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An implementation of scatter search to train neural networks for brain lesion recognition

Jeffrey Larson and Francis Newman

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In recent years, the use of computer aided diagnosis (CAD) has achieved acceptance in mammography and other areas. To facilitate automated detection of brain abnormalities, we propose a novel method for quickly training neural networks to classify brain images. Our method outperforms traditional neural network training methods by achieving a better balance between classification accuracy and training time.

1. Introduction

A variety of techniques have been implemented for lesion detection, including image filtering methods [Kotropoulos and Pitas 1992], support vector machines [Bilello et al. 2004], Markov random fields [Van Leemput et al. 2001], and a variety of artificial neural networks (ANNs) [Raff and Newman 1992; Wu et al. 1993; Yu and Guan 2000]. Nevertheless, automated pathology or lesion detection in most medical images has become somewhat dormant in recent years. Even in mammography, where computer-aided detection (CAD) has the greatest acceptance, the sensitivity (percentage of abnormal pathologies identified as "abnormal") is high, but the specificity (percentage of normal images where no abnormality is found) is poor. There are further difficulties for practical or commercial acceptance of CAD outside of mammography. These are often a combination of algorithmic and technical limitations. For example, when ANNs are used, backpropagation is frequently the method of choice for training. Backpropagation employs a multilayer feed forward architecture where error minimization is achieved via some form of gradient descent. Backpropagation (and its variants) can be slow to train, but a bigger problem for medical image diagnosis is the fact that the algorithm is likely to settle on an unsatisfactory local minimum [Gori and Tesi 1992; Sontag and Sussmann 1989].

MSC2000: primary 90C59, 92B20; secondary 90C90, 92C50.
Keywords: computer-aided diagnosis, scatter search, artificial neural networks, health care, diagnosis.

Though the problem is difficult, correct classification of pathology and lesions in medical images could offer great benefits in reducing diagnostic errors and labor costs. It is estimated that 10-30% of breast cancers that are retrospectively visible are missed by radiologists upon initial reading [Brake et al. 1998], and 17-21% of polyps in computed tomography (CT) colonography are given false negative diagnoses due to human perception errors [Fletcher et al. 2000]. Missed tumors in lung CTs are disturbingly common [White et al. 1996], and brain tumors especially are frequently misdiagnosed [Wang et al. 2003]. Recent studies have found that physicians' clinical diagnoses are proven wrong 10-15% of the time by autopsy findings [Shojania et al. 2003; Roulson et al. 2005]. In an effort to reduce the number of misdiagnoses, researchers in the radiological sciences have been pursuing CAD since the early ascendancy of the computer [Schwartz 1970; Raff and Newman 1992; Chan et al. 1987]. Nevertheless, CAD has had limited impact in the field of human radiology, aside from some impact in mammography, where commercial systems have been available since 1998 [Vyborny et al. 2000].

In this effort we pursue a novel method to minimize the classification error for a feed-forward ANN in the medical image diagnosis problem using a scatter search meta-heuristic. The method, which is trained on a small subset of images and then validated on a larger set of images, greatly outperforms traditional classification methods.

2. Background

Scatter search [Glover 1999; Glover et al. 2000] is a population-based meta-heuristic that uses a local search algorithm to find an optimum. In this application, the population to be optimized is a set of random weights. Diverse individuals from the ANN population are then combined to form a "best" set according to some metric. This best set is then incorporated into the next population to be evaluated. The implementation of scatter search discussed here permits comparative evaluation of various feature vectors extracted from medical images.

For ease of reference, we define the terms to be used in this paper. An artificial neural network (ANN) is a mathematical model that simulates the structure of biological neural networks. It consists of interconnected nodes, each of which is a functional that acts on a linear combination of its inputs. Supervised ANNs are adaptive models that adjust the weights on the network arcs during a training phase in order to match each individual input to its target.

For example, a person might wish to construct a neural network to determine whether individuals should be diagnosed with colon cancer after their first screening. One might train the model using data (age, weight, height, ..., race, etc.) collected for 100 patients, as well as an indicator of whether or not they were diagnosed with

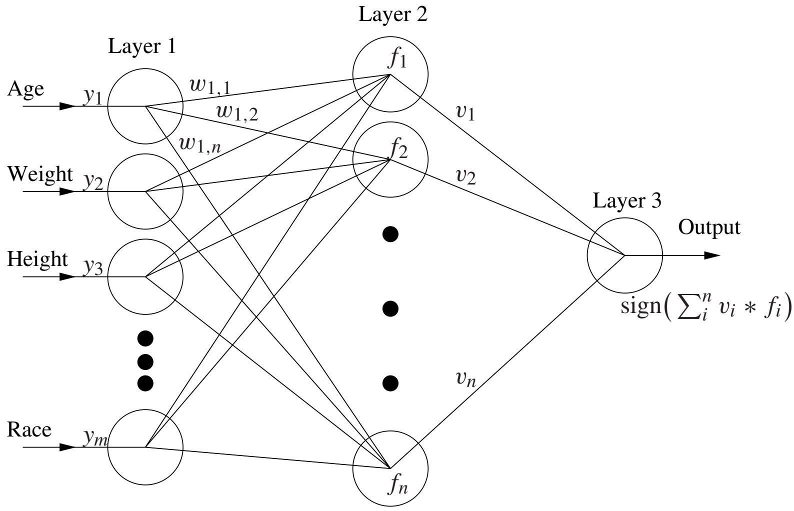

Figure 1. An example of an ANN.

colon cancer. To train the network, the weights on the arcs need to be adjusted until an acceptable percentage of the training set has the correct output (perhaps a network output of $+1$ for people who are diagnosed with cancer and -1 for those who are not). An example of such a three-layer network with m inputs and n hidden nodes is shown in Figure 1.

Our method employs a population of feed forward neural network architectures. According to the literature, three-layer networks are suitable for most problems, but the number of hidden nodes is frequently chosen by trial and error [Fausett 1994; Hassoun 1995, pp. 318–322]. It is important to note that our population consists of networks with identical three-layer architectures. Our process finds the network in the population that has the best weights and then uses these weights in an ANN to classify brain images. Though each network in our population has a fixed architecture with the same number of nodes, the method described in this paper permits easy and fast comparison of different hidden node architectures. Training a feed-forward ANN usually involves thousands of iterations (also known as epochs) to update the weights between layers in an effort to minimize the mean squared error. For a good treatment of neural networks, see [Fausett 1994; Hassoun 1995].

For a fixed architecture, training a neural network is a task in optimization. If a network has n different weights, training involves finding n weights that minimize the total error between the network output and the target values. Since backpropagation is a gradient descent method, its performance depends on initial network weights; if weights aren't well-initialized, backpropagation might perform poorly. Therefore, using a heuristic to find weights might yield a better neural network than backpropagation, in less time. Various heuristics have been proposed to solve the problem of training neural networks [Kelly et al. 1996; Ye et al. 2007]. A heuristic such as scatter search, which explores many basins of attraction, could drastically outperform backpropagation.

Since this ANN is a supervised learning scheme, the a priori target values are available, and the algorithm seeks to minimize the error between the targets and the outputs for the training set. When the training returns an error below a particular threshold, the training is halted and a validation set is used for testing. (The threshold for the error term is problem and user-dependent.)

3. Implementation

3.1. ***General scatter search implementation.*** Scatter search is a population-based meta-heuristic, where a collection of preferred solutions are maintained and recombined in order to generate new solutions. If the new solutions are preferred enough, they enter the population for the next iteration. For any given problem, the scatter search population has two subsets, good solutions and diverse solutions. The general framework for the algorithm is this:

(1) Generate a starting population.

(2) Perform a local search on every member of the population.

(3) Form a reference set of good solutions and diverse solutions using an appropriate metric.

(4) Form appropriate subsets from elements in the reference set.

(5) For each subset, generate new member(s) of the population.

(6) Return to step 2 and repeat until a satisfactory solution is found, or time runs out.

3.2. ***Our implementation.*** We attempt to find optimal weights for a neural network of fixed architecture; each instance is a fixed number of hidden nodes in a single layer. Each node in every implementation is a hyperbolic tangent activation function, which has been recommended as the best activation function for classification problems [Kalman and Kwasny 1992]. As with many meta-heuristics, any implementation allows for many degrees of freedom. For the purpose of classifying brain images as normal or abnormal, we adapted the scatter search algorithm as follows:

(1) Starting population: Generate 105 three layer networks with the same architecture. The weights (including biases) in each network are random numbers between -1 and 1.

(2) Local search: Perform a local search on the error function for each network using the Nelder–Mead method. From our initial weights, we are looking for a

local minimum of the total difference between the target output and the network output (for all training vectors).

(3) Form reference set: Select the 5 networks with the lowest total error to form the "good set". With this set fixed, we now create the "diverse set" by selecting networks that vary significantly from the good solutions. Select the network with the largest minimum Euclidean distance between its weights and the weights of the networks already in the reference set. Add this network to the reference set, and recalculate the minimum distance to the reference set for each network in the remaining population. Repeat this process until the reference set contains 10 networks: 5 good and 5 diverse.

(4) Form subsets of the reference set: For the 10 elements of the reference set, generate all possible unique pairs of networks. This creates $\binom{10}{2} = 45$ subsets.

(5) Generate new population: For each subset $\{x, y\}$, we generate three new elements of the population:

$$x_1 = x - v, \quad x_2 = x + v, \quad x_3 = y + v,$$

where $v = r(x - y)/2$ and r is a random number between 0 and 1. Thus, for each $\{x, y\}$ pair, we are create 3 points on the line through x and y. These 135 networks, along with 10 random networks to ensure diversity, form our new population. Also, the previous reference set is included in the population, though no local search needs to be performed on these networks.

4. Results

Normal and abnormal magnetic resonance images (MRIs) of the brain are used in this study (see Figure 2 for examples). Even with high-resolution images, representing individual images in a structure suitable for analysis is itself a considerable task. Though input of the entire image is desirable, the amount of data contained in a 256×256 gray-scale matrix is large. One widely used way to represent the data is to select regions of interest (ROIs) from an image; such an approach is widely used. Regions of interest are selected manually from each image and a single feature vector is generated for each region.

We examined a total of 250 normal images and 100 abnormal images. The neural network was trained on 10 of each type, while the remaining 330 were used as a validations set. To transform the images, we used the Haralick transform (a texture transform composed of second-order statistics) to generate feature vectors for each ROI. This transform uses the gray level co-occurrence matrix to uncover how often image pixel values appear adjacent to one another. If then computes quantities such as energy, correlation and homogeneity from that gray level co-occurrence matrix. For a complete discussion of the Haralick transform, see [Haralick et al. 1973].

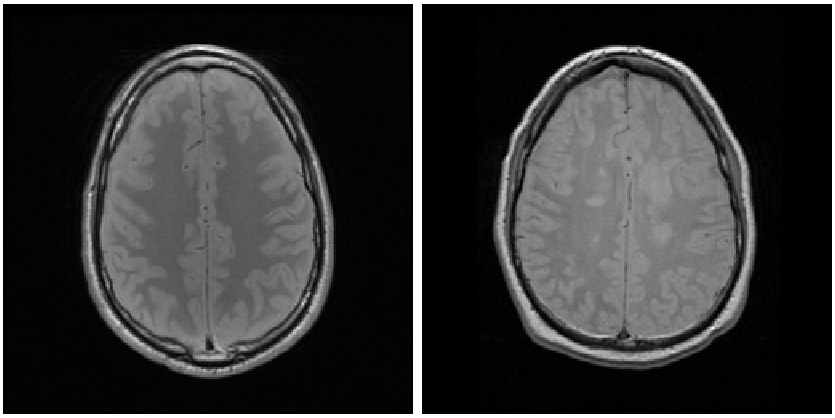

Figure 2. Normal (left) and abnormal (right) brain images.

Our training algorithm was used to classify 10 randomly selected normal images and 10 randomly selected abnormal images. Each network in the population consisted of 10 hidden nodes and 1 output node. With targets of -1 for normals and $+1$ for abnormals, a classification is considered successful when a given output is within 0.5 of the target. (Rounding is required since the output transfer function is a hyperbolic tangent function, and therefore ± 1 is only reached at infinity.)

For comparison, we used the same training vectors to train an identical neural network architecture 10,000 times using back propagation (from 10,000 different random starting weights). With the various random starting points, different networks converged to a variety of weights. The "best" network (the one that best classified the training vectors) was validated using the remaining vectors. These results are summarized in Table 1.

training set	training error	normals classified	abnormals classified	our classification rate	best back-propagation classification rate
random 10(a)	8.23×10^{-5}	199	56	77.27%	67.58%
random 10(b)	2.48	185	48	70.61%	59.39%
random 10(c)	0.49	209	56	80.06%	51.96%
random 10(d)	4.32	206	53	78.48%	55.76%
random 10(e)	3.50	210	57	80.91%	66.36%
random 10(f)	0.93	178	83	79.09%	62.12%

Table 1. Comparison of classification results between our training method and the best network (out of 10,000) trained using backpropagation for 6 different randomly selected training sets.

5. Discussion

The results reflect a dramatic improvement in the classification rate compared to that of backpropagation. This improvement is surprising given that the local search for our implementation is rudimentary. One would expect a rigorous local search, similar to what is used in backpropagation, to outperform Nelder–Mead. While there may not be a logical explanation for why taking a linear combination of the weights from two decent networks results in a worthwhile network, generating our population in the manner prescribed certainly allows us to identify many different basins of attraction. The fact that these results were achieved with only 20 training vectors is even more surprising since the number of training vectors usually used for the medical image recognition problem is orders of magnitude greater than 20 [Baum and Haussler 1989]. Expanding the training set would likely improve the classification rate.

