

# involve

a journal of mathematics

On the size of the resonant set for the products of  $2 \times 2$   
matrices

Jeffrey Allen, Benjamin Seeger and Deborah Unger

 mathematical sciences publishers

# On the size of the resonant set for the products of $2 \times 2$ matrices

Jeffrey Allen, Benjamin Seeger and Deborah Unger

(Communicated by Chi-Kwong Li)

For  $\theta \in [0, 2\pi)$  and  $\lambda > 1$ , consider the matrix  $h = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$  and the rotation matrix  $R_\theta$ . Let  $W_n(\theta)$  denote some product of  $m$  instances of  $R_\theta$  and  $n$  of  $h$ , with the condition  $m \leq \epsilon n$  ( $0 < \epsilon < 1$ ). We analyze the measure of the set of  $\theta$  for which  $\|W_n(\theta)\| \geq \lambda^{\delta n}$  ( $0 < \delta < 1$ ). This can be regarded as a model problem for the Bochi–Fayad conjecture.

## 1. Introduction

Avila and Roblin [2009] considered the following problem. Take the two matrices

$$H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \tag{1}$$

and

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in [0, 2\pi).$$

Fix  $\lambda > 1$  and let  $m, n \in \mathbb{N}$ . Consider words of the form

$$W_n(\theta) = H^{i_1} R_\theta^{j_1} \dots H^{i_k} R_\theta^{j_k},$$

where  $k$  is arbitrary and  $i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{N} \cup \{0\}$  are such that

$$i_1 + \dots + i_k = n, \quad j_1 + \dots + j_k = m.$$

Assume that  $m$  is much smaller than  $n$  and take a “generic” angle  $\theta$ . It is not unreasonable to conjecture that  $\|W_n\|$  grows geometrically with  $n$  regardless of the combinatorics of the word. Avila and Roblin proved the following theorem, where the norm is given by  $\|W\| = |a| + |b| + |c| + |d|$  if  $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .

*MSC2010:* primary 37H15; secondary 37H05, 37C85.

*Keywords:* Bochi–Fayad conjecture, resonant set, measure, rotation matrix, Fayad, Krikorian, exponential growth.

**Theorem 1.** *Assume that  $0 < \delta < 1$  is fixed. Then there is an  $n$ -independent set  $\Omega$  such that  $|\Omega| = 2\pi$  and for any  $\theta \in \Omega$  there is  $\epsilon > 0$  so that*

$$\min_{W_n} \|W_n(\theta)\| > \lambda^{\delta n}$$

provided  $m < \epsilon n (\ln n \ln \ln n)^{-1}$ .

Here the minimum is over all words  $W_n$  for  $n$  fixed and  $m$  as given. This theorem improved earlier results by Fayad and Krikorian [2008]. The special case of the Bochi–Fayad conjecture [Avila and Roblin 2009; Fayad and Krikorian 2008] deals with the similar situation when  $m < \epsilon n$  and  $\epsilon$  is small. One might expect that  $|\Omega| \rightarrow 2\pi$  as  $\epsilon \rightarrow 0$  in this case. Proving it seems to be quite hard. We investigate a simpler case. In (1), consider the matrix  $H$  when  $\lambda$  is large. Then  $\lambda^{-1} \rightarrow 0$  as  $\lambda \rightarrow \infty$  and one might wonder what happens if  $\lambda^{-1}$  is dropped. Thus we consider  $h = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \notin \text{SL}(2, \mathbb{R})$  instead of  $H$ . It turns out that a very precise analysis can be performed for this simpler model problem, as we shall see in the next section. Section 3 provides some numerical evidence and comparison of the model case with the real problem.

## 2. The model problem

In the previous setting, take  $h = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$  instead of  $H$  and fix  $\epsilon \in (0, 1)$ . Given  $n$ , set

$$f_n(\theta) = \min_{W_n} \|W_n(\theta)\|,$$

where the norm is again given by the sum of absolute values of matrix entries and the minimum is taken over all  $W_n$  with  $m \leq \epsilon n$ . Note that we can take the minimum because for a given  $n$  there are only finitely many possibilities for  $W_n$ .

Finally, we fix  $0 < \delta < 1$  and define the *resonant set*  $\mathcal{R}$  thus:  $\theta \in \mathcal{R}$  if there exists some  $n$  such that  $f_n(\theta) < \lambda^{\delta n}$ . We claim that  $|\mathcal{R}| < C\lambda^{-(1-\delta)/\epsilon}$ , where  $C$  is some constant that can be explicitly computed and  $|\mathcal{R}|$  denotes the Lebesgue measure of the set  $\mathcal{R}$ .

