolored

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vertex for which no legal color exists. The least m such that Alice has a winning strategy for this game is called the *game chromatic number of* G, and is denoted $\chi_g(G)$. In the k-relaxed version of the game [Chou et al. 2003; Dunn and Kierstead 2004a; 2004b; 2004c; He et al. 2004], a color is legal for a vertex if it does not result in any vertex with more than k neighbors of the same color. Said differently, a color $\alpha \in X$ is legal for an uncolored vertex u if once u is colored α, for every $\beta \in X$, the subgraph H induced by all vertices colored β satisfies $\Delta(H) \leq k$, where $\Delta(H)$ is the maximum degree of H. Alice wins if all the vertices of G are eventually colored. Bob wins if there is at least one uncolored vertex in G with no legal color. The least m such that Alice has a winning strategy for this game is called the *k-relaxed game chromatic number of* G, denoted $\chi_g^k(G)$. We will show how competitive coloring can be integrated with the definition of a clique-relaxed coloring.

2. Clique-relaxed coloring

A coloring of a graph G is a proper k-clique-relaxed coloring if G has no monochromatic $(k+1)$-cliques. For any graph G, the *k-clique-relaxed chromatic number of* G, denoted $\chi^{(k)}(G)$, is defined as the least number of colors that can be used to color the vertices of a graph G such that if H is a subgraph induced by one of the color classes, then $\omega(H) \leq k$, where $\omega(H)$ is the size of the largest clique in H. Notice that $\chi^{(1)}(G) = \chi(G)$ for all graphs G, and more generally that for every positive integer k, we have that $\chi^{(k)}(G) \leq \chi^{k-1}(G)$. The following theorem gives an upper bound for the k-clique-relaxed chromatic number of a graph G in terms of the standard chromatic number of G.

Theorem 1. *Let G be a graph. Then $\chi^{(k)}(G) \leq \left\lceil \dfrac{\chi(G)}{k} \right\rceil$ for any positive integer k.*

Proof. Let G be a graph with $\chi(G) = m$. Then G has a proper m-coloring. Let k be a positive integer. We know that there are unique nonnegative integers q and r, $r < k$, such that $m = qk + r$. We can thus divide the m colors into q groups of size k and one of size r if $r \neq 0$. This gives $\lceil m/k \rceil = n$ groups. Let $A_1, A_2, \ldots, A_n$ be these groups of colors. Now, using the proper m-coloring, we color a vertex v with a color β_i if $c(v) \in A_i$, where $c(v)$ denotes the color of v. The colors used in this new coloring are $\beta_1, \beta_2, \ldots, \beta_n$. Thus n colors are used. Notice that the vertices of any $(k+1)$-clique in the proper m-coloring must have been colored using $k+1$ different colors, and any set of $k+1$ colors from the proper m-coloring must be in at least two groups A_i and A_j where $i \neq j$. So in the new coloring, the vertices of any $(k+1)$-clique must include at least two colors. Therefore, the new coloring is a proper k-clique-relaxed coloring with n colors. So $\chi^{(k)}(G) \leq n = \lceil \chi(G)/k \rceil$. □

Using known characteristics of outerplanar and planar graphs it is easy to apply the result in Theorem 1 to these classes of graphs. By the 2-degeneracy of outerplanar graphs, $\chi(G) \leq 3$ for all outerplanar graphs G, and by the four-color theorem [Appel and Haken 1976], $\chi(H) \leq 4$ for all planar graphs H. We then have the following corollary.

Corollary 2. *If G is an outerplanar graph, then*

$$\chi^{(2)}(G) \leq 2 \qquad \text{and} \qquad \chi^{(k)}(G) = 1 \quad \text{for } k \geq 3.$$

Similarly, if H is a planar graph, then

$$\chi^{(2)}(H) \leq 2, \qquad \chi^{(3)}(H) \leq 2, \qquad \text{and} \qquad \chi^{(k)}(H) = 1 \quad \text{for } k \geq 4.$$

Observe that K_3 is an outerplanar graph with $\chi^{(2)}(K_3) = 2$, since if every vertex in K_3 is colored α, there is a 3-clique in the subgraph induced by the color α. Similarly, K_4 is a planar graph with $\chi^{(2)}(K_4) = 2$ and $\chi^{(3)}(K_4) = 2$, Since if every vertex in K_4 is colored α, there is a 4-clique and four 3-cliques in the subgraph induced by the color α.