The proposed method has many strengths. For example, the error function can easily be varied for different applications. In this implementation, false positives and false negatives were weighted equally. If sensitivity is more important than specificity, for example, it might be preferential to weight the error corresponding to missed abnormals higher than misclassified normals. Also, this training method has an ease of implementability. Although different network architectures (number of hidden nodes, layers, activation functions, etc.) might be better suited for different problem classes, this algorithm allows for quick testing of different networks. Changing any network parameter is simple, and training these different networks can be accomplished in a few minutes. Interested parties can receive the MATLAB code used in our implementation by emailing the corresponding author.

This success opens a number of avenues for further exploration. For example, the number of hidden nodes can be varied to determine whether a larger or smaller network better suits a given problem. A desire for the ability to meaningfully compare different feature vectors has been expressed in the literature [Duda et al. 2001; Egmont-Petersen et al. 2002], and the proposed network training algorithm can facilitate this well. Aside from feature vectors, different sizes of networks, different activation functions, and different network architectures can all easily be tested and compared with this algorithm.

6. Conclusion

In this paper, we propose a novel method for training neural networks for the specific task of classifying medical images as normal or abnormal. Our proposed method shows great promise for this task, but also has an ease of implementation that allows for quick training of neural networks for general classification problem.

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P_1 subalgebras of $M_n(\mathbb{C})$

Stephen Rowe, Junsheng Fang and David R. Larson

(Communicated by Charles R. Johnson)

A linear subspace B of $L(H)$ has the property P_1 if every element of its predual B_* has the form $x + B_\perp$ with $\mathrm{rank}(x) \le 1$. We prove that if $\dim H \le 4$ and B is a unital operator subalgebra of $L(H)$ which has the property P_1, then $\dim B \le \dim H$. We consider whether this is true for arbitrary H.

1. Introduction

The duality between the full algebra $L(H)$ of bounded linear operators on a Hilbert space H and its ideal L_* of trace class operators plays an important role in invariant subspace theory. Indeed, it is easy to use rank one operators in the preannihilator of an operator algebra B to construct nontrivial invariant subspaces for B and conversely (see [Larson 1982]). In his proof that subnormal operators are intransitive, S. Brown [1978] focused attention on a more subtle connection between rank one operators and invariant subspaces. He showed that certain linear subspaces B of $L(H)$ have the following property: every element of its predual B_* has the form $x + B_\perp$ with $\mathrm{rank}(x) \le 1$, where $B_\perp = \{a \in L_* : \mathrm{Tr}(ba) = 0, \text{ for all } b \in B\}$ is the preannihilator of B. This was called the P_1 *property* in [Larson 1982]. D. Hadwin and E. Nordgren [1982], and independently the third author, observed the connection between this property and *reflexivity*. Although neither property implies the other, if an algebra B has property P_1 and is also reflexive ($B = \mathrm{AlgLat}(B)$) then so are all of its ultra-weakly closed subalgebras.

Azoff obtained many results about linear subspaces of $L(H)$ which have the property P_1. Among them, he proved the following simple, but beautiful, result by using ideas from algebraic geometry. If $\dim H = n \in \mathbb{N}$ and a linear space $S \subset L(H) \equiv M_n(\mathbb{C})$ has the property P_1, then the dimension of S is no larger than $2n-1$. Furthermore, there exists a subspace $S \subset M_n(\mathbb{C})$ which has the property P_1 and $\dim S = 2n-1$. For an expository account of these and related results, we refer

MSC2000: primary 47L05, 47L75; secondary 47A15.

Keywords: property P_1, 2-reflexive.

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to [Azoff 1986], where linear spaces with the property P_1 are called *elementary spaces*. For this article the original term P_1 seemed more suitable because we want to work with the more general property P_k in the same context.

In this paper we consider the analogue of Azoff's result for the subcase of *unital operator subalgebras* in $L(H) \equiv M_n(\mathbb{C})$ (an operator algebra is unital if it contains the identity operator of $L(H)$). If B is the diagonal subalgebra of $L(H)$, it is easy to show that B has property P_1 and $\dim B = n$. In Section 5 we show that if $n \le 4$ and $B \subset M_n(\mathbb{C})$ is a unital subalgebra which has property P_1, then $\dim B \le n$. It is natural to conjecture that this is also true for arbitrary n. We make this formal:

Question 1. Suppose $\dim H = n \in \mathbb{N}$ and $B \subset L(H) \equiv M_n(\mathbb{C})$ is a unital operator algebra with property P_1. Must $\dim B \le n$?

Note that if the above conjecture is true, then we can deduce Azoff's result as a corollary. Indeed, if $S \subset L(H) \equiv M_n(\mathbb{C})$ is a linear space with property P_1, then

$$B = \left\{ \begin{pmatrix} \lambda & s \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C}, s \in S \right\} \subset L(H^{(2)}) \equiv M_{2n}(\mathbb{C})$$

is a unital operator algebra with property P_1 [Kraus and Larson 1986; 1985; Azoff 1986]. So $\dim B \le 2n$ implies $\dim S \le 2n - 1$.

An algebra $B \subset L(H)$ is called a P_1 *algebra* if A has property P_1. An algebra $B \subset L(H)$ is called a *maximal* P_1 *algebra* if whenever A is a subalgebra of $L(H)$ having property P_1 and $A \supset B$, then $A = B$. We consider a subquestion of Question 1.

Question 2. Suppose $\dim H = n \in \mathbb{N}$ and $B \subset L(H) \equiv M_n(\mathbb{C})$ is a unital operator algebra. If B has property P_1 and $\dim B = n$, is B a maximal P_1 algebra?

In Section 3 and Section 4, we prove that if a unital P_1 subalgebra $B \subset M_n(\mathbb{C})$ is semisimple or singly generated and $\dim B = n$, then B is a maximal P_1 algebra.

In [Larson 1982], the third author showed that if a weakly closed operator algebra B has property P_1, then B is 3-reflexive [Azoff 1973], that is, its three-fold ampliation $B^{(3)}$ is reflexive. (This result also holds for linear subspaces with the same proof). He raised the following problem: Suppose $\dim H = n \in \mathbb{N}$ and $B \subset L(H) \equiv M_n(\mathbb{C})$ is a unital operator algebra with property P_1. Is B 2-reflexive? Note that this question also makes sense for linear subspaces. Azoff [1986] showed that the answer to the above question is affirmative for $n = 3$ (for all linear subspaces of $M_3(\mathbb{C})$ with property P_1). Very little additional progress has been made on this problem since the mid 1980's. The purpose of the research project resulting in this article was to push further on this problem. In Section 6 of this paper, we will show that the answer to the above question for unital algebras is also affirmative for $n = 4$. The proof requires a detailed analysis of several subcases undertaken in the preceding sections.

We would like to pose the following subquestion.

Question 3. Suppose $\dim H = n \in \mathbb{N}$ and $B \subset L(H) \equiv M_n(\mathbb{C})$ is a unital operator algebra with property P_1 and $\dim B = n$. Is B 2-reflexive?

Throughout this paper, we will use the following notation. If H is a Hilbert space and n is a positive integer, then $H^{(n)}$ denotes the direct sum of n copies of H, that is, the Hilbert space $H \oplus \cdots \oplus H$. If a is an operator on H, then $a^{(n)}$ denotes the direct sum of n copies of a (regarded as an operator on $H^{(n)}$). However, we will use I_n instead of $I^{(n)}$ to denote the identity operator on $H^{(n)}$. If B is a set of operators on H, then $B^{(n)} = \{b^{(n)} : \ b \in B\}$.

This paper focuses on problems concerning operator algebras and linear subspaces of operators in finite dimensions. All of our results and proofs are given for finite dimensions. However, many of the definitions are given in the mathematics literature for infinite (as well as finite) dimensions, where the Hilbert space is assumed to be separable. The Hahn–Banach theorem and the Riesz representation theorem, the definitions of reflexive algebras and subspaces, the properties P_1 and P_k, are all given in the literature for infinite dimensions, but we will only use them here in the context of finite dimensions. In cases where proofs of known results are given for the sake of exposition, we will usually just give the proofs for finite dimensions. However, we will adopt the convention that if the statement of a result or definition in this article does not specify finite dimensions, then the reference we cite actually gives the infinite dimensional proof, or, if no reference is cited, then the proof we provide is in fact valid for infinite dimensions.

We will use some standard notation: If $A \in L(H)$, it is common to use $\mathrm{Alg}(A)$ to denote the algebra generated by A and I and $\mathrm{Alg}_0(A)$ to denote the algebra generated by A alone. If L is a lattice of subspaces, then it is also common to use $\mathrm{Alg}(L)$ to denote the algebra of operators that holds each element of L invariant. The meaning of the use of $\mathrm{Alg}(\cdot)$ will be clear from context so there will be no ambiguity.

2. Preliminaries

Let H be a Hilbert space with $\dim H = n$. Then $L(H) \equiv M_n(\mathbb{C})$. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of H. If $a \in L(H) \equiv M_n(\mathbb{C})$ is an arbitrary operator, then the trace of a is defined as

$$\mathrm{Tr}(a) = \sum_{i=1}^{n} \langle a e_i, e_i \rangle.$$

It is easy to show that $\mathrm{Tr}(a)$ does not depend on the choice of $\{e_i\}_{i=1}^n$. Moreover, the trace has the important property that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ for all $a, b \in L(H) \equiv M_n(\mathbb{C})$. In this case, the space of trace class operators on H, denoted L_*, can be identified algebraically with $M_n(\mathbb{C})$, and is equipped with the trace class norm

$$\|a\|_1 = \mathrm{Tr}((a^*a)^{1/2}).$$

Recall that the dual of a linear space is the space of all (continuous) linear functionals on the space. In the case of $L_* = M_n(\mathbb{C})$, every linear functional on L_* has the form $a \to \mathrm{Tr}(ab)$ for some $b \in L(H) \equiv M_n(\mathbb{C})$. In this way, $L(H)$ is identified as the dual space of L_*, and L_* is called the predual of $L(H)$. If $S \subset L(H)$ is a linear subspace, then as a linear space itself S can be identified as the dual of the quotient linear space $L_*/S_\perp$, where $S_\perp = \{a \in L_* | \mathrm{Tr}(ba) = 0 \text{ for all } b \in S\}$ is the preannihilator of S. Here, as usual, the quotient space $L_*/S_\perp$ means the set of all cosets of L_*, $\{x + S_\perp | x \in L_*\}$. We also write $x + S_\perp$ as $[x]$. We write $S_* = L_*/S_\perp$. The duality between S and S_* is that if $[x] \in S_*$ for some $x \in L_*$, and associate the linear functional on S given by

$$b \to \mathrm{Tr}(bx), \qquad \text{for all } b \in S.$$

This is well defined by the definition of $S_\perp$. In order to obtain S as exactly the dual of the space S_*, one needs to apply a version of the Hahn–Banach theorem [Han et al. 2007]. We say a linear subspace S of $L(H) \equiv M_n(\mathbb{C})$ has property P_1 if every element of its predual B_* has the form $x + B_\perp$ with $\mathrm{rank}(x) \leq 1$.

Let $B \subset L(H) \equiv M_n(\mathbb{C})$ be a unital operator subalgebra. If $z \in L(H)$ is an invertible operator, elementary computations yield $(zBz^{-1})_\perp = z^{-1}B_\perp z$ and $(zBz^{-1})_* = z^{-1}B_* z$, where the multiplication action of z on the quotient space B_* is given by

$$z^{-1}(x + B_\perp)z = z^{-1}xz + z^{-1}B_\perp z = z^{-1}xz + (zBz^{-1})_*.$$

From this it is easy to see that if B has property P_1, then so does zBz^{-1}. It is also true that B has property P_1 if and only if its adjoint algebra $B^* = \{b^* | b \in B\}$ has property P_1.

Lemma 2.1 [Larson 1982]. *An algebra B has property* P_1 *if and only if every element $b^* \in B^*$ has the form $x + B_\perp$ with* $\mathrm{rank}(x) \leq 1$.

Proof. Only if is trivial. Suppose every element $b^* \in B^*$ has the form $x + B_\perp$ with $\mathrm{rank}(x) \leq 1$. Note that for each $b \in B$ and each $b_\perp \in B_\perp$, $\mathrm{Tr}(bb_\perp) = 0$. This implies that $L(H) = B^* \oplus B_\perp$ with respect to the inner product $\langle x, y\rangle = \mathrm{Tr}(y^*x)$. So for each $a \in L(H)$, $a = b^* + b_\perp$ for some $b^* \in B^*$ and $b_\perp \in B_\perp$. Therefore, $a = x + B_\perp$ with $\mathrm{rank}(x) \leq 1$ by the assumption of the lemma. □

Lemma 2.2. *Let B be a subalgebra of $L(H)$. If B has property* P_1 *and $p \in B$ is a projection, then $pBp \subset L(pH)$ also has property* P_1.

Proof. Suppose $z \in B_\perp$ and $b \in B$. Then $\mathrm{Tr}(pbppzp) = \mathrm{Tr}(pbpz) = 0$. So $pzp \in (pBp)_\perp$. For each $a \in L(H)$, there exists a $b_\perp \in B_\perp$ such that the rank of

$a + b_\perp$ is at most 1. So the rank of $pap + pb_\perp p = p(a + b_\perp)p$ is at most 1. This proves the lemma. □

Recall that a vector $\xi \in \mathcal{H}$ is a separating vector of B if $b\xi = 0$ for some $b \in B$ then $b = 0$. We say that B has the separating vector property if it has a separating vector. A direct sum of subspaces with the separating vector property has the separating vector property (take the direct sum of the separating vectors). If B is similar to a subspace with a separating vector, then B has a separating vector. (If $B = TCT^{-1}$, and x separates C, then Tx separates B).

