involve
a journal of mathematics

Cost-conscious voters in referendum elections

Kyle Golenbiewski, Jonathan K. Hodge and Lisa Moats

mathematical sciences publishers

2011 vol. 4, no. 2

INVOLVE 4:2(2011)

Cost-conscious voters in referendum elections

Kyle Golenbiewski, Jonathan K. Hodge and Lisa Moats

(Communicated by Kenneth S. Berenhaut)

In referendum elections, voters are frequently required to register simultaneous yes/no votes on multiple proposals. The separability problem occurs when a voter's preferred outcome on a proposal or set of proposals depends on the known or predicted outcomes of other proposals in the election. Here we investigate cost-consciousness as a potential cause of nonseparability. We develop a mathematical model of cost-consciousness, and we show that this model induces nonseparable preferences in all but the most extreme cases. We show that when outcome costs are distinct, cost-conscious electorates always exhibit both a weak Condorcet winner and a weak Condorcet loser. Finally, we show that preferences consistent with our model of cost-consciousness are rare in randomly generated electorates. We then discuss the implications of our work and suggest directions for further research.

1. Introduction

In referendum elections, voters are often required to register simultaneous yes/no votes on multiple proposals. Recent research demonstrates that the outcomes of such elections can be unsatisfactory or even paradoxical. For example, Lacy and Niou show that the winning outcome can be the last choice of every voter; they argue that this and other troublesome behavior occurs because "referendum elections as currently practiced force people to separate their votes on issues that may be linked in their minds" [Lacy and Niou 2000, page 6].

The phenomenon to which Lacy and Niou allude is known as the *separability problem* [Brams et al. 1997]. What they and others have observed is that voter preferences often contain interdependencies that cannot be expressed through the

MSC2010: 91B12.

Keywords: referendum elections, cost-conscious, separability, separable preferences.

This research was supported by the National Science Foundation under grant no. DMS-0451254, which funds a Research Experiences for Undergraduates (REU) program at Grand Valley State University. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation (NSF).

standard simultaneous method of voting in a referendum. In other words, a voter's preferences on a proposal or a set of proposals may depend on the outcome of another proposal or a set of remaining proposals. Preferences that exhibit this kind of interdependence are said to be *nonseparable*.

Separability has been studied in a variety of contexts, with much of the most recent research focusing on the *structure* and *effects* of separable and nonseparable preferences. Here we take a different approach by investigating one of the underlying *causes* of nonseparability — namely, *cost-consciousness* within the electorate.

To illustrate, consider an election with multiple bond proposals, all competing for funds from the same tax base. In such an election, a voter who is cost-conscious — that is, who desires to limit the total expenditure of public funds — may vote no on a proposal that she supports in principle if she suspects that other proposals are more likely to pass. In doing so, the voter is acting based on predictions about the potential outcomes of these other proposals. If her predictions are wrong, then her voting strategy may also be wrong, or at least less than optimal. In other words, the voter's cost-consciousness complicates the decisions she must make about how to vote on each of the individual proposals. As we will see, these complications can have disastrous effects on the desirability of election outcomes.

Our goal in this paper is to formalize and investigate the consequences of cost-consciousness in referendum elections. Section 2 introduces a model of cost-conscious voter preferences, which we use to show how cost-consciousness induces nonseparability in voter preferences in Section 3. Section 4 demonstrates the existence of Condorcet winning and losing outcomes in certain cost-conscious electorates. Section 5 generalizes the original model by allowing voters to approve of outcomes that exceed their ideal maximum cost, provided that certain conditions are met. Section 6 explores the relative prevalence of cost-conscious voter preferences in randomly generated electorates. Finally, Section 7 summarizes our results and their implications.

2. Model for cost-conscious voters

For the purposes of our investigations, we assume the context of a referendum election on a set Q of $n \geq 2$ questions or proposals. Each potential outcome is represented by an ordered n-tuple of zeros and ones, with 1 typically representing passage of a proposal and 0 representing failure. We let X be the set of all 2^n possible election outcomes. For each $q \in Q$, we let $C(q)$ denote the cost of passing question q, where $C(q) \in \mathbb{R}^+$. The total cost incurred by an election outcome $x \in X$ is then given by

$$C(x) = \sum_{q=1}^{n} x_q C(q),$$

where $x_q = 1$ if question q passes in outcome x, and $x_q = 0$ if question q fails to pass in outcome x. For any subset S of Q, we let $C(S)$ denote the cost of passing all proposals in S; that is,

$$C(S) = \sum_{q \in S} C(q).$$

In general, we assume that each voter's preferences can be represented by a total order on X. This assumption simplifies our analysis and is consistent with prior research on the separability problem in referendum elections. We define a *cost-conscious voter* v to be one who, in principle, supports all of the proposals in Q, but in practice, wishes to limit total spending to some fixed amount M_v.

Definition 2.1. Let v be a voter whose preferences are represented by a total order $\succ$ on X. Then v is said to be *cost-conscious* if there exists some $M_v > 0$ (called the *cost ceiling* for v) such that for each $x, y \in X$, the following axioms hold:

Axiom 1. If $C(x), C(y) \leq M_v$ and $C(y) > C(x)$, then $y \succ x$.