We now make the convention that *there are no zero exponents in the expression of  $W_n(\theta)$* . Then, for words having precisely  $k$  blocks of rotation matrices, there are four possibilities, differing in which matrix ( $h$  or  $R_\theta$ ) begins the word and which matrix ends it:

$$W_n(\theta) = h^{i_1} R_\theta^{j_1} \cdots h^{i_k} R_\theta^{j_k}, \quad (2)$$

$$W_n(\theta) = R_\theta^{j_1} h^{i_1} \cdots R_\theta^{j_k} h^{i_k}, \quad (3)$$

$$W_n(\theta) = R_\theta^{j_1} h^{i_1} \cdots R_\theta^{j_{k-1}} h^{i_{k-1}} R_\theta^{j_k}, \quad (4)$$

$$W_n(\theta) = h^{i_1} R_\theta^{j_1} \cdots h^{i_k} R_\theta^{j_k} h^{i_{k+1}}. \quad (5)$$

For the word in (2), the product has this explicit form:

$$\begin{aligned}
 W_n(\theta) &= h^{i_1} R_\theta^{j_1} \cdots h^{i_k} R_\theta^{j_k} \\
 &= \begin{pmatrix} \lambda^{i_1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos j_1\theta & -\sin j_1\theta \\ \sin j_1\theta & \cos j_1\theta \end{pmatrix} \cdots \begin{pmatrix} \lambda^{i_k} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos j_k\theta & -\sin j_k\theta \\ \sin j_k\theta & \cos j_k\theta \end{pmatrix} \\
 &= \begin{pmatrix} \lambda^{i_1} \cos j_1\theta & -\lambda^{i_1} \sin j_1\theta \\ 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} \lambda^{i_k} \cos j_k\theta & -\lambda^{i_k} \sin j_k\theta \\ 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} \lambda^n \cos j_1\theta \cdots \cos j_k\theta & \lambda^n \cos j_1\theta \cdots \cos j_{k-1}\theta \sin j_k\theta \\ 0 & 0 \end{pmatrix}. \tag{6}
 \end{aligned}$$

Likewise, for (3), we obtain

$$\begin{aligned}
 W_n(\theta) &= R_\theta^{j_1} h^{i_1} \cdots R_\theta^{j_k} h^{i_k} \\
 &= \begin{pmatrix} \cos j_1\theta & -\sin j_1\theta \\ \sin j_1\theta & \cos j_1\theta \end{pmatrix} \begin{pmatrix} \lambda^{i_1} & 0 \\ 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} \cos j_k\theta & -\sin j_k\theta \\ \sin j_k\theta & \cos j_k\theta \end{pmatrix} \begin{pmatrix} \lambda^{i_k} & 0 \\ 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} \lambda^{i_1} \cos j_1\theta & 0 \\ \lambda^{i_1} \sin j_1\theta & 0 \end{pmatrix} \cdots \begin{pmatrix} \lambda^{i_k} \cos j_k\theta & 0 \\ \lambda^{i_k} \sin j_k\theta & 0 \end{pmatrix} \\
 &= \begin{pmatrix} \lambda^n \cos j_1\theta \cdots \cos j_k\theta & 0 \\ \lambda^n \sin j_1\theta \cos j_2\theta \cdots \cos j_k\theta & 0 \end{pmatrix}. \tag{7}
 \end{aligned}$$

Using the result in (7), we get for the word (4)

$$\begin{aligned}
 W_n(\theta) &= (R_\theta^{j_1} h^{i_1} \cdots R_\theta^{j_{k-1}} h^{i_{k-1}}) R_\theta^{j_k} \\
 &= \begin{pmatrix} \lambda^{i_1+\cdots+i_{k-1}} \cos j_1\theta \cos j_2\theta \cdots \cos j_{k-1}\theta & 0 \\ \lambda^{i_1+\cdots+i_{k-1}} \sin j_1\theta \cos j_2\theta \cdots \cos j_{k-1}\theta & 0 \end{pmatrix} \begin{pmatrix} \cos j_k\theta & -\sin j_k\theta \\ \sin j_k\theta & \cos j_k\theta \end{pmatrix} \\
 &= \begin{pmatrix} \lambda^n \cos j_1\theta \cos j_2\theta \cdots \cos j_k\theta & -\lambda^n \cos j_1\theta \cos j_2\theta \cdots \cos j_{k-1}\theta \sin j_k\theta \\ \lambda^n \sin j_1\theta \cos j_2\theta \cdots \cos j_k\theta & -\lambda^n \sin j_1\theta \cos j_2\theta \cdots \cos j_{k-1}\theta \sin j_k\theta \end{pmatrix}. \tag{8}
 \end{aligned}$$