We note that our discussion of clique-relaxed coloring can be reframed within the context of hypergraph colorings. For a given graph G we define the hypergraph $H = (V, E)$, where $V = V(G)$ and E is the set of hyperedges induced by the $(k+1)$-cliques in G. In this way, k-clique-relaxed coloring in G is equivalent to standard hypergraph coloring in H. However, for the simplicity of our arguments, we will remain within the context of graphs rather than hypergraphs.

3. Clique-relaxed coloring game

A natural extension of this relaxed coloring number is its application to competitive graph coloring. The *k-clique-relaxed n-coloring game* on a graph G is between two players, Alice and Bob, who take turns coloring uncolored vertices of G with colors from a set X of n colors. A color $\alpha \in X$ is *legal* for an uncolored vertex u if coloring u with α does not result in a monochromatic $(k+1)$-clique. At each step the players must color an uncolored vertex with a legal color. As before with the k-relaxed coloring game, we can restate this in terms of the subgraphs induced by the color classes. A color α is legal for u if once u is colored α, for every $\beta \in X$, the subgraph H induced by the vertices of color β satisfies $\omega(H) \leq k$. Alice always colors first, and she wins the game when all the vertices are colored. Hence, Bob wins when there is at least one uncolored vertex in G with no legal color. The *k-clique-relaxed game chromatic number of G*, denoted $\chi_{\mathrm{g}}^{(k)}(G)$, is the least n such that Alice has a winning strategy in the k-clique-relaxed n-coloring game on G.

Notice that $\chi_{\mathrm{g}}^{(1)}(G) = \chi_{\mathrm{g}}(G)$ for all graphs. Also, since outerplanar graphs have maximum clique size at most three, $\chi_{\mathrm{g}}^{(k)}(G) = 1$ for all outerplanar graphs G and $k \geq 3$. Therefore, we will be concerned only with the 2-clique-relaxed game on

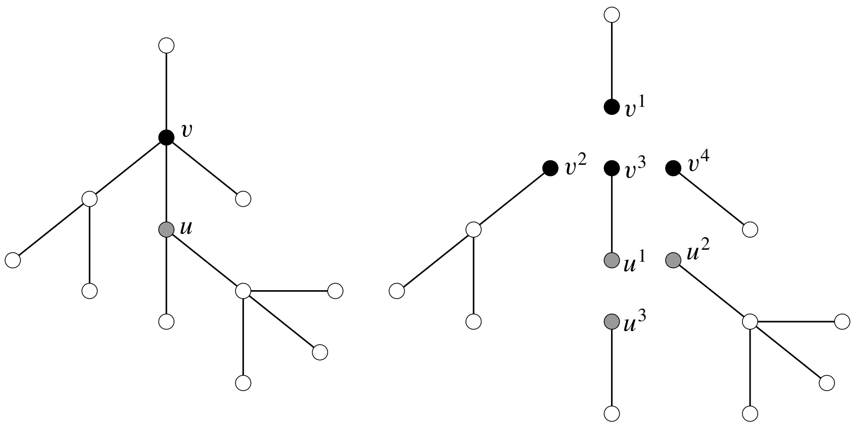

Figure 1. Alice will create trunks at the vertices v and u.

outerplanar graphs. Before proving an upper bound for the 2-clique-relaxed game chromatic number of outerplanar graphs, we will reprove Lemma 1 of [Guan and Zhu 1999] which is key to Alice's strategy.