Lemma 2.3. *If* $\mathrm{Alg}(A, I)$ *is a singly generated unital subalgebra of* $L(H)$ *with* H *finite dimensional, then* B *has a separating vector.*

Consider a Jordan block B. The vector

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

separates B. Since any matrix is similar to a finite direct sum of Jordan blocks, and each Jordan block has a separating vector, the result follows.

The following result is the finite-dimensional special case of Proposition 1.2 of [Herrero et al. 1991].

Theorem 2.4. *If* B *is a subalgebra of* $L(H)$, *with* H *finite dimensional, such that either* B *or* B^* *has a separating vector, then* B *has property* P_1.

Property P_k, a generalization of property P_1, was also introduced by the third author in [Larson 1982]. Recall that an algebra B has *property* P_k if every element of its predual B_* has the form $x + B_\perp$ with $\mathrm{rank}(x) \le k$.

Lemma 2.5 [Larson 1982]. *Let* B *be a subalgebra of* $L(H)$. *Then* B *has property* P_k *if and only if* $B^{(k)} = \{b^{(k)} | b \in B\} \subset L(H^{(k)})$ *has property* P_1.

Proof. "⇒". By Lemma 2.1, we need to show that each operator $(b^*)^{(k)}$, $b \in B$, can be written as $f + B_\perp$ with $\mathrm{rank}(f) \le 1$. Note that

$$B_\perp^{(k)} = \left\{(x_{ij})_{k\times k} | x_{11} + \cdots + x_{kk} \in B_\perp\right\} \supset \left\{(x_{ij})_{k\times k} | x_{11} \cdots, x_{kk} \in B_\perp\right\}.$$

By the assumption, B has property P_k. So there exists a $b_\perp \in B_\perp$ such that the rank of $b^* + b_\perp$ is at most k. We can write $b^* + b_\perp = \xi_1 \otimes \eta_1 + \cdots + \xi_k \otimes \eta_k$, where $\xi_i \otimes \eta_i$ is the rank one operator defined by $\xi_i \otimes \eta_i(\xi) = \langle \xi, \eta_i \rangle \xi_i$. Let

$z_{ii} = k\xi_i \otimes \eta_i - \sum_{1\le r\le k} \xi_r \otimes \eta_r$, $1 \le i \le k$, and let

$$z = \begin{pmatrix} z_{11} & k\xi_2 \otimes \eta_2 & \cdots & k\xi_k \otimes \eta_k \\ k\xi_1 \otimes \eta_1 & z_{22} & \cdots & k\xi_k \otimes \eta_k \\ \cdots & \cdots & \cdots & \cdots \\ k\xi_1 \otimes \eta_1 & k\xi_2 \otimes \eta_2 & \cdots & z_{kk} \end{pmatrix}.$$

Then $z \in B_\perp^{(k)}$ and

$$(b^*)^{(k)} + (b_\perp)^{(k)} + z = k \begin{pmatrix} \xi_1 \otimes \eta_1 & \xi_2 \otimes \eta_2 & \cdots & \xi_k \otimes \eta_k \\ \xi_1 \otimes \eta_1 & \xi_2 \otimes \eta_2 & \cdots & \xi_k \otimes \eta_k \\ \cdots & \cdots & \cdots & \cdots \\ \xi_1 \otimes \eta_1 & \xi_2 \otimes \eta_2 & \cdots & \xi_k \otimes \eta_k \end{pmatrix}$$

is a rank 1 matrix.

"$\Rightarrow$". By the assumption, for each $a \in L(H)$ there exists $z \in B_\perp^{(n)}$ such that the rank of $a^{(n)} + z$ is at most 1. Write $z = (z_{ij})_{k\times k}$. Then $z_{11} + \cdots + z_{kk} \in B_\perp$ and the rank of $a + z_{ii}$ is at most 1. So the rank of

$$a + \frac{1}{k}(z_{11} + \cdots + z_{kk}) = \frac{1}{k}((a + z_{11}) + \cdots + (a + z_{kk}))$$

is at most k. □

Corollary 2.6. *If B is a subalgebra of $L(H)$ and* $\dim H = k$, *then $B^{(k)} \subset L(H^{(k)})$ has property* P_1.

3. Semi-simple maximal P_1 algebras

Suppose B is a subalgebra of $M_n(\mathbb{C})$ which has property P_1. Recall that B is a maximal P_1 algebra of $M_n(\mathbb{C})$ if whenever A is a subalgebra of $M_n(\mathbb{C})$ having property P_1 and $A \supseteq B$, then $A = B$. The main result of this section is the following theorem.

Theorem 3.1. *Let $B \subseteq M_n(\mathbb{C})$ be a unital semisimple algebra. If B has property* P_1, *then* $\dim B \le n$. *Furthermore, if* $\dim B = n$, *then B is a maximal* P_1 *algebra.*

To prove this theorem, we will need the following lemmas:

Lemma 3.2. *Let $B \subseteq L(H) = M_n(\mathbb{C})$ be a semisimple algebra. If B has property* P_1, *then* $\dim B \le n$.

Proof. We will use induction on n. The case $n = 1$ is clear. Suppose this is true for $n \le k$ and let $B \subset M_{k+1}(\mathbb{C})$ be a semisimple algebra. We need to show $\dim B \le k+1$. Suppose B has a nontrivial central projection, p, $0 < p < 1$. Then, $B = pBp \oplus (1-p)B(1-p)$. By Lemma 2.1,

$$pBp \subset L(pH) \quad \text{and} \quad (1-p)B(1-p) \subset L((1-p)H),$$

are both semisimple algebras with property P_1. By the assumption of induction $\dim pBp \leq \dim(pH)$ and $\dim(1-p)B(1-p) \leq \dim(1-p)H$. Therefore,

$$\begin{aligned}\dim B &= \dim(pBp) + \dim((1-p)B(1-p)) \\ &\leq \dim pH + \dim(1-p)H \\ &= \dim H = k+1.\end{aligned}$$

Suppose B does not have a nontrivial central projection. Then, $B \cong M_r(\mathbb{C})$. Since B has property P_1, $r^2 \leq n+1$ by Lemma 2.5. So $r \leq n+1$. □

Lemma 3.3. *Suppose $0 \neq a \in M_n(\mathbb{C})$. Then there exists a finite set of operators $b_1, \ldots, b_k, c_1, \ldots, c_k$, such that $\sum_{i=1}^{k} b_i a c_i = I_n$.*

Proof. Note that $M_n(\mathbb{C})aM_n(\mathbb{C})$ is a two sided ideal of $M_n(\mathbb{C})$ and

$$M_n(\mathbb{C})aM_n(\mathbb{C}) \neq 0.$$

Since $M_n(\mathbb{C})$ is a simple algebra, $M_n(\mathbb{C})aM_n(\mathbb{C}) = M_n(\mathbb{C})$, which implies the lemma. □

The following well known lemma will be very helpful.

Lemma 3.4. *There are finitely many unitary matrices $u_1, u_2, \ldots, u_k \in M_n(\mathbb{C})$ such that $\frac{1}{k}\sum_{i=1}^{k} u_i a u_i^* = (\mathrm{Tr}(a)/n)I_n$ for all $a \in M_n(\mathbb{C})$.*

The following lemma is a special case of Lemma 3.6. However, we include its proof to illustrate our idea.

Lemma 3.5. *Suppose B is a unital subalgebra of $M_4(\mathbb{C})$ and $B \cong M_2(\mathbb{C})$, then B is a maximal* P_1 *algebra.*

Proof. We may write $M_4(\mathbb{C})$ as $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ and assume $B = M_2(\mathbb{C}) \otimes I_2$. Note that with respect to the matrix units of $I_2 \otimes M_2(\mathbb{C})$, each element of $B = M_2(\mathbb{C}) \otimes I_2$ has the following form $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, $a \in M_2(\mathbb{C})$. By Corollary 2.6, B has property P_1. Assume $B \subsetneq R \subseteq M_4(\mathbb{C})$ and R is an algebra with property P_1. We can write $R = R_1 + J$, where $R_1 \supset B$ is the semisimple part and J is the radical of R. Since R has property P_1, R_1 has property P_1. By Lemma 3.2, $\dim R_1 \leq 4$. Since $\dim B = 4$, we have $R_1 = B$.

Suppose $0 \neq x = (x_{ij})_{1 \leq i,j \leq 2} \in J$ with respect to the matrix units $I_2 \otimes M_2(\mathbb{C})$. Without loss of generality, we may assume $x_{11} \neq 0$. By Lemma 3.3, there are sets of operators $b_1, \ldots, b_k, c_1, \ldots, c_k \in M_2(\mathbb{C})$, such that

$$\sum_{i=1}^{k} b_i x_{11} c_i = I_2. \tag{1}$$

Let $y = (y_{ij})_{1\le i,j\le 2} = \sum_{i=1}^{k}(b_i \otimes I_2)x(c_i \otimes I_2) \in J$. By (1), we have $y_{11} = I_2$. Choose unitary matrices $u_1, \ldots, u_k$ as in Lemma 3.4. Let

$$z = (z_{ij}) = \sum_{i=1}^{k}(u_i \otimes I_2)y(u_i^* \otimes I_2) \in J.$$

Then, $z_{11} = I_2$ and $z_{ij} = \lambda_{ij} I_2$ for some $\lambda_{ij} \in \mathbb{C}$, $1 \le i, j \le 2$. So, $z \in I_2 \otimes M_2(\mathbb{C})$. Since $z \in J$, $z^2 = 0$, as elements in the radical are nilpotent. By the Jordan canonical theorem, there exists an invertible matrix $w \in I_2 \otimes M_2(\mathbb{C})$ such that

$$wzw^{-1} = I_2 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Replacing R by wRw^{-1}, we may assume that R contains B and $I_2 \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Furthermore, we may assume that R is the algebra generated by $M_2(\mathbb{C}) \otimes I_2$ and $I_2 \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in M_2(\mathbb{C}) \right\}.$$

Simple computation shows that R does not have property P_1. This is a contradiction. Therefore $J = 0$ and $R = B$. □

Lemma 3.6. *Let B be a unital subalgebra of $M_{n^2}(\mathbb{C})$ such that $B \cong M_n(\mathbb{C})$. Then B is a maximal* P_1 *algebra.*

Proof. We may write $M_{n^2}(\mathbb{C})$ as $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ and assume $B = M_n(\mathbb{C}) \otimes I_n$. Note that with respect to the matrix units of $I_n \otimes M_n(\mathbb{C})$, each element of $B = M_n(\mathbb{C}) \otimes I_n$ has the form

$$\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}, \quad a \in M_n(\mathbb{C}).$$

By Corollary 2.6, B has property P_1. Assume $B \subsetneq R \subseteq M_{n^2}(\mathbb{C})$ and R is an algebra with property P_1. We can write $R = R_1 + J$, where $R_1 \supset B$ is the semisimple part and J is the radical of R. Since R has property P_1, R_1 has property P_1. By Lemma 3.2, $\dim R_1 \le n^2$. Since $\dim B = n^2$, we have $R_1 = B$.

Suppose $0 \ne x = (x_{ij})_{1\le i,j\le n} \in J$ with respect to the matrix units $I_n \otimes M_n(\mathbb{C})$. Without loss of generality, we may assume $x_{11} \ne 0$. By Lemma 3.3, there are finite sets of operators $b_1, \ldots, b_k, c_1, \ldots, c_k \in M_n(\mathbb{C})$, such that

$$\sum_{i=1}^{k} b_i x_{11} c_i = I_n. \tag{2}$$

Let $y = (y_{ij})_{1\le i,j\le n} = \sum_{i=1}^{k}(b_i \otimes I_n)x(c_i \otimes I_n) \in J$. By (2), we have $y_{11} = I_n$. Choose unitary matrices $u_1, \ldots, u_k$ as in Lemma 3.4. Let

$$z = (z_{ij}) = \sum_{i=1}^{k}(u_i \otimes I_n)y(u_i^* \otimes I_n) \in J.$$

Then, $z_{11} = I_n$ and $z_{ij} = \lambda_{ij} I_n$ for some $\lambda_{ij} \in \mathbb{C}$, $1 \le i, j \le n$. So, $z \in I_n \otimes M_n(\mathbb{C})$.

Since $z \in J$, $z^n = 0$, as elements in the radical are nilpotent. By the Jordan Canonical theorem, there exists an invertible matrix $w \in I_n \otimes M_n(\mathbb{C})$ such that $0 \ne wzw^{-1} = \bigoplus_{i=1}^{k} z_i \in I_n \otimes M_n(\mathbb{C})$ and each z_i is a Jordan block with diagonal 0. Replacing R by wRw^{-1}, we may assume R contains B and $wzw^{-1} \in I_n \otimes M_n(\mathbb{C})$.

Suppose $r = \max\{\mathrm{rank} z_i : 1 \le i, \le k\}$. We may assume $\mathrm{rank} z_1 = \cdots = \mathrm{rank} z_s = r$ and rank $z_i < r$ for all $s < i \le k$. Then $z^{r-1} = I_n \otimes \left(\left(\bigoplus_{i=1}^{s} z^{r-1}\right) \oplus 0\right)$. Note that

$$z_i^{r-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We may assume R is the algebra generated by $M_n(\mathbb{C}) \otimes I_n$ and z^{r-1}.