Axiom 2. If $C(x) < C(y)$ and $C(y) > M_v$, then $x \succ y$.

Inherent in Definition 2.1 is the assumption that each voter derives a benefit from each passed proposal that is directly proportional to its cost. In fact, we assume that, for outcomes whose total cost is less than or equal to M_v, the total benefit outweighs the total cost, giving a nonnegative net utility. Furthermore, the utility of each outcome is an increasing function of its cost, provided that the cost does not exceed M_v. Outcomes whose costs exceed M_v have negative net utility, with the net utility decreasing as the cost increases further beyond M_v.

The sudden switch from positive to negative net utility creates a discontinuity in the utility function of each voter at M_v. This discontinuity is reasonable, since M_v marks a cost threshold beyond which outcomes can be thought of as being substantially less attractive, impractical, or even completely unacceptable. For instance, a consumer who has access to $40,000 of credit may attempt to purchase a new car that has as many options as possible, provided that the total cost remains at or below $40,000. Once the $40,000 threshold is exceeded, the consumer may have to go to great lengths in order to purchase the vehicle, if it is even possible for her to do so. In terms of negotiation theory, the $40,000 threshold can be viewed as a *resistance point* — that is, a point beyond which the negotiator would rather do nothing than incur further cost. We postulate that voters can have resistance points for a variety of reasons, both practical and psychological. For instance, a voter may simply be disinclined to approve any package of bond proposals whose total cost exceeds $1 million.

In our initial investigations, we assume that cost ceilings are absolute. That is, they cannot be exceeded without penalty for any reason. In Section 5, we relax this condition somewhat by allowing voters to exceed their cost ceilings when certain conditions are met.

To illustrate Definition 2.1, suppose

$$\begin{aligned} |Q| &= 3, & C(1) &= 200, \\ C(2) &= 400, & C(3) &= 500. \end{aligned}$$

Furthermore, suppose $M_v = 800$ for some voter v. We note that of the eight possible outcomes, only two have a total cost exceeding M_v— namely,

$$C(1)+C(2)+C(3) = 1100 \quad \text{and} \quad C(2)+C(3) = 900.$$

Thus, Axioms 1 and 2 induce the following ordering on the set of all possible outcomes: $101 \succ 110 \succ 001 \succ 010 \succ 100 \succ 000 \succ 011 \succ 111$. This ordering can also be represented by a preference matrix P_v, as shown below. (For a more detailed treatment of preference matrices, see [Bradley et al. 2005].)

$$P_v = \begin{pmatrix} 1&0&1\\ 1&1&0\\ 0&0&1\\ 0&1&0\\ 1&0&0\\ 0&0&0\\ 0&1&1\\ 1&1&1 \end{pmatrix}$$

Now, suppose v becomes more cost-conscious, decreasing M_v to 600. In this case, the outcome 101, which has a cost of 700, is no longer the voter's most preferred outcome. In fact, it becomes the voter's third to last choice. The new induced order is $110 \succ 001 \succ 010 \succ 100 \succ 000 \succ 101 \succ 011 \succ 111$, which corresponds to the preference matrix

$$P'_v = \begin{pmatrix} 1&1&0\\ 0&0&1\\ 0&1&0\\ 1&0&0\\ 0&0&0\\ 1&0&1\\ 0&1&1\\ 1&1&1 \end{pmatrix}.$$

Note that both P_v and P'_v are uniquely determined by Axioms 1 and 2, once v's cost ceiling and the proposal costs are specified. In particular, the axioms require each outcome whose cost exceeds M_v to be ranked lower than each outcome whose cost does not exceed M_v. Axiom 1 requires the outcomes whose costs do not exceed M_v to be ranked in descending order with respect to cost, whereas Axiom 2 requires the outcomes whose costs do exceed M_v to be ranked in ascending order with respect to cost. As long as no two outcomes have the same cost, these requirements are enough to induce a unique ordering on X.

Theorem 2.2. *Let v be a cost-conscious voter with cost ceiling M_v, and suppose $C(x) \neq C(y)$ for all distinct $x, y \in X$. Then there is exactly one total order on X that is consistent with Axioms* 1 and 2.

Note that Axioms 1 and 2 impose no restrictions on the ordering of outcomes whose costs are equal. As such, the requirement that no two outcomes have the same cost is essential to Theorem 2.2. To illustrate, consider the case in which $|Q| = 3$, $C(1) = 200$, $C(2) = 300$, and $C(3) = 500$. Since $C(001) = C(110)$, both $001 \succ 110$ or $110 \succ 001$ are permissible by Axioms 1 and 2, regardless of the value of M_v. Thus the conclusion of Theorem 2.2 fails to hold in this case.

3. Cost-consciousness and separability

In Section 1, we suggested that cost-consciousness is a cause of interdependence, or *nonseparability*, within voter preferences. In order to explore this assertion more, we must first define more what it means for a voter's preferences to be separable. Although a more formal treatment of separability can be found in a variety of sources (see, e.g., [Bradley et al. 2005]), the following informal definition will be sufficient for our purposes.

Definition 3.1. Let S be a proper, nonempty subset of Q, and let v be any voter. Then S is said to be *separable* with respect to v if v's preferences over the outcomes of questions within S do not depend on the known or predicted outcomes of questions outside of S.