The *separator strategy* on a tree T is defined as follows. Let $c(v)$ denote the color of a vertex v. At any point in the coloring game on T if a vertex v is colored and has degree d, Alice will imagine vertex v is replaced by d vertices, all colored $c(v)$, where each of these d vertices is incident with exactly one edge that was incident with v. We call these partially-colored subgraphs *trunks*. For example, consider the partially-colored tree on the left side of Figure 1. The vertices v and u are colored, so Alice creates trunks at these vertices as shown on the right side of the figure.

Lemma 3. *Using the separator strategy, Alice can ensure that after each of her turns each trunk has at most two colored vertices.*

Proof. Let T be a tree. It is clear that the property holds after Alice's first turn. Suppose this holds after Alice's k-th turn, and Bob colors a vertex u on a trunk. So at the end of Bob's turn there is at most one trunk with more than two colored vertices. If such a trunk exists, it is the trunk with vertex u, and this trunk has three colored vertices. If u lies on the path between the other two colored vertices, then according to Alice's view of the game, this trunk will be broken into two trunks, each with two colored vertices. Then, if possible, Alice will color on a trunk with only one colored vertex. If there are no such trunks, she can color a vertex on the distinct path between two colored vertices within a trunk with two colored vertices, separating the trunk at the vertex she just colored. If u does not lie on this path, Alice can color the unique vertex at which the paths between the three colored vertices intersect. Call this vertex v. As she is using the separator strategy, she then separates the unique trunk containing v of T at v into d trunks where $d = \deg(v)$. Now each of these d trunks has at most two colored vertices.

Suppose, instead, that Bob colors in a trunk with only one colored vertex. Then Alice plays as above in the case when Bob colored on the path between two colored vertices. Thus, in either case, the property holds after Alice's $(k+1)$-th turn. $\square$

Theorem 4. *Let G be an outerplanar graph. Then $\chi_g^{(2)}(G) \leq 4$.*

Proof. Let $G = (V, E)$ be an outerplanar graph. Alice will use a strategy for the 2-clique-relaxed 4-coloring game on G adapted from [Guan and Zhu 1999]. Alice begins by creating auxiliary graphs G' and T which she will use to determine which vertex she colors in the game on G.

To create $G' = (V', E')$, Alice adds edges to G so that G' is maximally outerplanar. Notice that $V = V'$, and $E \subseteq E'$. Guan and Zhu [1999] showed that for every maximally outerplanar graph, there is a linear ordering $L = v_1 v_2 \ldots v_n$ of the vertices of G' such that

- v_1 and v_2 are adjacent,
- $v_1 v_2$ is on the outer face of G', and
- for all $i \geq 3$, v_i is adjacent to exactly two vertices $v_{a(i)}$ and $v_{b(i)}$ such that $a(i) < i$ and $b(i) < i$.

We call $v_{a(i)}$ and $v_{b(i)}$ the *major parent* and *minor parent* of v_i, respectively, where $a(i) < b(i)$.

To create $T = (V_T, E_T)$, Alice deletes all edges of the form $v_i v_{b(i)}$. In other words, for each vertex u she deletes the edge between u and its minor parent. According to Lemma 1 of [Guan and Zhu 1999], each vertex is the minor parent of at most two vertices. Since each vertex also has at most one minor parent, every vertex in T is incident to at most three deleted edges from G'. Notice that $V_T = V' = V$ and $E_T \subseteq E'$.

We can see in T that v_1 and v_2 are still adjacent, and now for all $i \geq 3$, v_i is adjacent to exactly one vertex with a lower index, namely its major parent $v_{a(i)}$. So T is a tree. Alice will use the separator strategy on T to choose which vertex she will color. Let v be the vertex she chooses. She will look at the partially colored graph G and choose a legal color for v. We show that in the 2-clique-relaxed 4-coloring game, v will always have a legal color.