Finally, using (6), we get for the word (5) simply

$$\begin{aligned}
 W_n(\theta) &= (h^{i_1} R_\theta^{j_1} \cdots h^{i_{k-1}} R_\theta^{j_{k-1}} h^{i_k} R_\theta^{j_k}) h^{i_{k+1}} \\
 &= \begin{pmatrix} \lambda^{i_1+\cdots+i_k} \cos j_1\theta \cdots \cos j_k\theta & \lambda^{i_1+\cdots+i_k} \cos j_1\theta \cdots \cos j_{k-1}\theta \sin j_k\theta \\ 0 & 0 \end{pmatrix} \\
 &\quad \times \begin{pmatrix} \lambda^{i_{k+1}} & 0 \\ 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} \lambda^n \cos j_1\theta \cdots \cos j_k\theta & 0 \\ 0 & 0 \end{pmatrix}. \tag{9}
 \end{aligned}$$

Therefore  $\|W_n(\theta)\|$  is given by

$$\begin{aligned} \lambda^n |\cos j_1 \theta \cdots \cos j_{k-1} \theta| (|\cos j_k \theta| + |\sin j_k \theta|) & \quad \text{for } W_n \text{ of type (2),} \\ \lambda^n |\cos j_2 \theta \cdots \cos j_k \theta| (|\cos j_1 \theta| + |\sin j_1 \theta|) & \quad \text{for } W_n \text{ of type (3),} \\ \lambda^n |\cos j_2 \theta \cdots \cos j_{k-1} \theta| (|\cos j_1 \theta| + |\sin j_1 \theta|) \\ & \quad \times (|\cos j_k \theta| + |\sin j_k \theta|) \quad \text{for } W_n \text{ of type (4),} \\ \lambda^n |\cos j_1 \theta \cdots \cos j_k \theta| & \quad \text{for } W_n \text{ of type (5).} \end{aligned}$$

**Remark.** This shows that, among words with  $k$  rotation blocks,  $\min_{W_n} \|W_n(\theta)\|$  is reached by words of type (5).

**Theorem 2.** *Let*

$$\begin{aligned} S_\alpha &= \{ \theta \in [0, 2\pi) \mid |\cos \alpha \theta| < \lambda^{-(1-\delta)\alpha/\epsilon-1} \}, \\ \tilde{S}_\alpha &= \{ \theta \in [0, 2\pi) \mid |\cos \alpha \theta| < \lambda^{-(1-\delta)\alpha/\epsilon} \}. \end{aligned}$$

*Then the resonant set  $\mathcal{R}$  satisfies*

$$\bigcup_{\alpha \in \mathbb{N}} S_\alpha \subseteq \mathcal{R} \subseteq \bigcup_{\alpha \in \mathbb{N}} \tilde{S}_\alpha.$$

*Proof.* Suppose  $\theta \in \bigcup_{\alpha \in \mathbb{N}} S_\alpha$ . Then  $\theta \in S_\alpha$  for some  $\alpha \in \mathbb{N}$  and

$$|\cos \alpha \theta| < \lambda^{-(1-\delta)\alpha/\epsilon-1}.$$

Let  $n = \lceil \alpha/\epsilon \rceil + 1$ . Then,  $n - 1 \leq \alpha/\epsilon < n$  and  $\alpha < \epsilon n$ . Consider the word  $\omega_n(\theta) = h^{i_1} R_\theta^\alpha h^{i_2}$  where  $i_1 + i_2 = n$ . Since  $m = \alpha$ , we have  $m \leq \epsilon n$ . Then

$$f_n(\theta) = \min_{W_n(\theta)} \|W_n(\theta)\| \leq \|\omega_n(\theta)\| = \lambda^n |\cos \alpha \theta| < \lambda^n \cdot \lambda^{-(1-\delta)\alpha/\epsilon-1} \leq \lambda^{\delta n}.$$

Therefore  $\theta \in \mathcal{R}$ .