To illustrate this definition, consider again the preference matrix P_v (Section 2):

$$P_v = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Because $101 \succ 001$, we see that when the outcome on questions 2 and 3 is 01, v prefers 1 to 0 (passage to failure) on question 1. However, if the outcome on questions 2 and 3 is 11, then v prefers 0 to 1 (failure to passage) on question 1 (since $011 \succ 111$). In other words, voter v's preference on question 1 depends on the outcomes of questions 2 and 3. Because of this, we say that the set $\{1\}$ is nonseparable with respect to v. Note that, from a cost-consciousness standpoint, the nonseparability of $\{1\}$ with respect to v stems from the fact that v wants question 1 to pass if and only if the cost of the other passed proposals in the election is less than or equal to 600.

In contrast, note that regardless of whether question 1 passes or not, voter v always ranks the outcomes of questions 2 and 3 in the same order:

$$01 \succ 10 \succ 00 \succ 11.$$

This is because, for each of these outcomes, the additional passage or failure of question 1 has no bearing on whether the overall cost exceeds v's cost ceiling of 800. Thus for outcomes on $\{2, 3\}$ that cost less than 800 (01, 10, and 00), the more costly outcomes are preferred (by Axiom 1), regardless of whether question 1 passes or not. All of these outcomes are preferred to 11, which always yields a total cost of more than 800 — either with or without the passage of question 1. Because v's ordering of the outcomes on $\{2, 3\}$ does not depend on the outcome of question 1, we say that the set $\{2, 3\}$ is *separable* with respect to v.

The observations from the previous example generalize easily to the following theorem, whose proof is straightforward and thus omitted.

Theorem 3.2. *Let S be a nonempty, proper subset of Q.*

(i) *If $C(Q) > M_v$, then S is separable only if $C(S) > M_v$.*

(ii) *If $C(Q) \leq M_v$, then S is always separable.*

Theorem 3.2 guarantees that the preferences of cost-conscious voters will exhibit some degree of nonseparability, except in two extreme cases. The first is when each proposal, by itself, is more expensive than the voter's cost ceiling. In this case, the voter always prefers failure to passage. The second is when the total cost of all proposals is less than or equal to the voter's cost ceiling. In this case, cost-consciousness is a moot point, and the voter always prefers passage to failure. In every other case, the preferences of cost-conscious voters will exhibit at least some nontrivial interdependencies. The fact that these interdependencies can cause serious problems is illustrated by the following example.

Example 3.3. Consider again an election with $|Q| = 3$, $C(1) = 200$, $C(2) = 400$, and $C(3) = 500$. Suppose that the electorate is comprised of three voters, v_1, v_2, and v_3, for whom $M_{v_1} = 1000$, $M_{v_2} = 800$, and $M_{v_3} = 600$. Then Axioms 1 and 2 uniquely determine the voters' preferences, as follows:

$$P_{v_1} = \begin{pmatrix} 0&1&1\\1&0&1\\1&1&0\\0&0&1\\0&1&0\\1&0&0\\0&0&0\\1&1&1 \end{pmatrix}, \quad P_{v_2} = \begin{pmatrix} 1&0&1\\1&1&0\\0&0&1\\0&1&0\\1&0&0\\0&0&0\\0&1&1\\1&1&1 \end{pmatrix}, \quad P_{v_3} = \begin{pmatrix} 1&1&0\\0&0&1\\0&1&0\\1&0&0\\0&0&0\\1&0&1\\0&1&1\\1&1&1 \end{pmatrix}.$$

With these preferences, each question passes with two "yes" votes and one "no" vote. However, that this outcome (111) is the least preferred choice of every voter. This kind of paradoxical behavior was first observed by Lacy and Niou [2000], but here we have shown it to result from a set of realistic voter preferences — in particular, those consistent with a reasonable model of cost-consciousness.

4. Condorcet winners and losers

In Example 3.3, we saw how a collection of cost-conscious voters could inadvertently elect the worst possible outcome for each voter. It is interesting to note that, in that example, the outcome 101 is a Condorcet winner. The fact that such an outcome exists is not coincidental. In fact, the next theorem establishes that when outcome costs are distinct (as in Theorem 2.2), the assumption of cost-consciousness guarantees the existence of at least a weak Condorcet winner, which we define as follows:

Definition 4.1. Let V be a nonempty collection of voters, and for each $v \in V$, let $\succ_v$ denote a total order on X. An outcome $w \in X$ is said to be a *weak Condorcet winner* (with respect to V) provided that for each $y \in X$ with $y \neq w$,

$$|\{v \in V : w \succ_v y\}| \geq |\{v \in V : y \succ_v w\}|.$$

Theorem 4.2. *Suppose $C(x) \neq C(y)$ for all distinct $x, y \in X$, and let V be any nonempty collection of cost-conscious voters. Then X contains a weak Condorcet winner with respect to V.*