Without loss of generality, we assume $r = 2$, and $s = n/2$. The general case can be proved similarly. Then

$$R = \left\{ \begin{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \end{pmatrix}_{s\times s} : a, b \in M_n(\mathbb{C}) \right\}.$$

Simple computations show that

$$R_\perp = \left\{ \begin{pmatrix} \begin{pmatrix} x_1 & * \\ y_1 & x_2 \end{pmatrix} & & * \\ & \ddots & \\ * & & \begin{pmatrix} x_{n-1} & * \\ y_s & x_n \end{pmatrix} \end{pmatrix}_{s\times s} : x_i, y_i \in M_n(\mathbb{C}),\ \sum_{i=1}^{n} x_i = \sum_{i=1}^{s} y_i = 0 \right\}.$$

Let

$$m = \begin{pmatrix} 0_n & 0_n \\ I_n & 0_n \end{pmatrix}.$$

Since R has property P_1, we can write $m^{(s)} = x + R_\perp$ such that the rank of x is at most 1. This implies that $I_n + y_1$, $I_n + y_2, \ldots, I_n + y_s$ are all rank-1 matrices for some $y_1, \ldots, y_s \in M_n(\mathbb{C})$ with $y_1 + \cdots + y_s = 0$. Therefore, the rank of $I_n + y_1 + I_n + y_2 + \cdots + I_n + y_s = sI_n$ is at most $s = \frac{n}{2} < n$. This is a contradiction. So $J = 0$ and $R = B$. $\square$

The following is a key lemma to prove Theorem 3.1, which has an independent interest.

Lemma 3.7. *Let* $\lambda \neq 0$ *be a complex number, and let* $y_1, y_2, \ldots, y_n \in M_n(\mathbb{C})$ *satisfy* $y_1 + y_2 + \cdots + y_n = 0$. *Suppose* $\eta_1, \eta_2, \ldots, \eta_n \in \mathbb{C}^n$ *are linearly dependent vectors, and*

$$t = \begin{pmatrix} \lambda & * & * & * \cdots & * \\ \eta_1 & I_n + y_1 & * & * \cdots & * \\ \eta_2 & * & I_n + y_2 & * \cdots & * \\ \vdots & \vdots & \vdots & \vdots \ddots & \vdots \\ \eta_n & * & * & * \cdots & I_n + y_n \end{pmatrix}.$$

Then $\operatorname{rank} t > 1$.

Proof. We may assume that $\eta_1, \ldots, \eta_{k-1}$, $k \le n$, are linearly independent vectors, and each η_j, $k \le j \le n$, can be written as a linear combination of $\eta_1, \ldots, \eta_{k-1}$. Write

$$\eta_i = \begin{pmatrix} \sigma_{i1} \\ \vdots \\ \sigma_{in} \end{pmatrix}.$$

We may assume that the $(k-1) \times (k-1)$ matrix $(\sigma_{i,j})_{(k-1)\times(k-1)}$ is invertible. Using row reduction, we can transform t to a new matrix

$$\begin{pmatrix} \lambda & * & * & * \cdots & * \\ \eta_1' & I_n + y_1' & * & * \cdots & * \\ \eta_2' & * & I_n + y_2' & * \cdots & * \\ \vdots & \vdots & \vdots & \vdots \ddots & \vdots \\ \eta_n' & * & * & * \cdots & I_n + y_n' \end{pmatrix}$$

such that the k-th row of each η_j' is 0 for $1 \le j \le n$, and $y_1' + \cdots + y_n' = 0$. So the $(jk+1, 1)$-th entry of t' is zero for all $1 \le j \le n$.

Suppose t is a rank 1 matrix. Then t' is also a rank 1 matrix. By the assumption, $\lambda \neq 0$. This implies that each entry of the $(jk+1)$-th row of t' is zero for all $1 \le j \le n$. In particular, the (k, k)-th entry of $I_n + y_j'$ is 0 for all $1 \le j \le n$. Therefore, the (k, k)-th of $I_n + y_1' + I_n + y_2' + \cdots + I_n + y_n' = nI_n$ is zero. This is a contradiction. So $\operatorname{rank} t > 1$. $\square$

The following lemma is a special case of Lemma 3.10. However, we include its proof to illustrate our idea.

Lemma 3.8. *Suppose* $\dim H = 5$ *and*

$$B = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : \lambda \in \mathbb{C}, a \in M_2(\mathbb{C}) \right\} \subset L(H) = M_5(\mathbb{C}).$$

Then, B *is a maximal* P_1 *algebra.*

Proof. Since B has a separating vector, B has property P_1 by Theorem 2.4. Suppose $B \subset R \subseteq M_5(\mathbb{C})$ and R has property P_1. We can write $R = R_1 + J$, where $R_1 \supset B$ is the semisimple part and J is the radical part. By Lemma 3.2, $B = R_1$.

Suppose $0 \neq x \in J$. Let

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_2 & 0_2 \\ 0 & 0_2 & I_2 \end{pmatrix}.$$

Then $qBq \subseteq qRq \subset B(PH) = M_4(\mathbb{C})$. By Lemma 3.5, $qBq = qRq$. This implies that we may assume

$$0 \neq x = \begin{pmatrix} 0 & \xi^T & \eta^T \\ 0 & 0_2 & 0_2 \\ 0 & 0_2 & 0_2 \end{pmatrix}, \quad \text{where } \xi, \eta \in \mathbb{C}^2.$$

Case 1. ξ and η are linearly independent vectors. Note that

$$x \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & \xi^T a & \eta^T a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R.$$

Since ξ and η are linearly independent, and $a \in M_2(\mathbb{C})$ is arbitrary, this implies that

$$R \supseteq \left\{ \begin{pmatrix} \lambda & \xi^T & \eta^T \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : \lambda \in \mathbb{C}, \xi, \eta \in \mathbb{C}^2, a \in M_2(\mathbb{C}) \right\}.$$

Simple computation shows that

$$R_\perp \subseteq \left\{ \begin{pmatrix} 0 & * & * \\ 0 & y_1 & * \\ 0 & * & y_2 \end{pmatrix} : y_1, y_2 \in M_2(\mathbb{C}), y_1 + y_2 = 0 \right\}.$$

Since R has property P_1, we can write $I_5 = x + R_\perp$ such that the rank of x is at most 1. This gives us a rank 1 matrix x of the form

$$R_\perp = \begin{pmatrix} 1 & * & * \\ 0 & y_1 + I_2 & * \\ 0 & * & y_2 + I_2 \end{pmatrix}, \quad \text{where } y_1 + y_2 = 0.$$

This contradicts Lemma 3.7.

Case 2. ξ and η are linearly dependent. Without loss of generality, assume $\eta = t\xi$. So

$$x = \begin{pmatrix} 0 & \xi^T & t\xi^T \\ 0 & 0_2 & 0_2 \\ 0 & 0_2 & 0_2 \end{pmatrix} \quad \text{and} \quad x\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & \xi^T a & t\xi^T a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\xi \neq 0$, and $a \in M_2(\mathbb{C})$ is arbitrary, this implies that

$$R \supset \left\{ \begin{pmatrix} \lambda & \xi^T & t\xi^T \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : \lambda \in \mathbb{C}, \xi \in \mathbb{C}^2, a \in M_2(\mathbb{C}) \right\}.$$

Simple computation shows that

$$R_\perp \subset \left\{ \left(\begin{smallmatrix} 0 & * & * \\ \eta_1 & y_1 & * \\ \eta_2 & * & y_2 \end{smallmatrix}\right) y_1, y_2 \in M_2(\mathbb{C}) : y_1 + y_2 = 0,\ \eta_1, \eta_2 \in \mathbb{C}^2,\ \eta_1 + t\eta_2 = 0 \right\}. \quad (3)$$

Since R has property P_1, we can write $I_5 = x + R_\perp$ such that the rank of x is at most 1. This gives us a rank 1 matrix x of the form

$$R_\perp = \begin{pmatrix} 1 & * & * \\ \eta_1 & y_1 + I_2 & * \\ \eta_2 & * & y_2 + I_2 \end{pmatrix},$$

where $\eta_1 + t\eta_2 = 0$ and $y_1 + y_2 = 0$. This contradicts Lemma 3.7. □

Lemma 3.9. *Suppose* $\{z_{ij}\}_{1\le i\le s, 1\le j\le r} \subseteq M_{sr}(\mathbb{C})$ *and* $\{c_{ji}\}_{1\le i\le s, 1\le j\le r} \subseteq M_{rs}(\mathbb{C})$ *such that*

$$\sum_{i=1}^{s}\sum_{j=1}^{r} z_{ij} a c_{ji} b = 0, \qquad \textit{for all } a \in M_r(\mathbb{C}),\ \textit{for all } b \in M_s(\mathbb{C}).$$

If $c_{ji} \neq 0$ *for some* $1 \le i \le s$, $1 \le j \le r$, *then* z_{ij} *are linearly dependent.*

Proof. We may assume $c_{11} \neq 0$ and the (1, 1) entry of c_{11} is 1. Replacing c_{ji} by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} c_{ji} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

we may assume

$$c_{ji} = \lambda_{ij} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \text{where } \lambda_{11} = 1.$$

Let z_{ij}^k be the k-th column of z_{ij}. Simple computation shows that

$$\sum_{i=1}^{s} \sum_{j=1}^{r} z_{ij} c_{ji} = 0$$

is equivalent to $\sum_{i=1}^{s} \sum_{j=1}^{r} \lambda_{ij} z_{ij}^1 = 0$. Let

$$a = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Simple computation shows that $\sum_{i=1}^{s} \sum_{j=1}^{r} z_{ij} a c_{ji} = 0$ is equivalent to

$$\sum_{i=1}^{s} \sum_{j=1}^{r} \lambda_{ij} z_{ij}^2 = 0.$$

Choosing a appropriately, we have $\sum_{i=1}^{s} \sum_{j=1}^{r} \lambda_{ij} z_{ij}^k = 0$ for all $1 \le k \le n$. This implies $\sum_{i=1}^{s} \sum_{j=1}^{r} \lambda_{ij} z_{ij} = 0$. □

Lemma 3.10. *Suppose* $\dim H = (r^2 + s^2)$ *and*

$$B = \{a^{(r)} \oplus b^{(s)} : a \in M_r(\mathbb{C}), b \in M_s(\mathbb{C})\} \subset L(H) = M_{(r^2+s^2)}(\mathbb{C}).$$

Then B is a maximal P_1 *algebra.*

Proof. Since B has a separating vector, B has property P_1 by Theorem 2.4. Suppose $B \subseteq R \subseteq M_{(r^2+s^2)}(\mathbb{C})$ and R has property P_1. We can write $R = R_1 + J$, where $R_1 \supset B$ is the semisimple part and J is the radical part. By Lemma 3.2, $B = R_1$.

Suppose $0 \neq x \in J$. Let $p = I_r^{(r)} \oplus 0$ and $q = 0 \oplus I_s^{(s)}$. Then, $pBp \subseteq pRp \subseteq B(pH)$ and pRp has property P_1. By Lemma 3.6, $pRp = pBp$. Similarly, $qRq = qBq$. So we may assume

$$0 \neq x = \begin{pmatrix} 0_r^{(r)} & c \\ 0 & 0_s^{(s)} \end{pmatrix}.$$

Write $c = (c_{ij})_{1 \le i \le r, 1 \le j \le s}$. Note that $c \neq 0$.

Suppose

$$z=\begin{pmatrix} x_1 & * & \cdots & * & * & * & \cdots & * \\ * & x_2 & \cdots & * & * & * & \cdots & * \\ & & \ddots & & & & \ddots & \\ * & * & \cdots & x_r & * & * & \cdots & * \\ z_{11} & z_{12} & \cdots & z_{1r} & y_1 & * & \cdots & * \\ z_{21} & z_{22} & \cdots & z_{2r} & * & y_2 & \cdots & * \\ & & \ddots & & & & \ddots & \\ z_{s1} & z_{s2} & \cdots & z_{sr} & * & * & \cdots & y_s \end{pmatrix} \in R_\perp.$$

Since $R_\perp \subset B_\perp$, $x_1+x_2+\cdots+x_r=0_r$ and $y_1+y_2+\cdots+y_s=0_s$. Note that

$$x(a^{(r)}\oplus b^{(s)})=\begin{pmatrix} 0_r^{(r)} & cb^{(s)} \\ 0 & 0_s^{(s)} \end{pmatrix}.$$

Since $x\in R_\perp$ and $x(a^{(r)}\oplus b^{(s)})\in R$, we have

$$\operatorname{Tr}\left(\begin{pmatrix} z_{11} & \dots & z_{1r} \\ \vdots & & \\ z_{s1} & \dots & z_{sr} \end{pmatrix}\begin{pmatrix} c_{11} & \dots & c_{1s} \\ \vdots & & \\ c_{r1} & \dots & c_{rs} \end{pmatrix}\begin{pmatrix} b & & \\ & \ddots & \\ & & b \end{pmatrix}\right)=0.$$

Simple computation shows that $\operatorname{Tr}(\sum_{i=1}^{s}\sum_{j=1}^{r} z_{ij}c_{ji}b)=0$. Since $b\in M_s(\mathbb{C})$ is an arbitrary matrix, $\sum_{i=1}^{s}\sum_{j=1}^{r} z_{ij}c_{ji}=0$.

Note that

$$(a^{(r)}\oplus 0)x(0\oplus b^{(s)})=\begin{pmatrix} 0_r^{(r)} & a^{(r)}cb^{(s)} \\ 0 & 0_s^{(s)} \end{pmatrix}=\begin{pmatrix} 0_r^{(r)} & (ac_{ij}b)_{1\le i\le r,1\le j\le s} \\ 0 & 0_s^{(s)} \end{pmatrix}.$$

By similar arguments as above, we have $\sum_{i=1}^{s}\sum_{j=1}^{r} z_{ij}ac_{ji}b=0$ for all $a\in M_r(\mathbb{C})$ and $b\in M_s(\mathbb{C})$. By Lemma 3.9, this implies that $\{z_{ij}\}_{1\le i\le s,1\le j\le r}$ are linearly dependent matrices.