We proved in Lemma 3 that by using the separator strategy, Alice can ensure that after her turn each trunk has at most two colored vertices. After Bob's turn there may be one trunk with three colored vertices, so v is adjacent to at most three colored vertices in T. Since, as noted earlier, each vertex is incident to at most three deleted edges from G', the vertex v may be adjacent to three additional colored vertices in G'. Since $E_T \subseteq E'$, v is adjacent to at most six colored vertices

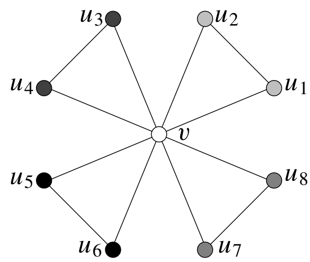

Figure 2. The vertex v is uncolorable in the 2-clique-relaxed 4-coloring game.

in G'. Also, because $E \subseteq E'$, we know that v is adjacent to at most six colored vertices in G. If v is uncolorable, then it must form a 3-clique with each of the four color classes (see Figure 2). Thus, it must be adjacent to at least eight colored vertices in G. Since v is only adjacent to six colored vertices, there is a legal color for v and Alice can win the 2-clique-relaxed 4-coloring-game on G. □

We do not yet know if the above bound is sharp. The theorem that follows gives an example of a graph G such that $\chi_g^{(2)}(G) \geq 3$. In order to prove this we show that Bob has a winning strategy in the 2-clique-relaxed 2-coloring game on G. This means that Alice would need three or more colors to have a winning strategy on G. We begin our proof with two lemmas which involve subgraphs of G.

Lemma 5. *Let H be the partially colored graph in Figure 3, where $c(v_1) = c(v_2)$, $c(v_3) \neq c(v_1)$, the vertices x, y, z, and w are uncolored, the color $c(v_1)$ is legal for both x and z, and the color $c(v_2)$ is legal for both y and w. If H is a subgraph of an outerplanar graph G at any point in the 2-clique-relaxed 2-coloring game on G, then Bob has a winning strategy.*

Proof. Assume v_1 and v_2 are colored α and v_3 is colored β. If it is Bob's turn he can color either y or w with β. Vertex z can then be colored neither α nor β, so Bob wins. Suppose instead that it is Alice's turn. If she does not color z, then either y or w is still uncolored after her turn (if not both). Suppose without loss of

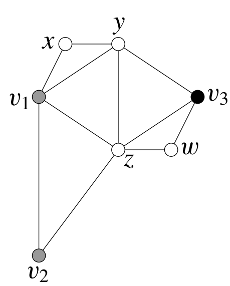

Figure 3. Bob can win the 2-clique-relaxed 2-coloring game.

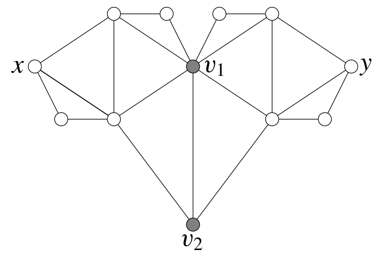

Figure 4. Bob can win the 2-clique-relaxed 2-coloring game.

generality that y is uncolored. Then Bob can color y with β leaving z uncolorable. If Alice does color z, she must color it β. Bob can then color x with α. Now y can be colored neither α nor β, so Bob has a winning strategy. □

Lemma 6. *Let H be the partially colored graph in Figure 4, where $c(v_1) = c(v_2) = \alpha$ and all other vertices in the subgraph are uncolored. Suppose Alice and Bob are playing the 2-clique-relaxed 2-coloring game on an outerplanar graph G with colors α and β. If H is a subgraph of G and β is legal for both $x$and y, then Bob has a winning strategy.*

Proof. Assume v_1 and v_2 are colored α. If it is Bob's turn he can color either x or y with β, and by Lemma 5 Bob can win. If instead it is Alice's turn, she can only play on one side of the line of symmetry. If she colors a vertex on the side with x, Bob can color y with β; if she colors a vertex on the side with y, Bob can color x with β. Either way, by Lemma 5, Bob can win. □

Theorem 7. *There exists an outerplanar graph G such that $\chi_g^{(2)}(G) \geq 3$.*

Proof. Consider the graph in Figure 5. If Alice colors v with α, then Bob can color u_1, u_2, or u_3 with α, and, by Lemma 6, Bob can win. If Alice does not color v,