Proof. Let $|Q| = n$. Then X contains 2^n distinct outcomes, which we denote by $x_1, x_2, \ldots, x_{2^n}$. Without loss of generality, assume that

$$C(x_{2^n}) > C(x_{2^n-1}) > \cdots > C(x_2) > C(x_1).$$

Then $x_1 = 00\cdots0$ and $x_{2^n} = 11\cdots1$. We claim that there are 2^n possible preference matrices consistent with Axioms 1 and 2, each determined by the size of M_v in

$$
\begin{array}{ccccccc}
\begin{pmatrix} 11\cdots1 \\ x_{2^n-1} \\ x_{2^n-2} \\ x_{2^n-3} \\ \vdots \\ x_4 \\ x_3 \\ x_2 \\ 00\cdots0 \end{pmatrix} &
\begin{pmatrix} x_{2^n-1} \\ x_{2^n-2} \\ x_{2^n-3} \\ x_{2^n-4} \\ \vdots \\ x_3 \\ x_2 \\ 00\cdots0 \\ 11\cdots1 \end{pmatrix} &
\begin{pmatrix} x_{2^n-2} \\ x_{2^n-3} \\ x_{2^n-4} \\ x_{2^n-5} \\ \vdots \\ x_2 \\ 00\cdots0 \\ x_{2^n-1} \\ 11\cdots1 \end{pmatrix} &
\cdots &
\begin{pmatrix} x_{2^n-i+1} \\ x_{2^n-i} \\ x_{2^n-i-1} \\ \vdots \\ 00\cdots0 \\ x_{2^n-i+2} \\ x_{2^n-i+3} \\ \vdots \\ 11\cdots1 \end{pmatrix} &
\cdots &
\begin{pmatrix} 00\cdots0 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{2^n-3} \\ x_{2^n-2} \\ x_{2^n-1} \\ 11\cdots1 \end{pmatrix} \\
P_1 & P_2 & P_3 & & P_i & & P_{2^n}
\end{array}
$$

Table 1. All possible preference matrices for cost-conscious voters, assuming distinct outcome costs.

comparison to the cost of the outcomes in X (see Table 1). In particular, if v is a voter with preference matrix P_v and cost-ceiling M_v, then

$$
\begin{aligned}
P_v &= P_1 \quad \text{if } M_v \geq C(x_{2^n}), \\
P_v &= P_2 \quad \text{if } C(x_{2^n}) > M_v \geq C(x_{2^n-1}),
\end{aligned}
$$

and in general,

$$P_v = P_i \qquad \text{if } C(x_{2^n-i+2}) > M_v \geq C(x_{2^n-i+1}).$$

Let $|V| = m$, and let m_j denote the number of voters in V with preference matrix P_j. Now suppose that, for some i,

$$\frac{1}{m}\sum_{j=1}^{i-1} m_j < 0.5 \quad \text{and} \quad \frac{1}{m}\sum_{j=1}^{i} m_j \geq 0.5.$$

Then, for each $k = (i+1), (i+2), \ldots, 2^n$, the outcome x_{2^n-i+1} is ranked higher than the outcome x_{2^n-k+1} by at least 50% of voters in V. Also, for each $k = 1, 2, \ldots, (i-2), (i-1)$, the outcome x_{2^n-i+1} is ranked lower than the outcome x_{2^n-k+1} by less than 50% of the voters in V. Since there must be a smallest i for which $(1/m)\sum_{j=1}^{i} m_j \geq 0.5$, the corresponding outcome x_{2^n-i+1} is a weak Condorcet winner with respect to V. □

To illustrate that Theorem 4.2 can fail when two outcomes in X have the same cost, consider the following example:

Example 4.3. Suppose that in an election with three proposals and three voters, $C(1) = C(2) = C(3) = 400$, and $M_v = 500$ for each v. In this case, each voter's preference matrix could be one of 36 distinct options. Suppose that the voters' preference matrices are as follows:

$$P_{v_1} = \begin{pmatrix} 0&0&1\\0&1&0\\1&0&0\\0&0&0\\0&1&1\\1&0&1\\1&1&0\\1&1&1 \end{pmatrix}, \quad P_{v_2} = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1\\0&0&0\\1&1&0\\0&1&1\\1&0&1\\1&1&1 \end{pmatrix}, \quad P_{v_3} = \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0\\0&0&0\\1&0&1\\1&1&0\\0&1&1\\1&1&1 \end{pmatrix}.$$

Since the outcomes 001, 010, and 100 comprise the top three choices for each voter, any Condorcet winner for this electorate must be one of these three outcomes. However, since the societal preference among these outcomes is cyclic (100 defeats 001, which defeats 010, which defeats 100), there can be no Condorcet winner.

Just as a weak Condorcet winner is guaranteed to exist when outcome costs are distinct, a weak Condorcet loser (defined analogously to Definition 4.1) can also be found in these circumstances.

Theorem 4.4. *Suppose $C(x) \neq C(y)$ for all distinct $x, y \in X$, and let V be any nonempty collection of cost-conscious voters. Then X contains a weak Condorcet loser with respect to V. Furthermore, this weak Condorcet loser is always either $00\cdots0$ or $11\cdots1$.*

Proof. By the same argument as in the proof of Theorem 4.2, each voter's preferences can be represented by one of the 2^n matrices in Table 1. The preference matrix P_1 is the only preference matrix that has the outcome $00\cdots0$ ranked as the least preferred outcome. Every other preference matrix has the outcome $11\cdots1$ ranked as the least preferred outcome. Consider three cases:

Case 1: Less than 50% of voters in V have preference matrix P_1. In this case, more than 50% of voters have preference matrices P_2 through P_{2^n}. Since $11\cdots1$ is the least preferred outcome in P_2 through P_{2^n}, $11\cdots1$ is ranked as the lowest outcome by more than 50% of the voters in V. Thus $11\cdots1$ is a Condorcet loser.