Since R has property P_1, $I_{r^2+s^2}=x+R_\perp$ for some x such that the rank of x is at most 1. So x is a matrix of the form

$$\begin{pmatrix} I_r+x_1 & * & \cdots & * & * & * & \cdots & * \\ * & I_r+x_2 & \cdots & * & * & * & \cdots & * \\ & & \ddots & & & & \ddots & \\ * & * & \cdots & I_r+x_r & * & * & \cdots & * \\ z_{11} & z_{12} & \cdots & z_{1r} & I_s+y_1 & * & \cdots & * \\ z_{21} & z_{22} & \cdots & z_{2r} & * & I_s+y_2 & \cdots & * \\ & & \ddots & & & & \ddots & \\ z_{s1} & z_{s2} & \cdots & z_{sr} & * & * & \cdots & I_s+y_s \end{pmatrix}.$$

Since x is a rank 1 matrix, $(z_{ij})_{1\le i\le s,1\le j\le r}$ are rank 1 matrices. So there are $\xi_1,\dots,\xi_s\in\mathbb{C}^s$, $\eta_1,\dots,\eta_r\in\mathbb{C}^r$ such that $z_{ij}=\xi_i\otimes\eta_j$ for $1\le i\le s$ and $1\le j\le r$. Since $\{z_{ij}\}_{1\le i\le s,1\le j\le r}$ are linearly dependent matrices, either $\{\xi_i\}_{i=1}^s$ are linearly dependent or $\{\eta_j\}_{j=1}^r$ are linearly dependent. Without loss of generality, assume $\{\xi_i\}_{i=1}^s$ are linearly dependent. Now, x is a matrix of the form

$$\begin{pmatrix}
I_r+x_1 & * & \cdots & * & * & * & \cdots & * \\
* & I_r+x_2 & \cdots & * & * & * & \cdots & * \\
 & & \ddots & & & & \ddots & \\
* & * & \cdots & I_r+x_r & * & * & \cdots & * \\
\xi_1\otimes\eta_1 & \xi_1\otimes\eta_2 & \cdots & \xi_1\otimes\eta_r & I_s+y_1 & * & \cdots & * \\
\xi_2\otimes\eta_1 & \xi_2\otimes\eta_2 & \cdots & \xi_2\otimes\eta_r & * & I_s+y_2 & \cdots & * \\
 & & \ddots & & & & \ddots & \\
\xi_s\otimes\eta_1 & \xi_s\otimes\eta_1 & \cdots & \xi_s\otimes\eta_r & * & * & \cdots & I_s+y_s
\end{pmatrix}.$$

Since $x_1+\cdots+x_r=0$, one entry of I_r+x_i is not zero for some $1\le i\le r$. We may assume the (1, 1) entry of I_r+x_1 is $\lambda\ne 0$. Let

$$\eta_1=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_r\end{pmatrix}.$$

Then the matrix

$$\begin{pmatrix}
\lambda & * & \cdots & * \\
\alpha_1\xi_1 & I_s+y_1 & \cdots & * \\
\vdots & & \ddots & \\
\alpha_1\xi_s & * & \cdots & I_s+y_s
\end{pmatrix}$$

has rank 1 since it is a submatrix of x. This contradicts Lemma 3.7. So $R=B$. $\square$

Proof of Theorem 3.1. By Lemma 3.2, if B has P_1, then $\dim B\le n$. Assume B has property P_1, and $\dim B=n$. We claim $B=\bigoplus_{i=1}^r M_{n_i}(\mathbb{C})^{(n_i)}$ and $n=\sum_{i=1}^r n_i^2$. We will proceed by induction on n. If $n=1$, this is clear. Assume our claim is true for $n\le k$. Let $B\subseteq M_{k+1}(\mathbb{C})$ be a semisimple P_1 algebra and $\dim B=k+1$. Suppose B has a nontrivial central projection p, $0<p<1$. Then, $B=pBp\oplus(1-p)B(1-p)$. By Lemma 2.1, $pBp\subseteq B(pH)$ and $(1-p)B(1-p)\subseteq B((1-p)H)$ are both semisimple algebras with property P_1. By Lemma 3.2, $\dim(pBp)=\dim(pH)$ and $\dim((1-p)B(1-p))=\dim((1-p)H)$. By induction, $pBp=\bigoplus_{i=1}^{r_1} M_{n_i}(\mathbb{C})^{(n_i)}$, $(1-p)B(1-p)=\bigoplus_{i=1}^{r_2} M_{m_i}(\mathbb{C})^{(m_i)}$, and $\sum_{i=1}^{r_1} n_i^2+\sum_{i=1}^{r_2} m_i^2=k+1$. Suppose B does not have a nontrivial central projection. Then $B=M_r(\mathbb{C})\subseteq M_{n+1}(\mathbb{C})$ and $\dim B=r^2=n+1$ by Lemma 2.5.

Suppose $B \subsetneq R \subseteq M_k(\mathbb{C}) \in L(H)$ and R is an algebra with property P_1. Let $0 \neq x \in R \setminus B$. Note that $B = \bigoplus_{i=1}^r M_{n_i}(\mathbb{C})^{(n_i)}$. Let p_i be the projection of B that corresponds to the summand $M_{n_i}(\mathbb{C})^{(n_i)}$. Then, we have $p_i B p_i \subseteq p_i R p_i \subseteq L(p_i H)$ and $p_i R p_i$ has property P_1. By Lemma 3.6, $p_i R p_i = p_i B p_i$. So we may assume

$$0 \neq x = \begin{pmatrix} 0_{n_1}^{(n_1)} & x_{12} & x_{13} & \cdots & x_{1n_r} \\ & 0_{n_2}^{(n_2)} & x_{23} & \cdots & x_{2n_r} \\ & & \ddots & & \vdots \\ & & & 0_{n_{r-1}}^{(n_{r-1})} & x_{r-1r} \\ 0 & & & & 0_{n_r}^{(n_r)} \end{pmatrix}.$$

We may assume that $x_{12} \neq 0$. Then

$$(p_1 + p_2)x(p_1 + p_2) \in (p_1 + p_2)R(p_1 + p_2) \setminus (p_1 + p_2)B(p_1 + p_2).$$

By Lemma 2.1, $(p_1 + p_2)R(p_1 + p_2)$ has property P_1. By Lemma 3.10,

$$(p_1 + p_2)B(p_1 + p_2) = M_{n_1}(\mathbb{C})^{(n_1)} \oplus M_{n_2}(\mathbb{C})^{(n_2)}$$

is a maximal P_1 algebra. This is a contradiction. So B is a maximal P_1 algebra. □

4. Singly generated maximal P_1 algebras

In this section, we prove the following result.

Theorem 4.1. *Suppose B is a singly generated unital subalgebra of $M_n(\mathbb{C})$ and* $\dim B = n$. *Then B is a maximal* P_1 *algebra.*

To prove Theorem 4.1, we need several lemmas. Let J_n be the $n \times n$ Jordan block.

Lemma 4.2. *Let B be the unital subalgebra of $M_n(\mathbb{C})$ generated by the Jordan block J_n. If $N \supset B$ is a subalgebra of the upper-triangular algebra of $M_n(\mathbb{C})$ and N has property* P_1, *then $N = B$.*

Proof. Suppose $N \supsetneq B$ is a subalgebra of the upper-triangular algebra and N has property P_1. Note that

$$B = \left\{ \sum_{k=0}^{n-1} \lambda_k (J_n)^k : \lambda_0, \ldots, \lambda_{n-1} \in \mathbb{C} \right\}.$$

A special case. Suppose N contains an operator x of the following form

$$x = \begin{pmatrix} 0 & \cdots & 0 & \lambda & 0 \\ & 0 & \cdots & 0 & \eta \\ & & 0 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 0 \end{pmatrix}, \tag{4}$$

where $\lambda \neq \eta$. Then N contains the algebra generated by B and x. Therefore,

$$N \supset \left\{ \begin{pmatrix} \lambda_1 & \cdots & \lambda_{n-2} & \alpha & \gamma \\ & \lambda_1 & \cdots & \lambda_{n-2} & \beta \\ & & \lambda_1 & \cdots & \lambda_{n-2} \\ & & & \ddots & \vdots \\ & & & & \lambda_1 \end{pmatrix} : \lambda_1, \ldots, \lambda_{n-2}, \alpha, \beta, \gamma \in \mathbb{C} \right\}.$$

Simple computation shows that

$$N_\perp \subset \left\{ \begin{pmatrix} * & \cdots & * & 0 & 0 \\ & * & \cdots & * & 0 \\ & & * & \cdots & * \\ & & & \ddots & \vdots \\ & & & & * \end{pmatrix} \right\}.$$

It is easy to see that the operator $(J_n)^{n-2}$ can not be written as a sum of a rank one operator and an operator in $N_\perp$. This contradicts the assumption that N has property P_1.

The general case. Suppose $z \in N \setminus B$. By the assumption of the lemma, $z = (z_{i,j})_{n\times n}$ is an upper-triangular matrix. Since $z \notin B$, we may assume that

$$z_{j,j+k-1} \neq z_{j+r,j+r+k-1}$$

for some positive integers j, k, r, and $z_{s,t} = 0$ for $t < s + k - 1$. Without loss of generality, we assume that $z_{1,k} \neq z_{2,1+k}$ and $1 \leq k \leq n-1$. If $k = n-1$, then this implies that N contains an x as in (4). If $k < n-2$, then $(J_n)^{k+1}z$ (or consider $z(J_n)^{k+1}$ if $z_{n-1,n-1} \neq z_{n,n}$) is a matrix in N. If we write

$$(J_n)^{k+1}z = (y_{ij})_{n\times n}.$$

Then $y_{1,k+1} \neq y_{2,k+2}$ and $y_{s,t} = 0$ for $t < s + k$. Repeating the above arguments, we can see that N contains an x as in (4). This completes the proof. □

Lemma 4.3. *Let B be the unital subalgebra of $M_n(\mathbb{C})$ generated by the Jordan block J_n. Then B is a maximal* P_1 *algebra.*

Proof. Suppose $N \supset B$ is a subalgebra of $M_n(\mathbb{C})$ and N has property P_1. By Wedderburn's theorem,

$$N = M_{n_1}(\mathbb{C}) \oplus \cdots M_{n_s}(\mathbb{C}) \oplus J,$$

where J is the radical of N.

Case 1. $n_1 = \cdots = n_s = 1$. Then N is triangularizable, that is, there exists a unitary matrix $u \in M_n(\mathbb{C})$ such that uNu^* is contained in the algebra of upper-triangular matrices (see [Christensen 1999, Proposition 2.5]). Since $J_n \in B \subset N$, uJ_nu^* is a strictly upper-triangular matrix. Simple computation shows that u has to be a diagonal matrix. Therefore, $N = u^*(uNu^*)u$ is contained in the algebra of upper-triangular matrices. Since N has property P_1, $N = B$ by Lemma 4.2.

Case 2. Suppose $n_i \geq 2$ for some i, $1 \leq i \leq s$. Choose a nonzero partial isometry $v \in M_{n_i}(\mathbb{C})$ such that $v^2 = 0$. Then either $v \notin B$ or $v^* \notin B$ since B does not contain any nontrivial projections. We may assume that $v \notin B$. Consider the subalgebra $\tilde{N}$ generated by v and B. An element of $\tilde{N}$ can be written as $b_1vb_2v\cdots vb_n$, where $b_i \in J$ for $2 \leq i \leq n-1$, $b_1 = 1$ or $b_1 \in J$, $b_n = 1$ or $b_n \in J$. By Lemma 2.1 of [Christensen 1999], $\tilde{N} = \mathbb{C}1 \oplus \tilde{J}$, where $\tilde{J}$ is the radical part of $\tilde{N}$ such that $v \in \tilde{J}$. Note that $\tilde{N}$ also has property P_1. By Case 1, $\tilde{N} = B$. So $v \in B$. This is a contradiction. □

Lemma 4.4. *Let $B_i \subset M_{n_i}(\mathbb{C})$ be the unital subalgebra generated by the Jordan block J_{n_i} for $i = 1, 2$. Then $B = B_1 \oplus B_2$ is a maximal* P_1 *subalgebra of $M_{n_1+n_2}(\mathbb{C})$.*

Proof. Suppose $B \subsetneq N \subset M_{n_1+n_2}(\mathbb{C})$ and N has property P_1. Let p_i be the central projections of B corresponding to B_i. Then $B_1 \subset p_1Np_1 \subset M_{n_1}(\mathbb{C})$ and p_1Np_1 has property P_1. By Lemma 4.3, $p_1Np_1 = B_1$. Similarly, $p_2Np_2 = B_2$. Suppose $x \in N \setminus B$. Then we may assume that $0 \neq x = p_1xp_2$. With respect to matrix units of $M_{n_1}(\mathbb{C})$ and $M_{n_2}(\mathbb{C})$, we can write x as

$$x = \begin{pmatrix} 0 & (x_{ij})_{n_1 \times n_2} \\ 0 & 0 \end{pmatrix},$$

where $(x_{ij})_{n_1 \times n_2}$ is a nonzero matrix. Multiplying on the left by a suitable matrix of B, we may assume that $x_{ij} = 0$ for all $i \geq 2$ (which can be easily seen for the case $n_2 = 1$, other cases are similar). Multiplying on the right by another suitable matrix of B, we may further assume that $x_{1,n_2} = 1$ and $x_{1,j} = 0$ for $1 \leq j \leq n_2 - 1$.