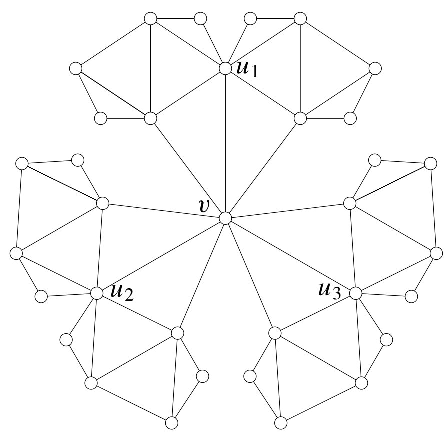

Figure 5. Bob has a winning strategy on this graph.

then Bob can color v on his first turn with α. On Bob's second turn at least one of the three identical trunks adjacent to v has no colored vertices since Alice has only played twice. Suppose without loss of generality that the part containing u_1 has no colored vertices. Bob can color u_1 with α, and by Lemma 6 he can win. $\square$

4. Family representation for outerplanar graphs

In this section, we present a representation for outerplanar graphs such that each component of the graph is rooted, and its vertices are organized into generations. Recall that a graph is outerplanar if and only if it has no $K_{2,3}$ or K_4 minor. Let G be an outerplanar graph with m components, and let $G_1, G_2, \ldots, G_m$ be the components of G.

- For each G_i choose any vertex r_i to be the root.
- Partition $V(G_i)$ into $V_0^i, V_1^i, \ldots, V_k^i$ such that
$$V_j^i = \{x \in V(G) \mid d(x, r_i) = j\},$$
where $d(x, r_i)$ is the distance between x and r_i. Define $V_j = \bigcup_{i=1}^m V_j^i$. Each V_j is the *j-th generation* of G.

Since the vertex set of any outerplanar graph can be partitioned according to the distance of a vertex from a fixed root and the edge set remains unchanged, all outerplanar graphs have a family representation.

Let $v \in V_j$ for some $j \geq 1$. Then u is a *parent* of v if $u \in V_{j-1} \cap N(v)$, where $N(v)$ is the set of neighbors of v. Likewise, u is a *child* of v if $u \in V_{j+1} \cap N(v)$. We call a vertex u a *descendant* of v if there is a shortest (nonempty) path from u to the root that includes v. We note that if the following properties of the family representation are true for each component of G, then they are true for G; thus, we may assume that G is connected.

Proposition 8. *All vertices in G have at most two parents.*

Proof. Assume that a vertex $x \in V_j$ has three parents in V_{j-1}. Note, $j \neq 1$ since V_0 has only one vertex. Call the three parents a, b, and c. Let $M = \{V_i \mid i < j-1\}$. Clearly, $G[M]$, the graph induced by M, is connected. The vertices a, b, and c each have at least one parent in M. Let $X = \{a, b, c\}$ and let $Y = \{x, G[M]\}$. These bipartite sets and the edges that connect them form a minor of $K_{2,3}$, contradicting the fact that G is outerplanar. Thus, each vertex has at most two parents. $\square$

Proposition 9. *For each $v \in V_j$, $|N(v) \cap V_j| \leq 2$.*

Proof. Assume that a vertex $x \in V_j$ has three neighbors in V_j. Call the three neighbors a, b, and c. Let $M = \{V_i \mid i < j\}$. As in the previous proof, we see that with $X = \{a, b, c\}$ and $Y = \{x, G[M]\}$, we have a $K_{2,3}$ minor, contradicting the fact that G is outerplanar. So, each vertex in the j-th generation has at most two neighbors in the j-th generation. □

5. The coloring game on certain outerplanar graphs

It is known [Guan and Zhu 1999] that for the class $\mathcal{G}$ of outerplanar graphs, that

$$6 \leq \max_{G \in \mathcal{G}} \chi_g(G) \leq 7.$$

In this section we consider a specific subclass of outerplanar graphs for which we can improve this upper bound. We consider outerplanar graphs for which there exists a family representation such that each vertex u has at most one parent $p(u)$. This means that for each vertex $v \in V(G_i)$ there is a unique shortest path from v to root r_i. We call this class $\mathcal{F}$. See Figure 6.