Case 2: Exactly 50% of the voters in V have preference matrix P_1. Then exactly 50% of the voters in V have preference matrices P_2 through P_{2^n}. Since $00\cdots0$ is the least preferred outcome in P_1 and $11\cdots1$ is the least preferred outcome in P_2 through P_{2^n}, $00\cdots0$ is ranked lower than every other outcome by 50% of the voters and $11\cdots1$ is ranked lower than every other outcome by 50% of voters. Thus, both $00\cdots0$ and $11\cdots1$ are weak Condorcet losers.

Case 3: More than 50% of the voters in V have preference matrix P_1. Then, since $00\cdots0$ is the least preferred outcome in P_1, $00\cdots0$ is ranked as the lowest outcome by more than 50% of voters in V. Thus, $00\cdots0$ is a Condorcet loser.

In each case, either $00\cdots0$ or $11\cdots1$ is a weak Condorcet loser, as desired. $\square$

It is worth noting that the proof of Theorem 4.4 depends only on the placement of the outcomes $00\cdots0$ and $11\cdots1$ within the matrices $P_1, P_2, \ldots, P_{2^n}$, and not on the relative rankings of other outcomes. Since $00\cdots0$ and $11\cdots1$ will always be the unique least expensive and most expensive outcomes, respectively, the proof would still be valid even without the assumption of distinct outcome costs. Thus, the conclusion of Theorem 4.4 holds even when some of these costs are equal.

5. Weak cost-consciousness

Up to this point, we have assumed that cost-conscious voters are universally resistant to exceeding their cost ceilings. That is, outcomes whose costs exceed M_v are necessarily less preferred than those whose costs do not exceed M_v.

There may, however, be circumstances in which a voter can gain a significant additional benefit by exceeding his or her cost ceiling by a small amount. In this section, we modify our original model of cost-consciousness to allow for such deviations. Our modifications assume that voters are willing to exceed their cost ceiling only when (i) the excess is bounded within a specified tolerance; and (ii) all other options for increasing the voter's total benefit also cause the voter's cost ceiling to be exceeded.

To formulate these conditions more precisely, we must first introduce some new terminology. First, for any outcome $x \in X$, we define the *support set* of x, denoted $S(x)$ to be the set of all questions passed in x. That is,

$$S(x) = \{q \in Q : x_q = 1\}.$$

For all $x, y \in X$, if $S(x) \subset S(y)$, we say that y *augments* x. If $|S(x)| = 1$, then x is said to be a *singleton*. An outcome x is said to be *cost-maximal* if $C(x) \leq M_v$ and there does not exist an outcome $y \in X$ such that y augments x and $C(y) \leq M_v$.

Definition 5.1. Let v be a voter whose preferences are represented by a total order $\succ$ on X. Then v is said to be *weakly cost-conscious* if there exists some $M_v > 0$ (called the *cost ceiling* for v) and some nonnegative $\tau \leq M_v$ (called the *tolerance* for v) such that for each $x, y \in X$, the following axioms hold:

Axiom 1. If $C(x), C(y) \leq M_v$ and $C(y) > C(x)$, then $y \succ x$.

Axiom 2′. If $C(x) < C(y)$ and $C(y) > M_v + \tau$, then $x \succ y$.

Axiom 3. If x is cost-maximal, y augments x, and $M_v < C(y) \leq M_v + \tau$, then $y \succ x$.

Note that when $\tau = 0$, Definition 5.1 is equivalent to Definition 2.1. The next example illustrates the effect of allowing τ to be nonzero.

Example 5.2. Consider an election with three proposals in which $C(1) = 200$, $C(2) = 400$, and $C(3) = 501$. Suppose also that for some voter v, $M_v = 700$ and

$\tau = 0$. Then Theorem 2.2 (which applies since $\tau = 0$ and all outcome costs are unique) guarantees a unique preference matrix consistent with Axioms 1 and 2. In this case, the matrix is

$$P_v = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Note that the outcome 101, with a cost of 701, is the voter's third least preferred outcome. Note, however, that 101 augments three other outcomes: 000, 100, and 001. Of these three outcomes, only the latter is cost-maximal. Thus, if $\tau = 1$, then Axiom 3 requires $101 \succ 001$. This leaves two possibilities for v's now weakly cost-conscious preferences:

$$P_v = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{or} \quad P_v = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Note that the first matrix can be obtained by simply increasing M_v to 701, keeping τ fixed at 0. However, the second matrix cannot be obtained in this way and is in fact inconsistent with our original definition of cost-consciousness. This contrast demonstrates that the flexibility afforded by allowing τ to be nonzero cannot be accomplished by simply increasing M_v.