So we may assume that

$$x = \begin{pmatrix} 0_{n_1\times n_1} & \begin{pmatrix} 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix}_{n_1\times n_2} \\ 0 & 0_{n_2\times n_2} \end{pmatrix}.$$

Let $\tilde{N}$ be the algebra generated by B and x above. Then

$$\tilde{N} = \left\{ \begin{pmatrix} \begin{pmatrix} \lambda_1 & \cdots & \lambda_{n_1} \\ & \ddots & \vdots \\ 0 & & \lambda_1 \end{pmatrix} & \begin{pmatrix} 0 & \cdots & \alpha \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix}_{n_1\times n_2} \\ 0 & \begin{pmatrix} \eta_1 & \cdots & \eta_{n_2} \\ & \ddots & \vdots \\ 0 & & \eta_1 \end{pmatrix} \end{pmatrix} : \lambda_i, \eta_j, \alpha \in \mathbb{C} \right\}.$$

Simple computation shows that

$$\tilde{N}_\perp \subset \left\{ \begin{pmatrix} \begin{pmatrix} * & \cdots & 0 \\ & \ddots & \vdots \\ * & & * \end{pmatrix} & \begin{pmatrix} * & \cdots & 0 \\ * & \cdots & * \\ \cdots & \cdots & \cdots \\ * & \cdots & * \end{pmatrix} \\ * & \begin{pmatrix} * & \cdots & 0 \\ & \ddots & \vdots \\ * & & * \end{pmatrix} \end{pmatrix} \right\}.$$

Let

$$y = \begin{pmatrix} \begin{pmatrix} 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} & 0_{n_1\times n_2} \\ 0 & \begin{pmatrix} 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} \end{pmatrix}.$$

It is easy to see that the operator y cannot be written as a sum of a rank one operator and an operator in $\tilde{N}_\perp$. This contradicts the fact that $\tilde{N}$ has property P_1. $\square$

Proof of Theorem 4.1. Suppose B is generated by a matrix T. By the Jordan canonical form theorem, we may assume that $T = \bigoplus_{i=1}^{r}(\lambda_i + J_{n_i})$ and $\sum_{i=1}^{r} n_i = n$. Note that $\dim(B) = n$ if and only if $\lambda_i \neq \lambda_j$ for $i \neq j$, and if and only if

$B = \bigoplus_{i=1}^{r} B_i$, where each B_i is the subalgebra of $M_{n_i}(\mathbb{C})$ generated by the Jordan block J_{n_i}.

Suppose $B \subsetneq N \subset M_n(\mathbb{C})$ and N has property P_1. Let p_i be the central projection of B corresponding to B_i. Then $B_i \subset p_i N p_i \subset M_{n_i}(\mathbb{C})$ and $p_i N p_i$ has property P_1. By Lemma 4.3, $B_i = p_i N p_i$. Since $B \neq N$, there is an element $0 \neq x \in N$ such that $x = p_i x p_j$ for some $i \neq j$. Without loss of generality, we may assume that $0 \neq x = p_1 x p_2$. Now we have $B_1 \oplus B_2 \subsetneq (p_1 + p_2) N (p_1 + p_2) \subseteq M_{n_1+n_2}(\mathbb{C})$ and $(p_1 + p_2) N (p_1 + p_2)$ also has property P_1. On the other hand, by Lemma 4.4, $B_1 \oplus B_2 = (p_1 + p_2) N (p_1 + p_2)$. This is a contradiction. □

5. P_1 algebras in $M_n(\mathbb{C})$, $n \leq 4$

Let B be a subalgebra of $M_n(\mathbb{C})$. Then $B = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C}) \oplus J$, where J is the radical part of B. If $n_1, \ldots, n_s = 1$, then B is upper-triangularizable, that is, there exists a unitary matrix u such that uBu^* is a subalgebra of the upper-triangular algebra of $M_n(\mathbb{C})$ (see [Christensen 1999, Proposition 2.5] or [Humphreys 1972, Corollary A, page 17]). The following lemma will be useful.

Lemma 5.1. [Azoff] *Let S be a subspace of $L(H)$ and consider the subalgebras of $L(H^{(2)})$ defined by*

$$B = \left\{ \begin{pmatrix} \lambda e & a \\ 0 & \lambda e \end{pmatrix} : \lambda \in \mathbb{C}, a \in S \right\}, \quad C = \left\{ \begin{pmatrix} \lambda e & a \\ 0 & \mu e \end{pmatrix} : \lambda, \mu \in \mathbb{C}, a \in S \right\}.$$

(1) *B has property* P_1 *if and only if S has property* P_1.

(2) *C has property* P_1 *if and only if S has property* P_1 *and is intransitive.*

Proposition 5.2. *Let B be a unital subalgebra of $M_2(\mathbb{C})$ with property* P_1. *Then B is unitarily equivalent to one of the following three subalgebras*:

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \eta \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}.$$

Proof. It is easy to verify that the above algebras have property P_1. Suppose B has property P_1. Then the semisimple part of B must be abelian. Conjugating by a unitary matrix, we may assume that B is a subalgebra of the algebra of upper-triangluar matrices. Note that the algebra of upper-triangular matrices does not have property P_1. So B must be one of the algebras listed in the lemma. □

Proposition 5.3. *Let B be a unital subalgebra of $M_3(\mathbb{C})$ with property* P_1. *Then either B or B^* has a separating vector. Therefore*, $\dim B \leq 3$. *Furthermore*, *if*

$\dim B = 3$, *then* B *is similarly conjugate to one of the following algebras*

$$A_1 = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}, \ A_2 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\},$$

$$A_3 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_3 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}, \ A_4 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\},$$

$$A_5 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}, \ A_6 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 \\ 0 & \lambda_1 & \lambda_3 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

Proof. Suppose B has property P_1. Then the semisimple part of B must be abelian. Conjugating by a unitary matrix, we may assume that B is a subalgebra of the algebra of upper-triangluar matrices. We consider the following cases.

Case 1. Suppose the semisimple part of B is $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. Then $B = A_1$ by Theorem 3.1.

Case 2. Suppose the semisimple part of B is $\mathbb{C} \oplus \mathbb{C}$. We may assume that the semisimple part of B consists of matrices

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

We consider two subcases.

Subcase 2.1. Suppose B is contained in the following algebra

$$B_1 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_1 & \lambda_4 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\}.$$

Simple computation shows that B_1 does not have property P_1 (the identity matrix can not be written as $x + (B_1)_\perp$ such that the rank of x is at most 1). So B is a proper subalgebra of B_1. This implies that there exist α, β such that

$$B_1 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_3\alpha \\ 0 & \lambda_1 & \lambda_3\beta \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

If $\alpha \neq 0$, let

$$s = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Simple computation shows that $sA_2s^{-1} = B$, that is, $s^{-1}Bs = A_2$. If $\alpha = 0$, $\beta \neq 0$, let

$$s = \begin{pmatrix} 0 & 1 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $sA_2s^{-1} = B$, that is, $s^{-1}Bs = A_2$. If $\alpha = \beta = 0$, then clearly B has a separating vector.

Subcase 2.2. Suppose B is not contained in B_1. Since B is an algebra, B contains A_3. It is easy to see that A_3 is the algebra generated by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\dim A_3 = 3$. So $B = A_3$ by Theorem 4.1.

Case 3. Suppose the semisimple part of B is $\mathbb{C}$. Then B is contained in the following algebra

$$B_3 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & \lambda_4 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that B_3 does not have property P_1. So B is a proper subalgebra of B_3. We consider the following subcases.

Subcase 3.1. Suppose B contains an element

$$b = \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix},$$

such that $\alpha \neq 0$ and $\beta \neq 0$. Conjugating by an invertible upper-triangular matrix, we may assume that $b = J_3$ is the Jordan block. So B contains A_4. By Theorem 4.1, $B = A_4$.

Subcase 3.2. Suppose B does not contain an element b as in subcase 3.2. Then $B \subseteq A_5$ or $B \subseteq A_6$. Note that A_5^* has a separating vector and A_6 has a separating vector. So both A_5 and A_6 have property P_1. $\square$

Lemma 5.4. *Let*

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\} \subset M_4(\mathbb{C}).$$

Then B is a maximal P_1 *algebra.*

Proof. Note that B^* has a separating vector. So B has property P_1. Suppose $A \supsetneq B$ is a P_1 algebra. Suppose A contains a matrix

$$a_1 = \begin{pmatrix} 0 & \alpha & * & * \\ 0 & 0 & \beta & * \\ 0 & 0 & \lambda_1 & \gamma \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix},$$

such that $\gamma \neq 0$. Since $B \subset A$, we may assume that $\alpha \neq 0$ and $\beta \neq 0$. Conjugating by an upper-triangular invertible matrix, we may assume that A contains the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So A is the algebra generated by the Jordan block by Theorem 4.1 and $\dim A = 4$. However, $\dim B = 4$ and $B \subsetneq A$. This is a contradiction.

Therefore, A is contained in

$$\left\{ \begin{pmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_1 & * & * \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1 \in \mathbb{C} \right\}.$$

Since A is an algebra containing B and $A \neq B$, we may assume that A contains a matrix of the following form

$$a_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix},$$

where either $s \neq 0$ or $t \neq 0$. Furthermore, we can assume that $s = 1$ and $t \neq 0$. Let A_1 be the algebra generated by B and a_2. Then

$$A_1 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & \lambda_2 + \lambda_5 & t\lambda_5 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \ldots, \lambda_5 \in \mathbb{C} \right\}.$$

Simple computation shows that the predual space of A_1 is

$$\left\{ \begin{pmatrix} \eta_1 & * & * & * \\ t\eta_5 & \eta_2 & * & * \\ 0 & -t\eta_5 & \eta_3 & 0 \\ 0 & \eta_5 & 0 & \eta_4 \end{pmatrix} : \eta_1, \dots, \eta_4 \in \mathbb{C}, \eta_1 + \eta_2 + \eta_3 + \eta_4 = 0 \right\}.$$

It is easy to show that the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -t & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

cannot be written as $x + (A_1)_\perp$ such that the rank of x is at most 1. This is a contradiction. So B is a maximal P_1 algebra. □

Proposition 5.5. *Let B be a unital subalgebra of $M_4(\mathbb{C})$ with property* P_1. *Then B satisfies one of the following conditions*:

(i) *B has a separating vector.*

(ii) *B^* has a separating vector.*

(iii) *B is similarly conjugate to an algebra of the form*

$$\left\{ \begin{pmatrix} \lambda I_2 & s \\ 0 & \eta I_2 \end{pmatrix} : \lambda, \eta \in \mathbb{C}, s \in S \right\},$$

where S is a subspace of $M_2(\mathbb{C})$ with dimension 2.

In particular, $\dim B \le 4$.

Proof. Suppose B has property P_1. Then the semisimple part of B must be $M_2(\mathbb{C})$ or abelian. If the semisimple part of B is $M_2(\mathbb{C})$, then $B = M_2(\mathbb{C})^{(2)}$ by Theorem 3.1. So B has a separating vector. Suppose the semisimple part of B is abelian. Conjugating by a unitary matrix, we may assume that B is a subalgebra of the algebra of upper triangluar matrices. We consider the following cases.

Case 1. Suppose the semisimple part of B is $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. Then

$$B = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}$$

by Theorem 3.1. So B has a separating vector.

Case 2. Suppose the semisimple part of B is $\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$. We may assume that the semisimple part of B consists of matrices

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}.$$

Let

$$e_1 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}.$$

By Lemma 2.1, $(e_2+e_3)B(e_2+e_3) \subset M_2(\mathbb{C})$ has property P_1. By Theorem 3.1 and the assumption of Case 2,

$$(e_2+e_3)B(e_2+e_3) = \left\{ \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix} : \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

We consider two subcases.

Subcase 2.1. Suppose B is contained in the following algebra

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & \lambda_6 \\ 0 & \lambda_1 & \lambda_5 & \lambda_7 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \ldots, \lambda_7 \in \mathbb{C} \right\}.$$

By Lemma 2.1, $(e_1+e_2)B(e_1+e_2) \subset M_3(\mathbb{C})$ has property P_1. Note that

$$(e_1+e_2)B(e_1+e_2) \subseteq \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 \\ 0 & \lambda_1 & \lambda_5 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \ldots, \lambda_5 \in \mathbb{C} \right\}.$$

By the proof of Subcase 2.1 of Proposition 5.3, there exists an invertible matrix

$$s = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix},$$

such that

$$s^{-1}[(e_1+e_2)B(e_1+e_2)]s \subseteq \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

Conjugating by $(s \oplus 1)^{-1} \in M_4(\mathbb{C})$, we may assume that B is contained in the algebra

$$B_1 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & \lambda_5 \\ 0 & \lambda_1 & 0 & \lambda_6 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \ldots, \lambda_6 \in \mathbb{C} \right\}.$$

It is easy to see that B_1 is similarly conjugate to the algebra

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_5 & 0 \\ 0 & \lambda_1 & \lambda_6 & \lambda_4 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \ldots, \lambda_6 \in \mathbb{C} \right\}.$$

So we may assume that

$$B_1 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & 0 \\ 0 & \lambda_1 & \lambda_5 & \lambda_6 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \ldots, \lambda_6 \in \mathbb{C} \right\}.$$

Repeating the above arguments, we may assume that B is contained in the algebra

$$B_2 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & 0 \\ 0 & \lambda_1 & 0 & \lambda_5 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \ldots, \lambda_5 \in \mathbb{C} \right\}.$$

Simple computation shows that B_2 does not have property P_1 (the identity matrix can not be written as $x + (B_2)_\perp$ such that the rank of x is at most 1). So B is a proper subalgebra of B_2. Therefore, there exist α, β such that

$$B = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4\alpha & 0 \\ 0 & \lambda_1 & 0 & \lambda_4\beta \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\}.$$

If $\alpha = \beta = 0$, then clearly B has a separating vector.