Now suppose  $\theta \notin \bigcup_{\alpha \in \mathbb{N}} \tilde{S}_\alpha$ . Then  $|\cos \alpha \theta| \geq \lambda^{-(1-\delta)\alpha/\epsilon}$  for all  $\alpha \in \mathbb{N}$ . Choose an arbitrary  $n \in \mathbb{N}$ . Then

$$\begin{aligned} f_n(\theta) &= \min_{W_n(\theta)} \|W_n(\theta)\| = \|\omega_n(\theta)\| & \quad (\text{for some word } \omega_n(\theta)) \\ &= \lambda^n |\cos j_1 \theta \cdots \cos j_k \theta| & \quad (\text{by the remark above}) \\ &= \lambda^n |\cos \alpha_1 \theta^{m_1} \cdots \cos \alpha_l \theta^{m_l}|, \end{aligned}$$

where  $\alpha_1 < \cdots < \alpha_l$  and  $m_1 \alpha_1 + \cdots + m_l \alpha_l = m \leq \epsilon n$ . Then

$$\begin{aligned} f_n(\theta) &= \lambda^n |\cos \alpha_1 \theta^{m_1} \cdots \cos \alpha_l \theta^{m_l}| \\ &\geq \lambda^n \cdot \lambda^{-(1-\delta)(m_1 \alpha_1 + \cdots + m_l \alpha_l)/\epsilon} = \lambda^n \cdot \lambda^{-m(1-\delta)/\epsilon} \geq \lambda^{\delta n}, \end{aligned}$$

and therefore  $\theta \notin \mathcal{R}$ . □

We claim that  $\mathcal{R}$  is a dense open set. To show that  $\mathcal{R}$  is open, we show that for each  $n$ ,  $f_n$  is continuous. For each  $n$ ,

$$R_n = \{\theta \in [0, 2\pi) \mid f_n(\theta) < \lambda^{\delta n}\} = f_n^{-1}((-\infty, \lambda^{\delta n})),$$

which is open as the preimage of a continuous function of an open set. Note that  $\mathcal{R} = \bigcup_{n=1}^{\infty} R_n$ , a union of open sets, so  $\mathcal{R}$  is open.

To show that  $f_n$  is continuous, we note that  $f_n$  is the minimum of a finite number of continuous functions (the norms of a finite number of words). Denote these functions by  $F_1, F_2, \dots, F_M$ ,  $M \in \mathbb{N}$ . Fix arbitrary  $\theta \in [0, 2\pi)$ , fix  $\zeta > 0$ , and let  $\eta > 0$  be such that whenever  $|\theta - \tilde{\theta}| < \eta$ ,  $|F_k(\theta) - F_k(\tilde{\theta})| < \zeta$  for all  $k = 1, \dots, M$ . Consider arbitrary  $\tilde{\theta} \in (\theta - \eta, \theta + \eta)$ . For some  $i, j$ ,

$$f_n(\theta) = F_i(\theta) \quad \text{and} \quad f_n(\tilde{\theta}) = F_j(\tilde{\theta}).$$

By the definition of  $f_n$ ,

$$F_i(\theta) \leq F_j(\theta) \quad \text{and} \quad F_j(\tilde{\theta}) \leq F_i(\tilde{\theta}).$$

Notice that if  $F_i(\theta) = F_j(\tilde{\theta})$ , then  $|f_n(\theta) - f_n(\tilde{\theta})| = 0 < \zeta$  and we are done. Suppose that  $F_i(\theta) > F_j(\tilde{\theta})$ . Then  $|f_n(\theta) - f_n(\tilde{\theta})| = F_i(\theta) - F_j(\tilde{\theta}) \leq F_j(\theta) - F_j(\tilde{\theta}) < \zeta$ . Otherwise, if  $F_i(\theta) < F_j(\tilde{\theta})$ , then

$$|f_n(\theta) - f_n(\tilde{\theta})| = F_j(\tilde{\theta}) - F_i(\theta) \leq F_i(\tilde{\theta}) - F_i(\theta) < \zeta.$$

To see that  $\mathcal{R}$  is dense, let  $I$  be any open interval in  $[0, 2\pi)$ . The collection of points  $R_\alpha = \{\pi/2\alpha + (\pi/\alpha)k : k \in \{1, \dots, 2\alpha - 1\}\}$  is in  $S_\alpha$ ; indeed, for any  $\phi \in R_\alpha$ ,  $\cos \alpha\phi = 0 < \lambda^{-(1-\delta)\alpha/\epsilon-1}$ . If we choose  $\alpha > |I|/\pi$ , then there must be some element  $\phi$  of  $R_\alpha$  in  $I$ . Since  $\phi \in \bigcup_{\alpha \in \mathbb{N}} S_\alpha \subseteq \mathcal{R} \subseteq \bigcup_{\alpha \in \mathbb{N}} \tilde{S}_\alpha$ , we see that every open interval in  $[0, 2\pi)$  contains a point in  $\mathcal{R}$ .