6. Prevalence of cost-conscious voters

As we showed in Section 3, cost-consciousness can be a significant cause of nonseparability in voter preferences over multiple issues. Hodge and TerHaar [2008] have also shown that the vast majority of randomly selected preference matrices correspond to completely nonseparable preferences — that is, preferences for which every nonempty, proper subset of Q is nonseparable. In light of these observations, it is natural to consider how prevalent cost-conscious preferences are among all possible preference orders. In this section, we will show that the proportion of total orders on X that are consistent with the axioms of weak cost-consciousness approaches 0 asymptotically. In particular, we will prove the following theorem:

Theorem 6.1. *Let Ω_n denote the set of all total orders on X that are consistent with Axioms 1, $2'$, and 3. Then*

$$\lim_{n\to\infty} \frac{|\Omega_n|}{2^n!} = 0.$$

To prove Theorem 6.1 we establish several lemmas, each of which assumes that $\succ$ represents the preferences of a weakly cost-conscious voter. Lemmas 6.2 and 6.3 follow immediately from Axioms 1 and $2'$, respectively.

Lemma 6.2. *If $00\cdots0 \succ x$ for some $x \in X$, then $C(x) > M_v$.*

Lemma 6.3. *If $x \succ 00\cdots0$ for some $x \in X$, then $C(x) \le M_v + \tau$.*

Lemma 6.4. *Let $x, y \in X$ with $S(x) \cap S(y) = \varnothing$. If $x \succ 11\cdots1 \succ 00\cdots0$, then $11\cdots1 \succ y \succ 00\cdots0$.*

Proof. By assumption, there is an outcome $x \in X$ such that $x \succ 11\cdots1$. Consequently, Axiom 1 implies that $C(11\cdots1) > M_v$. Since $11\cdots1 \succ 00\cdots0$, Lemma 6.3 implies that $C(11\cdots1) \le M_v + \tau$. Since $\tau \le M_v$, it follows that

$$M_v < C(11\cdots1) \le M_v + \tau \le 2M_v.$$

Since $S(x) \cap S(y) = \varnothing$, we know that $C(x) + C(y) \le C(11\cdots1) \le 2M_v$. Thus, either $C(x) \le M_v$ or $C(y) \le M_v$.

Suppose $C(x) \le M_v$. Then either x is cost-maximal or there exists a cost-maximal outcome that augments x. To account for either of these cases, let x' denote a cost-maximal element that is either equal to x or augments x. Note that since $C(11\cdots1) > M_v$, $x' \ne 11\cdots1$. Thus, $11\cdots1$ augments x', which implies by Axiom 3 that $11\cdots1 \succ x'$. But since $C(x) \le C(x') \le M_v$, Axiom 1 implies that $x' \succeq x$. So $11\cdots1 \succ x' \succeq x$, a contradiction.

Since it cannot be the case that $C(x) \le M_v$, it must be that $C(y) \le M_v$. But then an argument similar to that in the preceding paragraph establishes that $11\cdots1 \succ y$. Since $C(y) \le M_v$, we know also that $y \succ 00\cdots0$ (by Axiom 1. Thus,

$$11\cdots1 \succ y \succ 00\cdots0,$$

as desired. □

Lemma 6.5. *If $00\cdots0 \succ 11\cdots1$, then $C(11\cdots1) > M_v + \tau$.*

Proof. Assume, to the contrary, that $00\cdots0 \succ 11\cdots1$ and $C(11\cdots1) \le M_v + \tau$. By Lemma 6.2, $M_v < C(11\cdots1)$. Thus, $M_v < C(11\cdots1) \le M_v + \tau$. Since $\tau \le M_v$, there exists a cost-maximal $x \in X$ such that $0 < C(x) \le M_v$. Since

$11\cdots 1$ augments x, it follows by Axioms 1 and 3 that $11\cdots 1 \succ x \succ 00\cdots 0$, a contradiction to the assumption that $00\cdots 0 \succ 11\cdots 1$. $\square$

Lemma 6.6. *If* $00\cdots 0 \succ 11\cdots 1$, *then* $x \succ 11\cdots 1$ *for all* $x \in X$.

Proof. By Lemma 6.5, $C(11\cdots 1) > M_v + \tau$. But for all $x \in X$, $C(x) < C(11\cdots 1)$. Therefore, $x \succ 11\cdots 1$ by Axiom $2'$. $\square$

Lemma 6.7. *If* $11\cdots 1 \succ 00\cdots 0$, *then there exists* $x \in X$ *such that*

$$11\cdots 1 \succ x \succ 00\cdots 0.$$

Proof. If $11\cdots 1 \succ 00\cdots 0$, then $C(11\cdots 1) \leq M_v + \tau$ by Lemma 6.3. Now consider two cases:

Case 1: If $C(11\cdots 1) \leq M_v$, then there exists $x \in X$ such that

$$C(00\cdots 0) < C(x) < C(11\cdots 1) \leq M_v.$$

So, by Axiom 1, $11\cdots 1 \succ x \succ 00\cdots 0$.

Case 2: If $M_v < C(11\cdots 1) \leq M_v + \tau$, then $\tau \leq M_v$ implies that there exists a cost-maximal $x \in X$ such that $0 < C(x) \leq M_v$. Since $11\cdots 1$ augments x, it follows by Axioms 1 and 3 that $11\cdots 1 \succ x \succ 00\cdots 0$. $\square$

Lemma 6.7 can be stated more concisely by simply noting that $11\cdots 1$ cannot *cover* $00\cdots 0$. In general x is said to cover z (with respect to $\succ$) if $x \succ z$ and there does not exist y such that $x \succ y \succ z$.