If $\alpha \neq 0$ and $\beta \neq 0$, let

$$t = \begin{pmatrix} \alpha^{-1} & 0 & 0 & 0 \\ 0 & \beta^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Simple computation shows that

$$tBt^{-1} = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & 0 \\ 0 & \lambda_1 & 0 & \lambda_4 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

So B has a separating vector.

If $\alpha \neq 0$, $\beta = 0$ or $\alpha = 0$, $\beta \neq 0$, then B is similarly conjugate to the algebra

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

So B has a separating vector.

Subcase 2.2. Suppose B is not contained in B_1. Since B is an algebra, B contains the algebra

$$B_3 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_4 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that B_3 is the algebra generated by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

and $\dim B_3 = 4$. So $B = B_3$ by Theorem 4.1 and B has a separating vector.

Case 3. Suppose the semisimple part of B is $\mathbb{C} \oplus \mathbb{C}$.

Subcase 3.1. Suppose B contains the following subalgebra

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{C} \right\}.$$

Let

$$f_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 2.1, $f_i B f_i \subset M_2(\mathbb{C})$ has property P_1. By Proposition 5.2,

$$f_i B f_i = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\} \quad \text{or} \quad f_i B f_i = \left\{ \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}.$$

We consider the following subsubcases.

Subsubcase 3.1.1. Suppose

$$f_1 B f_1 = f_2 B f_2 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\}.$$

This implies that

$$B \subset \left\{ \begin{pmatrix} \lambda I_2 & * \\ 0 & \eta I_2 \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}.$$

By Lemma 5.1,

$$B = \left\{ \begin{pmatrix} \lambda I_2 & S \\ 0 & \eta I_2 \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\},$$

where S has property P_1 and is intransitive. By [Azoff 1973, Table 5A, page 34], S is equivalent to one of the following spaces: zero space, or

$$\left\{ \begin{pmatrix} \zeta & 0 \\ 0 & 0 \end{pmatrix} : \zeta \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} : \zeta \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \zeta & \xi \\ 0 & 0 \end{pmatrix} : \zeta, \xi \in \mathbb{C} \right\},$$
$$\left\{ \begin{pmatrix} \zeta & 0 \\ \xi & 0 \end{pmatrix} : \zeta, \xi \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \xi \end{pmatrix} : \zeta, \xi \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \zeta & \xi \\ 0 & \zeta \end{pmatrix} : \zeta, \xi \in \mathbb{C} \right\}.$$

Note that in the last four cases, neither B nor B^* has a separating vector.

Subsubcase 3.1.2. Suppose

$$f_1 B f_1 = f_2 B f_2 = \left\{ \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}.$$

This implies that B contains the following subalgebra

$$B_4 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_4 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that B_4 is the algebra generated by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\dim B_4 = 4$. So $B = B_4$ by Theorem 4.1, and B has a separating vector.

Subsubcase 3.1.3. Suppose

$$f_1Bf_1 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\} \quad \text{and} \quad f_2Bf_2 = \left\{ \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}.$$

If $\dim B > 3$, then B contains a nonzero matrix

$$b = \begin{pmatrix} 0_2 & a \\ 0_2 & 0_2 \end{pmatrix}.$$

Let B_5 be the subalgebra generated by f_1Bf_1, f_2Bf_2 and b. Then $\dim B_5 = 4$ and B_5 is the algebra generated by the matrix

$$\begin{pmatrix} 0_2 & a \\ 0_2 & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

So $B = B_5$ by Theorem 4.1 and

$$B = \left\{ \begin{pmatrix} \lambda_1 I_2 & \lambda_4 a \\ 0_2 & \begin{pmatrix} \lambda_2 & \lambda_3 \\ 0 & \lambda_2 \end{pmatrix} \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\},$$

where a is a 2×2 matrix. Let

$$t = \begin{pmatrix} b & 0 \\ 0_2 & I_2 \end{pmatrix}.$$

Then

$$tBt^{-1} = \left\{ \begin{pmatrix} \lambda_1 I_2 & \lambda_4 ba \\ 0_2 & \begin{pmatrix} \lambda_2 & \lambda_3 \\ 0 & \lambda_2 \end{pmatrix} \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\}.$$

So we can choose b appropriately such that $ba = 0_2$, or $ba = I_2$, or

$$ba = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad ba = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad ba = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad ba = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

In each case, B has a separating vector.

Subcase 3.2. Suppose B contains the following subalgebra

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{C} \right\}.$$

Let

$$p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 2.1, $pBp \subset M_3(\mathbb{C})$ has property P_1. By Proposition 5.2,

$$pBp = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}$$

or

$$pBp = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{C} \right\}.$$

We consider the following subsubcases.

Subsubcase 3.2.1. Suppose

$$pBp = \left\{ \begin{pmatrix} \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_2 & \lambda_3 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C} \right\}.$$

Then B contains the following subalgebra

$$B_6 = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & 0 & \lambda_2 & \lambda_3 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that B_6 is the algebra generated by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\dim B_6 = 4$. So $B = B_6$ by Theorem 4.1, and B has a separating vector.

Subsubcase 3.2.2. Suppose

$$pBp = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{C} \right\}.$$

If $\dim B > 3$, then B contains a nonzero matrix

$$b = \begin{pmatrix} 0 & a \\ 0 & 0_3 \end{pmatrix}.$$

Let B_7 be the subalgebra generated by $(1-p)B(1-p)$, pBp and b. Then $\dim B_7 = 4$ and B_7 is the algebra generated the matrix

$$\begin{pmatrix} 0 & a \\ 0 & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}.$$

So $B = B_7$ by Theorem 4.1 and

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_4 a \\ 0 & \begin{pmatrix} \lambda_2 & 0 & \lambda_3 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\}.$$

Conjugating by an appropriate invertible matrix

$$t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & * & * \\ 0 & 0 & \eta & * \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

we have

$$tBt^{-1} = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 & \lambda_3 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\},$$

$$tBt^{-1} = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 & 0 \\ 0 & \lambda_2 & 0 & \lambda_3 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\},$$

or

$$tBt^{-1} = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & \lambda_2 \\ 0 & \lambda_2 & 0 & \lambda_3 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\}.$$

In each case, B^* has a separating vector.

Case 4. Suppose the semisimple part of B is $\mathbb{C}$. Consider matrices in B with the form

$$b = \begin{pmatrix} 0 & \alpha & * & * \\ 0 & 0 & \beta & * \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Subcase 4.1. B contains a matrix b with $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$. Conjugating by an upper-triangular invertible matrix, we may assume that B contains the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So B is the algebra generated by the Jordan block by Theorem 4.1. Note that B has a separating vector.

Subcase 4.2. B does not contain a matrix b as in Subcase 4.1 and B contains a matrix b with two elements of α, β, γ nonzero. We may assume that $\alpha \neq 0$ and $\beta \neq 0$. Conjugating by an upper-triangular invertible matrix, we may assume that B contains the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and therefore} \quad B \supseteq \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

By the assumption of Subcase 4.2, we have

$$B \subset \left\{ \begin{pmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_1 & * & * \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1 \in \mathbb{C} \right\}. \tag{5}$$

Subsubcase 4.2.1. Suppose the $(2,4)$-entry of every matrix in B is zero. Then B is contained in the algebra

$$B_8 \subset \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_4 & \lambda_5 \\ 0 & \lambda_1 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \ldots, \lambda_5 \in \mathbb{C} \right\}.$$

Simple computation shows that B_8 does not have property P_1. So B is a proper subalgebra of B_8. By (5), there exist α, β such that

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4\alpha \\ 0 & \lambda_1 & \lambda_2 + \lambda_4\beta & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\}.$$

If $\alpha = 0$ and $\beta \neq 0$, then B does not have property P_1. So we may assume that $\alpha \neq 0$. It is easy to see that B^* has a separating vector.

Subsubcase 4.2.2. Suppose the $(2, 4)$-entry of a matrix in B is not zero. By (5), B contains an element

$$b = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \beta & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\gamma \neq 0$. Since B is an algebra, B contains

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} b = \begin{pmatrix} 0 & 0 & \beta & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By (5), B contains

$$\begin{pmatrix} 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since B is an algebra, B contains the subalgebra

$$B_9 \subseteq \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\}.$$

By Lemma 5.4, B_9 is a maximal P_1 algebra. Hence, $B = B_9$ and B^* has a separating vector.

Subcase 4.3. B does not contain a matrix b as in subcase 4.1, subcase 4.2, and B contains a matrix b with one element of α, β, γ nonzero. We may assume that $\alpha \neq 0$. Conjugating by an upper-triangular invertible matrix, we may assume that

B contains the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By the assumption of subcase 4.3, B is contained in the algebra

$$B_{10} = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & 0 & \lambda_5 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_5 \in \mathbb{C} \right\}.$$

Simple computation shows that B_{10} does not have property P_1. So B is a proper subalgebra of B_{10}. We consider the following subsubcases.

Subsubcase 4.3.1. . If the $(1, 3)$ entry of each element of B is zero, then B is contained in the algebra

$$B_{11} = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & \lambda_3 \\ 0 & \lambda_1 & 0 & \lambda_4 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

Simple computation shows that B_{11} does not have property P_1. So there exist α, β such that

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & \lambda_3\alpha \\ 0 & \lambda_1 & 0 & \lambda_3\beta \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

If $\beta = 0$, then B^* has a separating vector. If $\beta \neq 0$, then B has a separating vector.

Subsubcase 4.3.2. If the $(2, 4)$ entry of each element of B is zero, then B is contained in the algebra

$$B_{12} = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

Note that B_{12}^* has a separating vector and hence B^* has a separating vector.

Subsubcase 4.3.3. Suppose B contains an element

$$b=\begin{pmatrix}0&0&\alpha&\beta\\0&0&0&\gamma\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

where $\alpha\neq 0$ and $\gamma\neq 0$. Let

$$t=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\alpha^{-1}&-\frac{\beta}{\alpha\gamma}\\0&0&0&\gamma^{-1}\end{pmatrix}.$$

Then

$$t^{-1}bt=\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Conjugating by t^{-1} if necessary, we may assume that $\alpha=\gamma=1$ and $\beta=0$. Since B is a proper subalgebra of B_{10}, B is the algebra,

$$B=\left\{\begin{pmatrix}\lambda_1&\lambda_2&\lambda_3&\lambda_4\\0&\lambda_1&0&\lambda_3\\0&0&\lambda_1&0\\0&0&0&\lambda_1\end{pmatrix}:\lambda_1,\dots,\lambda_4\in\mathbb{C}\right\}.$$

It is easy to see that B^* has a separating vector.

Subcase 4.4. B does not contain a matrix B as in subcase 4.1, subcase 4.2, and subcase 4.3. Then

$$B\subset\left\{\begin{pmatrix}\lambda_1&0&\lambda_2&\lambda_3\\0&\lambda_1&0&\lambda_4\\0&0&\lambda_1&0\\0&0&0&\lambda_1\end{pmatrix}:\lambda_1,\dots,\lambda_4\in\mathbb{C}\right\}.$$

Combining Lemma 5.1 [Azoff 1973, Table 5A, page 34], and similar arguments as in Subsubcase 3.1.1,

$$B=\left\{\begin{pmatrix}\lambda_1&\lambda_2&\lambda_3&0\\0&\lambda_1&0&0\\0&0&\lambda_1&0\\0&0&0&\lambda_1\end{pmatrix}:\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}\right\},$$

or

$$B = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & \lambda_2 \\ 0 & \lambda_1 & 0 & \lambda_3 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\},$$

or

$$B = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & 0 & \lambda_2 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\},$$

or

$$B = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 & 0 \\ 0 & \lambda_1 & 0 & \lambda_3 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

It is easy to show that in each case either B or B^* has a separating vector. □

6. 2-reflexivity and property P_1

Let H be a Hilbert space. The usual notation $\mathrm{Lat}(B)$ will denote the lattice of invariant subspaces (or projections) for a subset $B \subseteq L(H)$, and $\mathrm{Alg}(L)$ will denote the algebra of bounded linear operators leaving invariant every member of a family L of subspaces (or projections). An algebra B is called reflexive if $B = \mathrm{AlgLat}(B)$. An algebra B is called n-reflexive if the n-fold inflation $B^{(n)} = \{b^{(n)} : b \in B\}$, acting on $\mathscr{H}^{(n)}$, is reflexive [Azoff 1986]. In [Larson 1982], the third author proved the following result: An algebra B is n-reflexive if and only if $B_\perp$, the preannihilator of B, is the trace class norm closed linear span of operators of rank$\leq n$. In [Larson 1982], the third author also showed the following connection between n-reflexivity and the P_1 property: If an algebra B has property P_1, then B is 3-fold reflexive. (This result also holds for linear subspaces with the same proof). He raised the following problem: Suppose $\dim H = n \in \mathbb{N}$ and $B \subset L(H) \equiv M_n(\mathbb{C})$ is a unital operator algebra with property P_1. Is B 2-reflexive? Note that this question also makes sense for linear subspaces. Azoff [1986] showed that the answer to the above question is affirmative for $n = 3$ (for all linear subspaces of $M_3(\mathbb{C})$ with property P_1). In this section, we prove the following result.