Now we are ready to estimate the size of  $\mathcal{R}$ . Consider  $\tilde{S}_\alpha$  for arbitrary  $\alpha \in \mathbb{N}$ . The measure of this set is

$$\begin{aligned} |\tilde{S}_\alpha| &= 4\alpha \left( \frac{\pi}{2\alpha} - \frac{1}{\alpha} \cos^{-1}(\lambda^{-(1-\delta)\alpha/\epsilon}) \right) \\ &= 2\pi - 4 \cos^{-1}(\lambda^{-(1-\delta)\alpha/\epsilon}) \\ &\approx 2\pi - 2\pi + 4\lambda^{-(1-\delta)\alpha/\epsilon} \\ &= 4\lambda^{-(1-\delta)\alpha/\epsilon}. \end{aligned}$$

Then our estimate for the size of  $\mathcal{R}$  is

$$|\mathcal{R}| \leq \left| \bigcup_{\alpha \in \mathbb{N}} \tilde{S}_\alpha \right| \leq \sum_{\alpha=1}^{\infty} |\tilde{S}_\alpha| \approx 4 \sum_{\alpha=1}^{\infty} \lambda^{-(1-\delta)\alpha/\epsilon} = \frac{4\lambda^{-(1-\delta)/\epsilon}}{1 - \lambda^{-(1-\delta)/\epsilon}} \approx 4\lambda^{-(1-\delta)/\epsilon},$$

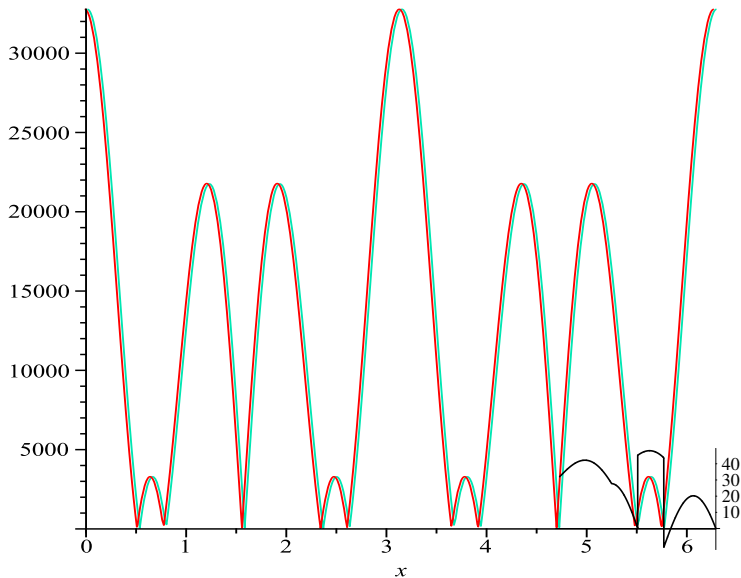
as  $\epsilon \sim 0$  and  $\lambda, \delta$  are fixed.

### 3. Some numerical evidence

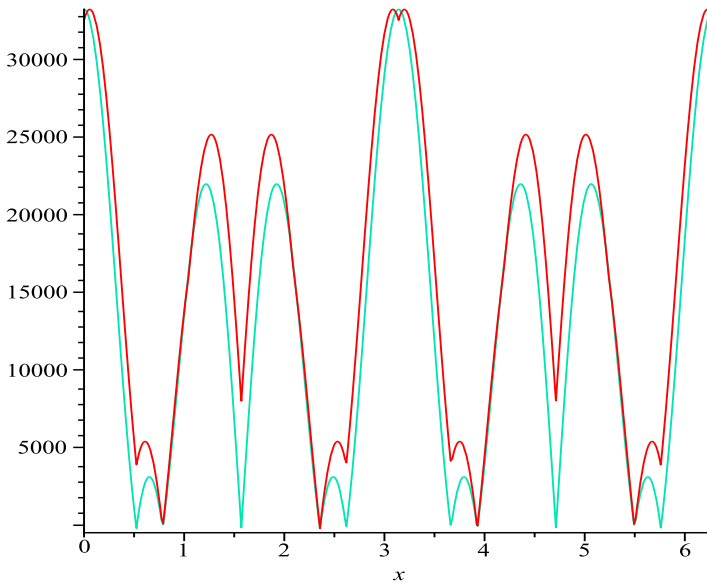
We provide some numerical and graphical evidence of what was proved. We see how the graphs of the model case and the real case compare for fixed  $n$  and  $m$ . In addition, it is shown graphically that changing the multiplicities of  $H$  affects the word's norm, but in the model case the word's norm is invariant under these changes. The graphs in this section were plotted with Maple 14.

Based on the similarities between the pictures of the real case and the model case, we conjecture that the resonant set in the model case is, in some sense, the limiting set of the resonant set in the real case as  $\lambda$  grows large, since the  $\lambda^{-1}$  term goes to 0 as  $\lambda$  goes to infinity. Of course, since here we take  $\lambda$  relatively small ( $\lambda = 2$ ) for graphing convenience, this is a rough conjecture; in fact, proving it seems to be rather difficult.