We are now able to prove Theorem 6.1.

Proof of Theorem 6.1. Let A and B to be the collections of total orders on X defined as follows:

$$A = \{\succ : 00\cdots 0 \succ 11\cdots 1 \text{ and } 11\cdots 1 \succ x \text{ for some } x \in X\}.$$

$$B = \{\succ : 11\cdots 1 \text{ covers } 00\cdots 0 \text{ with respect to } \succ\}.$$

Furthermore, let C be the collection of total orders $\succ$ on X that satisfy all of the following conditions:

1. $11\cdots 1 \succ 00\cdots 0$.
2. $11\cdots 1$ covers some nonsingleton element z of X, where $z \neq 00\cdots 0$.
3. For some singletons $x, y \in X$, either

$$x \succ y \succ 11\cdots 1 \quad \text{or} \quad x \succ 11\cdots 1 \succ 00\cdots 0 \succ y.$$

Note that $A \nsubseteq \Omega_n$, $B \nsubseteq \Omega_n$, and $C \nsubseteq \Omega_n$ by Lemmas 6.6, 6.7, and 6.4, respectively. Note also that A, B, and C are pairwise disjoint. Thus,

$$|\Omega_n| \leq 2^n! - |A \cup B \cup C| = 2^n! - |A| - |B| - |C|.$$

It can be easily shown that

$$|A| = \binom{2^n - 1}{2}(2^n - 2)! \quad \text{and} \quad |B| = (2^n - 1)(2^n - 2)!.$$

Thus,

$$\begin{aligned} |A| + |B| &= \frac{(2^n - 1)!}{2!(2^n - 3)!}(2^n - 2)! + (2^n - 1)(2^n - 2)! \\ &= \frac{(2^n - 2)}{2}(2^n - 1)! + (2^n - 1)! \\ &= 2^{n-1}(2^n - 1)!. \end{aligned}$$

To count the elements of C, we note that every order $\succ$ from C can be constructed via a sequence of five choices.

First, we choose z, the nonsingleton element of X that is covered by $11 \cdots 1$. There are $2^n - n - 2$ possible choices (excluding $11 \cdots 1$, $00 \cdots 0$, and the n singleton outcomes).

Next, we divide the singleton elements of X into three groups according to their ranking relative with respect to $11 \cdots 1$ and $00 \cdots 0$. In particular, let X' denote the set of singleton elements of X, and let

$$\begin{aligned} i &= |\{x \in X' : x \succ 11 \cdots 1\}|, \\ j &= |\{x \in X' : 11 \cdots 1 \succ z \succ x \succ 00 \cdots 0\}|, \\ k &= |\{x \in X' : 00 \cdots 0 \succ x\}|. \end{aligned}$$

Note that $i + j + k = n$. Furthermore, the definition of C requires that $i \neq 0$, and if $i = 1$, $k \neq 0$. Any values of i, j, and k that satisfy these conditions will yield a grouping consistent with the definition of C. Thus, there are

$$\binom{n+2}{2} - (n+1) - 1 = \frac{(n+2)(n-1)}{2}$$

such groupings.

Next, we choose an ordering for the n singletons. There are $n!$ such choices.

Our first three steps produce a unique ordering of the singleton elements of X along with the elements $11 \cdots 1$, z, and $00 \cdots 0$. Now we must choose which of the 2^n positions in the ranking induced by $\succ$ will be occupied by these $n+3$ outcomes. Since $11 \cdots 1$ must cover z, we have $\binom{2^n-1}{n+2}$ choices.

Once the positions and ordering of the singletons, $11 \cdots 1$, z, and $00 \cdots 0$ are determined, we must choose an ordering for the remaining $2^n - n - 3$ elements of X. There are $(2^n - n - 3)!$ such choices.

Putting all of this together, we obtain:

$$\begin{aligned}
|C| &= n!(2^n-n-2)\frac{(n+2)(n-1)}{2}\binom{2^n-1}{n+2}(2^n-n-3)! \\
&= n!(2^n-n-2)\frac{(n+2)(n-1)}{2}\frac{(2^n-1)!}{(n+2)!(2^n-n-3)!}(2^n-n-3)! \\
&= \frac{n!(2^n-n-2)(n+2)(n-1)(2^n-1)!}{2(n+2)!} \\
&= \frac{(n-1)(2^n-n-2)(2^n-1)!}{2(n+1)}.
\end{aligned}$$

From this it follows that

$$\begin{aligned}
\lim_{n\to\infty}\frac{|\Omega_n|}{2^n!} &\le \lim_{n\to\infty}\frac{2^n!-|A|-|B|-|C|}{2^n!} \\
&= \lim_{n\to\infty}\left(1-\frac{2^{n-1}(2^n-1)!}{2^n!}-\frac{(n-1)(2^n-n-2)(2^n-1)!}{2(n+1)(2^n)!}\right) \\
&= \lim_{n\to\infty}\left(1-\frac{1}{2}-\frac{(n-1)(2^n-n-2)}{2^{n+1}(n+1)}\right) \\
&= \frac{1}{2}-\lim_{n\to\infty}\frac{(n-1)(2^n-n-2)}{2^{n+1}(n+1)} = \frac{1}{2}-\frac{1}{2}=0.
\end{aligned}$$

But since $\dfrac{|\Omega_n|}{2^n!} \ge 0$ for all n, it follows that $\lim_{n\to\infty}\dfrac{|\Omega_n|}{2^n!}=0$. □

At first glance, the conclusion of Theorem 6.1 may seem rather surprising. Indeed, one might expect cost-conscious voters to be more prevalent than the theorem suggests. There are a number of reasonable explanations for this apparent discrepancy, all of which warrant further investigation.