Alice will use an *activation strategy* to win the usual 6-coloring game and the 2-clique-relaxed 3-coloring game on graphs in $\mathcal{F}$. At any point in the game, we define U to be the set of uncolored vertices, and C to be the set of colored vertices. Alice maintains a set of active vertices, A. Any colored vertex is automatically active, and once a vertex is active it remains active. Therefore $C \subseteq A$.

Activation strategy: On Alice's first turn, she colors a vertex in V_0. Suppose Bob colors vertex $v \in V(G_i)$.

(1) Search stage:

- If v is not a root and $p(v)$ is uncolored, Alice begins activating vertices along the shortest path from v to root r_i. As she does this, there are four possible cases for each vertex x she reaches.
  - If x is active and uncolored, she lets $u = x$ and moves to the coloring stage.
  - If x is inactive and is the root r_i, she activates x, chooses $u = x$, and moves to the coloring stage.

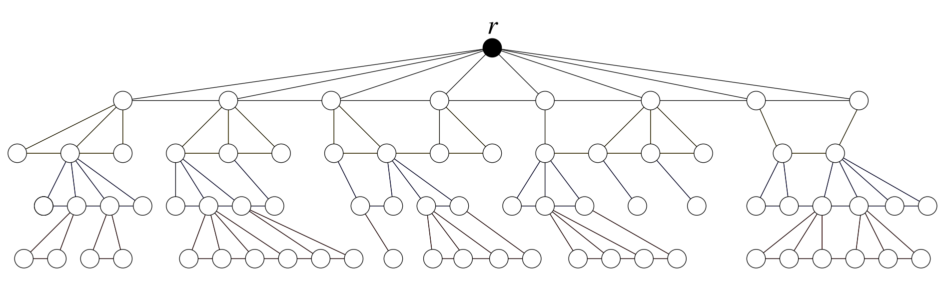

Figure 6. An example of a graph in $\mathcal{F}$.

   - If x is inactive and $p(x)$ is colored, she activates x, chooses $u = x$, and moves to the coloring stage.
   - If x is inactive and $p(x)$ is uncolored, she activates x and continues up the path.
 - If v is a root or $p(v)$ is colored, Alice chooses an arbitrary uncolored vertex $u \in V_j$, where j is the least index such that V_j has an uncolored vertex, and moves to the coloring stage.

(2) Coloring stage:
 - On each turn, Alice chooses a legal color for u.

We now prove an important lemma which will help bound the parameters of interest.

Lemma 10. *If Alice uses the activation strategy, at any point in the game any uncolored vertex u has at most two active children.*

Proof. Consider the case where u has no active children. The strategy ensures that Alice will not color a descendant of any inactive vertex. Thus, if Alice activates a child of u, it must be the direct result of Bob coloring a descendant of u. When Alice activates a child of u, she activates u as well. Now consider Alice activating a second child of u. Again, this must be a result of Bob coloring a descendant of u by the argument above. After Alice activates the second child of u, she will take action at u. Since u is active, Alice colors u. Therefore, an uncolored vertex u has at most two active children. □

Theorem 11. *For all graphs G in $\mathscr{F}$, $\chi_{\mathrm{g}}(G) \leq 6$.*

Proof. Consider an uncolored vertex u. Note that u has at most one parent $p(u)$, and by Proposition 9, u has at most two adjacent siblings, say u^* and u'. See Figure 7. It is easy to see that if Alice uses the activation strategy, it may be the case that $p(u)$, u^*, and u' are all colored with u remaining uncolored. Since u has at most two active children, it has at most two colored children. Therefore, u has at most five colored neighbors. This means that Alice needs at most six colors available to win the original game on graphs in $\mathscr{F}$. □

Now we prove a similar result for the 2-clique-relaxed game. Recall that in Theorem 4 and Theorem 7 we showed that

$$3 \leq \max_{G \in \mathscr{G}} \chi_{\mathrm{g}}^{(2)}(G) \leq 4$$

for the class of outerplanar graphs $\mathscr{G}$. With the following result, we provide an improved upper bound for the class $\mathscr{F}$.