Figure 1 shows a red curve and a blue curve. The blue curve is the graph of  $\|h^{i_1} R_\theta^2 h^{i_2} R_\theta^3 h^{i_3}\|$  where  $i_1, i_2, i_3 \in \mathbb{N}$  and  $i_1 + i_2 + i_3 = 15$ . Recall that  $h$  is the matrix we use in the model case, where  $\lambda^{-1}$  is replaced by 0. With these combinatorics, varying  $i_1, i_2, i_3$  does not change the graph, as long as their sum is 15. Specifically, Figure 1 is the model case of  $n = 15$ ,  $m = 5$  ( $j_1 = 2, j_2 = 3$ ), and  $\lambda = 2$ .



**Figure 1.** Graphs of the functions  $\|H^5 R_\theta^2 H^5 R_\theta^3 H^5\|$  (red curve) and  $\|h^5 R_\theta^2 h^5 R_\theta^3 h^5\|$  (blue curve) when  $\lambda = 2$ . (The curves have been slightly offset horizontally; otherwise they would coincide at this resolution.) The thin black curve on the bottom right is the difference between the first and second functions, with the y-axis expanded 1000 times.



**Figure 2.** Graphs of the functions  $\|H^5 R_\theta^2 H^9 R_\theta^3 H^1\|$  (red curve) and  $\|h^5 R_\theta^2 h^9 R_\theta^3 h^1\|$  (blue curve) when  $\lambda = 2$ .

The red curve in Figure 1 shows the case where we replace  $h$  with  $H$  and set  $i_1 = i_2 = i_3 = 5$ . Explicitly, we are graphing  $\|H^5 R_\theta^2 H^5 R_\theta^3 H^5\|$ .

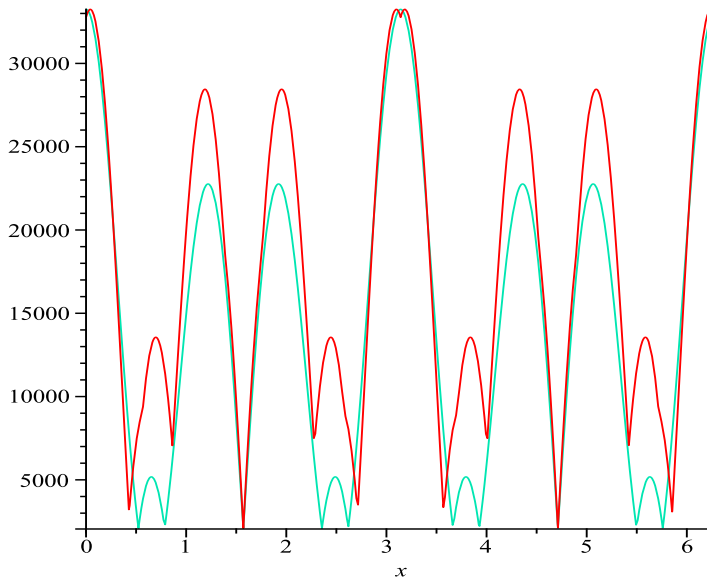
In Figure 2 we change only two parameters relative to Figure 1: the lengths of the last two blocks of  $H$ 's (or  $h$ 's) are now  $i_2 = 9$ , and  $i_3 = 1$ . As already seen, the  $h$  curve (blue) remains the same, but the  $H$  curve (red)—that is, the graph of the function  $\|H^5 R_\theta^2 H^9 R_\theta^3 H^1\|$ —changes significantly as a result of changing the order of multiplication in the word.

By comparing the red curves in Figures 1 and 2, we observe that a greater disparity between the multiplicities of  $H$  (the  $i_k$ 's) is correlated with a smaller resonant set (the set of points  $\theta$  between 0 and  $2\pi$  such that the norm of the word is within a certain distance of zero). The slope of the word's norm is steeper in Figure 2 than it is in Figure 1 and the peaks in Figure 2 are associated with larger values of the word's norm than in the case depicted by Figure 1. Both conditions lead to fewer points  $\theta$  that are mapped to a norm of the word that is close to zero.

Figure 3 shows the graph of  $\|H^1 R_\theta^2 H^1 R_\theta^3 H^{13}\|$ . Comparing this graph with Figure 2 provides further evidence that a greater disparity between the multiplicities of  $H$  results in a smaller resonant set.