First, it may be the case that random samples of preference orders do not accurately represent the preferences of electorates in actual elections. Perhaps some orders are unrealistic and should be eliminated from the start. If this is the case, then among all *realistic* preference orders, however that notion is defined, cost-conscious preferences may be more prevalent. Since random preferences have been used in past research to simulate referendum elections [Hodge and Schwallier 2006], a more careful look at their ability to model actual electorates seems appropriate.

Second, it could be the case that as the number of questions increases, other factors in addition to cost-consciousness have more of an opportunity to play a role in the formation of voter preferences. In other words, while *purely* cost-conscious preferences may become increasingly rare, the presence of some form of cost-consciousness may still be found, perhaps in abundance.

Finally, our model may not account for all forms of cost-consciousness. In particular, there may be ways of generalizing our model that would allow for a broader range of preferences to be classified as cost-conscious. One direction for further research would be formulate a model based on penalty functions that decrease a voter's net utility in some predictable way when the voter's cost ceiling is exceeded.

7. Summary and conclusions

Cost-consciousness is one cause of nonseparability within voter preferences in multiple-question referendum elections. In fact, cost-consciousness induces preference nonseparability in all but the most trivial of cases. This nonseparability can lead to undesirable election outcomes under the typical method of simultaneous voting.

We have shown that in electorates consisting entirely of cost-conscious voters, a weak Condorcet winner is guaranteed to exist whenever outcome costs are distinct. Furthermore, a weak Condorcet loser is guaranteed to exist whether outcome costs are distinct or not, and this weak Condorcet loser is always either $11\cdots 1$ or $00\cdots 0$.

Even with a relaxed model of cost-consciousness that allows cost ceilings to be exceeded when certain conditions are met, we showed that preference orders consistent with the axioms of cost-consciousness comprise an arbitrarily small proportion of all possible preferences as the number of questions increases without bound. We discussed several possible explanations for this result, all of which suggest directions for further research.

This research is one of the first attempts to formally model a practical cause of nonseparability in voter preferences over multiple issues. There are certainly other underlying causes of nonseparability, and further investigation of these other causes could eventually lead to the development of a scheme for classifying voter preferences according to the types of interdependence they exhibit.

Our work here has focused on modeling the preferences of cost-conscious voters, but we have not investigated or proposed methods for choosing better election outcomes when electorates are cost-conscious. This direction seems like a natural next step, and one that could potentially have practical implications for the implementation of direct democracy via referendum elections.

References

[Bradley et al. 2005] W. J. Bradley, J. K. Hodge, and D. M. Kilgour, "Separable discrete preferences", *Math. Social Sci.* **49**:3 (2005), 335–353. MR 2005k:91093 Zbl 1114.91030

[Brams et al. 1997] S. J. Brams, D. M. Kilgour, and W. S. Zwicker, "Voting on referenda: The separability problem and possible solutions", *Electoral Studies* **16**:3 (1997), 359–377.

[Hodge and Schwallier 2006] J. K. Hodge and P. Schwallier, "How does separability affect the desirability of referendum election outcomes?", *Theory and Decision* **61**:3 (2006), 251–276. MR 22688736 Zbl 1101.91320

[Hodge and TerHaar 2008] J. K. Hodge and M. TerHaar, "Classifying interdependence in multidimensional binary preferences", *Math. Social Sci.* **55**:2 (2008), 190–204. MR 2009a:91029 Zbl 1143.91012

[Lacy and Niou 2000] D. Lacy and E. M. S. Niou, "A problem with referendums", *Journal of Theoretical Politics* **12**:1 (2000), 5–31.

Received: 2010-09-17 Revised: 2011-02-14 Accepted: 2011-02-16

kyle@math.utk.edu *Department of Mathematics, University of Tennessee, 227 Ayres Hall, 1403 Circle Drive, Knoxville, TN 37996-1320, United States*

hodgejo@gvsu.edu *Department of Mathematics, Grand Valley State University, Allendale, Michigan 49401, United States*

s-lmoats1@math.unl.edu *Department of Mathematics, University of Nebraska, 203 Avery Hall, PO Box 880130, Lincoln, NE 68588-0130, United States*

involve

pjm.math.berkeley.edu/involve

See inside back cover or http://pjm.math.berkeley.edu/involve for submission instructions.

The subscription price for 2011 is US $100/year for the electronic version, and $130/year (+$35 shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94704-3840, USA.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

http://msp.org/

A NON-PROFIT CORPORATION

Typeset in LaTeX

involve

2011 vol. 4 no. 2

1944-4176(2011)4:2;1-D