Corollary 12. *For all graphs G in $\mathscr{F}$, $\chi_{\mathrm{g}}^{(2)}(G) \leq 3$.*

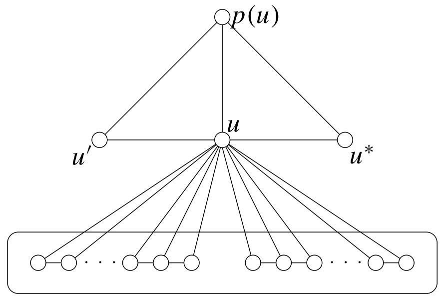

Figure 7. An uncolored vertex u and its neighbors.

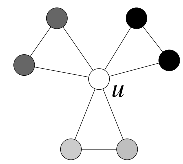

Figure 8. The vertex u is uncolorable in the 2-clique-relaxed 3-coloring game.

Proof. Suppose Alice and Bob are playing the 2-clique-relaxed coloring game on a graph in $\mathcal{F}$ with three colors. For Bob to win the game, he requires an uncolored vertex u, with neighbors as in Figure 8. This would require that six of the neighbors of u be colored while u remains uncolored. By the proof for Theorem 11, u has at most five colored neighbors. Hence, three colors are sufficient for Alice to win the 2-clique relaxed game on any graph in $\mathcal{F}$. □

6. Future work

At present, we do not know whether the bounds in Theorem 11 and Corollary 12 are tight. In the case of the latter, it is clear that the graph in Theorem 7 is not in $\mathcal{F}$. Showing this bound is tight would require providing an example of a graph in $\mathcal{F}$ such that Bob has a winning strategy with 2 colors. However, it may be the case that Alice has a winning strategy with 2 colors. We are certain that the strategy we have provided will not suffice; however, it is possible that a modification could yield an upper bound of 2.

We now have an upper bound, $\chi_g^{(2)}(G) \leq 4$, for outerplanar graphs G and an example of an outerplanar graph such that $\chi_g^{(2)}(G) \geq 3$. The next question is whether there exists an outerplanar graph G such that $\chi_g^{(2)}(G) = 4$. If there is, then such an example must lie outside of the subclass $\mathcal{F}$ of outerplanar graphs. In particular, Proposition 8 guarantees that such an example would require a vertex with two parents.

Another area for further investigation is the clique-relaxed game chromatic number of planar graphs. All planar graphs have maximum clique size at most four. For this reason, with a k-clique relaxation, where $k \geq 4$, planar graphs can always be completely colored with one color. The games of interest are then the 2- and 3-clique-relaxed games on planar graphs.

More broadly, as we noted earlier in Section 2, much of this work can be reframed in terms of hypergraph coloring. We have presented competitive coloring results for a specific class of hypergraphs. This could lead to more questions in the area of competitive hypergraph coloring.

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cdunn@linfield.edu *Department of Mathematics, Linfield College, 900 SE Baker Street, Unit A468, McMinnville, OR 97128, United States*

jfirkins@linfield.edu *Department of Mathematics, Linfield College, 900 SE Baker Street, Unit A468, McMinnville, OR 97128, United States*

cnaymie@gmail.com *University of Waterloo, 200 University Avenue West, Waterloo, Ontario N2L 3G1, Canada*

epitney@mac.com *Meadow Park Middle School, Beaverton School District, 16550 SW Merlo Road, Beaverton, OR 97006, United States*

william.sehorn@gmail.com *Whitworth University, 300 W. Hawthorne Road, Spokane, WA 99251, United States*

suerchaj@gmail.com *Department of Mathematics, University of Louisville, Louisville, KY 40292, United States*

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