To further justify our use of the model case, consider the Figure 4, which treats the case of a word of the form (4). Specifically, the blue curve shows the function  $\|R_\theta h^{i_1} R_\theta h^{i_2} R_\theta\|$ , where  $i_1 + i_2 = 15$  and  $\lambda = 2$ . We take  $i_1 = 7$  and  $i_2 = 8$  and replace





**Figure 3.** Graphs of the functions  $\|H^1 R_\theta^2 H^1 R_\theta^3 H^{13}\|$  (red curve) and  $\|h^1 R_\theta^2 h^1 R_\theta^3 h^{13}\|$  (blue curve) when  $\lambda = 2$ .

$h$  by  $H$  to obtain the red curve. As in Figure 1, the two curves are indistinguishable to within the plot's resolution.

Note that the blue curve in Figure 4 is not comparable to that of Figures 1–3. Both show the model case, but with different combinatorics on the word: expression (5) for the earlier figures, and (4) for Figure 4. Both graphs still have a small resonant set.

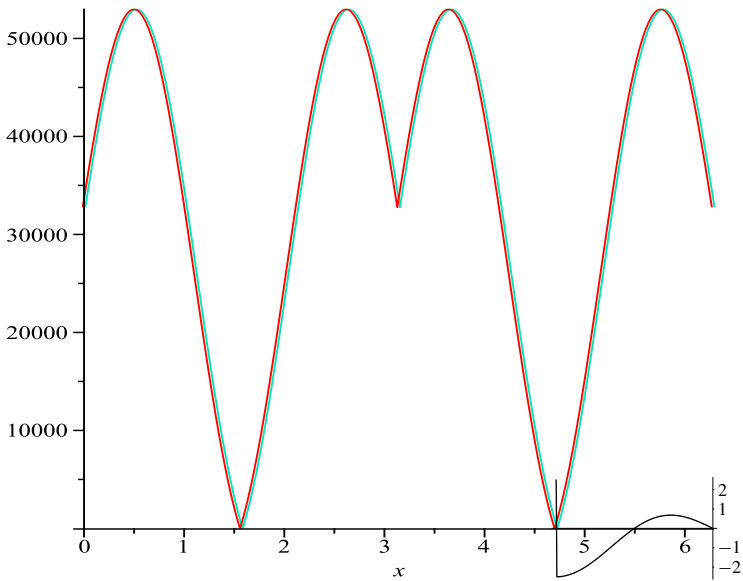
Figure 5 shows the graphs of  $\|R_\theta H^{14} R_\theta H R_\theta\|$  (red) and of  $\|R_\theta h^{14} R_\theta h R_\theta\|$  (blue); the latter of course is the same as the blue curve of Figure 4. Comparing Figure 4 with Figure 5, again we see that a greater disparity between the multiplicities of  $H$  results in a smaller resonant set.

#### 4. Conclusion

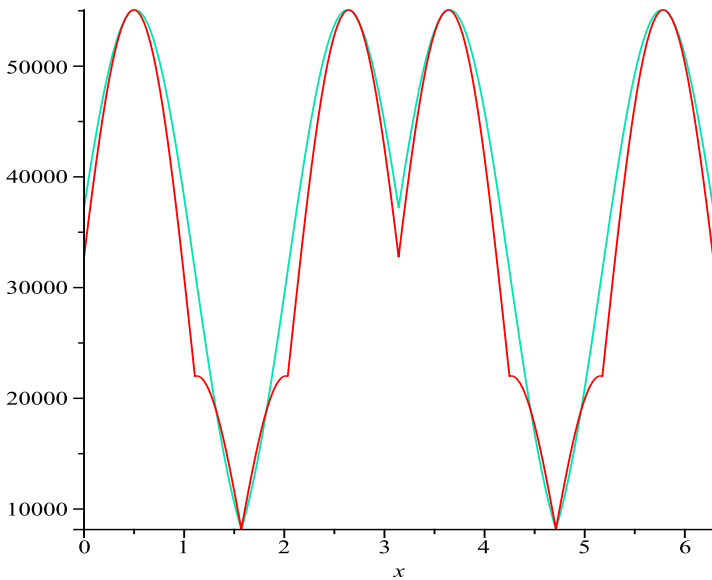
We hope that our model problem is a viable approximation for what happens when the matrix  $H$  is used. The next step might be to express the the Bochi–Fayad problem in terms of the model problem. One way to do this might be to write  $H = h + e$ , where  $e$  is the matrix

$$e = \begin{pmatrix} 0 & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

and then express a product of  $H$ 's and  $R_\theta$ 's in terms of a product of  $h$ 's and  $R_\theta$ 's, and some other, hopefully small, error terms. The numerical evidence above suggests



**Figure 4.** Graphs of the functions  $\|R_\theta H^7 R_\theta H^8 R_\theta\|$  (red curve) and  $\|R_\theta h^7 R_\theta h^8 R_\theta\|$  (blue curve) when  $\lambda = 2$ . (The curves have been slightly offset horizontally; otherwise they would coincide at this resolution.) The thin black curve on the bottom right is the difference between the first and second functions, with the y-axis expanded 2500 times.



**Figure 5.** Graphs of  $\|R_\theta h^{14} R_\theta h R_\theta\|$  and  $\|R_\theta H^{14} R_\theta H R_\theta\|$  when  $\lambda = 2$ .

that the resonant set for words using the matrix  $H$  would actually be smaller than that for words using the matrix  $h$ , especially if the distribution of  $H$ 's is in some sense irregular. This behavior might become more apparent when  $\lambda$  is taken much larger, and  $\epsilon$  much smaller, than in the experiments above.

### Acknowledgements

We would like to thank Professors Serguei Denisov and Alexander Kiselev for their constant support throughout the research and writing processes. We are grateful to them for suggesting the topic and providing us with the opportunity to apply the concepts we have learned throughout our mathematical careers. We thank them both for their excellent advice and encouragement.

### References

- [Avila and Roblin 2009] A. Avila and T. Roblin, “Uniform exponential growth for some  $SL(2, \mathbb{R})$  matrix products”, *J. Mod. Dyn.* **3**:4 (2009), 549–554. MR 2011f:37099 Zbl 1189.37060
- [Fayad and Krikorian 2008] B. Fayad and R. Krikorian, “Exponential growth of product of matrices in  $SL(2, \mathbb{R})$ ”, *Nonlinearity* **21**:2 (2008), 319–323. MR 2009d:37041 Zbl 1142.37024

Received: 2010-12-10

Revised: 2011-02-17

Accepted: 2011-04-03

jhallen2@wisc.edu

*University Of Wisconsin – Madison, Madison, WI 53706,  
United States*

bseeger@wisc.edu

*University Of Wisconsin – Madison, Madison, WI 53706,  
United States*

dunger@wisc.edu

*University Of Wisconsin – Madison, Madison, WI 53715,  
United States*

# involve

pjm.math.berkeley.edu/involve

## EDITORS

### MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhs@wfu.edu

### BOARD OF EDITORS

John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Pietro Cerone	Victoria University, Australia pietro.cerone@vu.edu.au	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Ken Ono	University of Wisconsin, USA ono@math.wisc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	Robert J. Plemmons	Wake Forest University, USA plemmons@wfu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Karen Kafadar	University of Colorado, USA karen.kafadar@cudenver.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
David Larson	Texas A&M University, USA larson@math.tamu.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu

## PRODUCTION

Silvio Levy, Scientific Editor

Sheila Newbery, Senior Production Editor

Cover design: ©2008 Alex Scorpan

See inside back cover or <http://pjm.math.berkeley.edu/involve> for submission instructions.

The subscription price for 2011 is US \$100/year for the electronic version, and \$130/year (+\$35 shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94704-3840, USA.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY  
 **mathematical sciences publishers**  
<http://msp.org/>

A NON-PROFIT CORPORATION

Typeset in L<sup>A</sup>T<sub>E</sub>X

Copyright ©2011 by Mathematical Sciences Publishers

# involve

2011

vol. 4

no. 2

The visual boundary of $\mathbb{Z}^2$	103
KYLE KITZMILLER AND MATT RATHBUN	
An observation on generating functions with an application to a sum of secant powers	117
JEFFREY MUDROCK	
Clique-relaxed graph coloring	127
CHARLES DUNN, JENNIFER FIRKINS NORDSTROM, CASSANDRA NAYMIE, ERIN PITNEY, WILLIAM SEHORN AND CHARLIE SUER	
Cost-conscious voters in referendum elections	139
KYLE GOLENBIEWSKI, JONATHAN K. HODGE AND LISA MOATS	
On the size of the resonant set for the products of $2 \times 2$ matrices	157
JEFFREY ALLEN, BENJAMIN SEEGER AND DEBORAH UNGER	
Continuous $p$ -Bessel mappings and continuous $p$ -frames in Banach spaces	167
MOHAMMAD HASAN FAROUGHI AND ELNAZ OSGOOEI	
The multidimensional Frobenius problem	187
JEFFREY AMOS, IULIANA PASCU, VADIM PONOMARENKO, ENRIQUE TREVIÑO AND YAN ZHANG	
The Gauss–Bonnet formula on surfaces with densities	199
IVAN CORWIN AND FRANK MORGAN	



1944-4176(2011)4:2;1-D