Proposition 6.1. *If* $\dim H = 4$ *and* $B \subset L(H) \equiv M_4(\mathbb{C})$ *is a unital operator algebra with property* P_1, *then* B *is* 2-*reflexive.*

Proof. By Proposition 5.5, either B or B^* has a separating vector or B is similarly conjugate to an algebra of the form

$$\left\{\begin{pmatrix}\lambda I_2 & s\\ 0 & \eta I_2\end{pmatrix} : \lambda, \eta \in \mathbb{C}, s \in S\right\},$$

where S is a subspace of $M_2(\mathbb{C})$ with dimension two. If B has a separating vector or B^* has a separating vector, then the fact that B is 2-reflexive follows from the proofs of Corollary 7 of [Larson 1982] and Proposition 1.2 of [Herrero et al. 1991]. If B is similarly conjugate to an algebra of the form

$$\left\{\begin{pmatrix}\lambda I_2 & s\\ 0 & \eta I_2\end{pmatrix} : \lambda, \eta \in \mathbb{C}, s \in S\right\},$$

where S is a subspace of $M_2(\mathbb{C})$ with dimension two, then the fact that B is 2-reflexive follows from Proposition 1 of [Kraus and Larson 1985]. □

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On three questions concerning groups with perfect order subsets

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In a finite group, an *order subset* is a maximal set of elements of the same order. We discuss three questions about finite groups G having the property that the cardinalities of all order subsets of G divide the order of G. We provide a new proof to one of these questions and evidence to support answers to the other two questions.

1. Introduction

Let G be a finite group. Carrie E. Finch and the first author [Finch and Jones 2002; 2003] defined the *order subset of* G *determined by* $x \in G$ to be the set of elements in G with the same order as x. They defined G to have *perfect order subsets* — in short, to be a *POS group* — if the number of elements in each order subset of G divides the order $|G|$. It is easy to see that any nontrivial POS group has even order.

The next three theorems, whose proofs are given in [Finch and Jones 2002], allow us to refine the search for abelian POS groups to a particular class of groups.

Theorem 1.1. *Let* $G \simeq (\mathbb{Z}_{p^a})^t \times M$ *and* $\widehat{G} \simeq (\mathbb{Z}_{p^{a+1}})^t \times M$, *where* M *is an abelian group and* p *is a prime not dividing* $|M|$. *If* G *is a POS group, then so is* $\widehat{G}$.

Theorem 1.2. *Suppose* $G \simeq \mathbb{Z}_{p^{a_1}} \times \mathbb{Z}_{p^{a_2}} \times \cdots \times \mathbb{Z}_{p^{a_{s-1}}} \times (\mathbb{Z}_{p^{a_s}})^t \times M$, *where* M *is an abelian group*, p *is a prime not dividing* $|M|$, *and* $a_1 \le a_2 \le \ldots \le a_{s-1} < a_s$. *If* G *is a POS group, then so is* $\widehat{G} \simeq (\mathbb{Z}_{p^{a_s}})^t \times M$.

Theorem 1.3. *If* G *is a POS group with* $G \simeq (\mathbb{Z}_{p^a})^t \times M$, *where* M *is an abelian group and* p *is a prime not dividing* $|M|$, *then* $\widehat{G} \simeq (\mathbb{Z}_p)^t \times M$ *is also a POS group.*

The previous theorems provide motivation for the following definition.

Definition 1.4. Let $G \simeq (\mathbb{Z}_2)^t \times M$, where $|M|$ is odd, be a POS group. We say that G is *minimal* if $(\mathbb{Z}_2)^t \times \widehat{M}$ is not a POS group for any subgroup $\widehat{M}$ of M.

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Theorem 1.5 [Finch and Jones 2002]. *Let $G \cong (\mathbb{Z}_2)^t \times M$, where $t \geq 1$ and M is a cyclic group of odd square-free order. If G is a POS group and $G \cong (\mathbb{Z}_2)^t \times \widehat{M}$ is not a POS group for any subgroup $\widehat{M}$ of M, then G is isomorphic to one of*

$$\begin{aligned}
&\mathbb{Z}_2,\\
&(\mathbb{Z}_2)^2 \times \mathbb{Z}_3,\\
&(\mathbb{Z}_2)^3 \times \mathbb{Z}_3 \times \mathbb{Z}_7,\\
&(\mathbb{Z}_2)^4 \times \mathbb{Z}_3 \times \mathbb{Z}_5,\\
&(\mathbb{Z}_2)^5 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{31},\\
&(\mathbb{Z}_2)^8 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{17},\\
&(\mathbb{Z}_2)^{16} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{17} \times \mathbb{Z}_{257},\\
&(\mathbb{Z}_2)^{17} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{17} \times \mathbb{Z}_{257} \times \mathbb{Z}_{131071},\\
&(\mathbb{Z}_2)^{32} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{17} \times \mathbb{Z}_{257} \times \mathbb{Z}_{65537}.
\end{aligned}$$

Various authors have investigated nonabelian groups in search of POS groups. For example, certain special linear groups were considered in [Finch and Jones 2003], the dihedral groups in [Libera and Tlucek 2003], and certain semidirect products and the alternating groups in [Das 2009]. In this article, our focus will be on the symmetric groups and on certain abelian groups, and specifically on three questions posed in [Finch and Jones 2002]:

Question 1.6. *Is S_3 the only symmetric group that is a POS group*?

Question 1.7. *If G is a POS group and $|G|$ is not a power of* 2, *then must $|G|$ be divisible by* 3?

Question 1.8. *Are there only finitely many minimal POS groups that contain noncyclic Sylow p-subgroups of odd order*?

Tuan and Hai [2010] answered Question 1.6 in the affirmative. We provide here an alternative proof that is shorter and more direct. The techniques used in our proof are similar to those of Tuan and Hai, but whereas they use a theorem of Chebyshev [1852], we resort to a more refined version of that result [Nagura 1952].

Walter Feit (personal communication; see also [Finch and Jones 2003]) answered Question 1.7 in the negative, by providing counterexamples: if p is a Fermat prime, the Frobenius group of order $p(p-1)$, with Frobenius complement $\mathbb{Z}_{p-1}$ and Frobenius kernel $\mathbb{Z}_p$, is a POS group but its order is not divisible by 3. Other counterexamples to Question 1.7 were constructed in [Das 2009].

All these counterexamples are nonabelian. This leads to a modified version of the question, for which we will show evidence of an affirmative answer:

Question 1.9 (modified Question 1.7). *If G is an abelian POS group and $|G|$ is not a power of* 2, *then must $|G|$ be divisible by* 3?

Concerning Question 1.8, the only known abelian POS group with a noncyclic Sylow p-subgroup is

$$(\mathbb{Z}_2)^{11} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times (\mathbb{Z}_{11})^2 \times \mathbb{Z}_{23} \times \mathbb{Z}_{89}, \tag{1-1}$$

found in [Finch and Jones 2002]. Theorem 4.3 below shows that this is, in fact, the only such POS group whose order has exactly 5 distinct odd prime divisors and exactly one odd square prime factor.

To summarize, these are the main results of this paper:

Theorem 1.10. *The symmetric group S_n is a POS group if and only if $n \leq 3$.*

Theorem 1.11. *Suppose that G is an abelian POS group and $|G|$ is not a power of 2. If $|G|$ is not divisible by 3, then $|G| > 4.48 \cdot 10^{457008}$, and $|G|$ has at least 57097 distinct prime factors.*

Theorem 1.12. *Let G be a minimal abelian POS group such that*

$$G \simeq (\mathbb{Z}_2)^t \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{k-1}} \times (\mathbb{Z}_{p_k})^2 \times \mathbb{Z}_{p_{k+1}} \times \cdots \times \mathbb{Z}_{p_m},$$

where $p_1 < p_2 < \cdots < p_m$ are odd primes. If $1 \leq m \leq 5$, then

$$G \simeq (\mathbb{Z}_2)^{11} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times (\mathbb{Z}_{11})^2 \times \mathbb{Z}_{23} \times \mathbb{Z}_{89}.$$

2. The proof of Theorem 1.10

The proof is based on a result of Nagura, which refines a theorem of Chebyshev [1852] (also known as Bertrand's postulate) to the effect that for every integer $x \geq 4$, there exists a prime p such that $x < p < 2x - 2$.

Theorem 2.1 [Nagura 1952]. *If $x \geq 25$, then there exists a prime p such that*

$$x < p < \tfrac{6}{5}x.$$

Proof of Theorem 1.10. It is easy to verify that S_n is a POS group when $n \leq 3$. Suppose that $n \geq 60$. By Theorem 2.1, there exists a prime p such that $\frac{5}{12}n < p < \frac{1}{2}n$. Note that $n \geq 60$ and $p > \frac{5}{12}n$ imply that $p \geq 29$. Also, since $\frac{5}{12}n < p < \frac{1}{2}n$, it follows that $2p < n < 3p$, so an element of order p in S_n is either a p-cycle or the product of 2 disjoint p-cycles. Thus, the number of elements of order p in S_n is

$$C := \frac{n(n-1)(n-2)\cdots(n-p+1)}{p} + \frac{\frac{n(n-1)(n-2)\cdots(n-p+1)}{p} \cdot \frac{(n-p)(n-p-1)\cdots(n-2p+1)}{p}}{2}.$$

Then

$$\frac{n!}{C} = \frac{2p^2(n-p)!}{2p + (n-p)\cdots(n-2p+1)}.$$

Define

$$A := 2p^2(n-p)! \quad \text{and} \quad B := 2p + (n-p)\cdots(n-2p+1).$$

We show that B does not divide A. Let q be a prime divisor of B. We consider four ranges for q:

Case 1: $q \le p$. Since $B-2p$ is a product of $p \ge q$ consecutive integers, at least one of its factors is divisible by q. Thus, q divides $B-(B-2p)=2p$, so that $q=2$ or p.

Case 2: $p<q<n-2p+1$. Impossible, since $n<3p$ implies $(n-2p+1)-p<1$.

Case 3: $n-2p+1 \le q \le n-p$. Then q appears as a factor in $B-2p$. So again, $q=2$ or p.

Case 4: $n-p<q$. Clearly q does not divide $A=2p^2(n-p)!$. Thus, $B=2^k p^m$. Observe that B is divisible by 2, but not by 4. Also, since $p<n-p<2p$, we have that p^3 is the exact power of p that divides A. Hence, $k=1$ and $m \le 3$. Therefore, $B \le 2p^3$. It follows that

$$\begin{aligned} 2p(p-1)(p+1) = 2p^3 - 2p \ge B - 2p &= (n-p)(n-p-1)\cdots(n-2p+1) \\ &> p(p-1)(p-2)(p-3)\cdots 3\cdot 2, \end{aligned}$$

since $n>2p$. But this is impossible since $p \ge 29$.

Finally, to complete the proof, we need the number a_n of elements of order 2 in S_n, for $4 \le n \le 59$. By a result of Chowla, Herstein and Moore [Chowla et al. 1951], this number satisfies (for any n) the recurrence relation

$$a_n = a_{n-1} + (a_{n-2}+1)(n-1).$$

All that remains is to verify with a computer that $n!$ is never divisible by a_n for these values of n. $\square$

3. The Proof of Theorem 1.11

In light of Theorems 1.2 and 1.3, it is enough to focus on groups all of whose Sylow subgroups are elementary abelian. Thus, throughout this section, we let

$$G \simeq (\mathbb{Z}_2)^t \times (\mathbb{Z}_{p_1})^{t_1} \times \cdots \times (\mathbb{Z}_{p_m})^{t_m},$$

where $p_1 < p_2 < \cdots < p_m$ are odd primes, and $m \ge 1$. Let

$$n = |G| = 2^t \prod_{i=1}^{m} p_i^{t_i} \quad \text{and} \quad f(n) = (2^t - 1)\prod_{i=1}^{m}(p_i^{t_i} - 1).$$

The following lemma is a direct consequence of the definition of a POS group.

Lemma 3.1. *The group G is a POS group if and only if $n/f(n)$ is an integer.*

Lemma 3.2. *If $m = 1$ and G is a POS group then $p_1 = 3$.*

Proof. Since $m = 1$, we have that $n = 2^t p_1^{t_1}$ and $f(n) = (2^t - 1)(p_1^{t_1} - 1)$. Then, since G is a POS group, $n/f(n)$ is an integer by Lemma 3.1. Thus, there exist positive integers a and b such that

$$a(2^t - 1) = p_1^{t_1} \qquad \text{and} \qquad b(p_1^{t_1} - 1) = 2^t. \tag{3-1}$$

Hence,

$$p_1^{t_1} - 2 \le 2^t - 1 \le p_1^{t_1}.$$

Thus, there are two cases to consider:

Case 1: $2^t - 1 = p_1^{t_1} - 2$. Then $p_1^{t_1} = 2^t + 1$, and so from (3-1) we conclude that $a = 1 + 2/(2^t - 1)$. Hence, $t = 1$, since a is an integer, which implies that $p_1 = 3$.

Case 2: $2^t - 1 = p_1^{t_1}$. We deduce from (3-1) that $p_1^{t_1} + 1 = 2^t$ and $p_1^{t_1} - 1 = 2^c$, for some $c < t$. Subtracting one equation from the other gives $2^c(2^{t-c} - 1) = 2$, which implies that $c = 1$ and $p_1 = 3$. □

Proof of Theorem 1.11. By way of contradiction, assume $p_1 > 3$. By Lemma 3.2, we may assume that $m \ge 2$. Let q be an arbitrary prime divisor of n. Since all prime divisors of $q-1$ divide n, we have that $q \equiv 2 \pmod 3$ and all prime divisors of $q - 1$ are congruent to 2 modulo 3. Thus, we can recursively construct the list S of viable prime divisors of n as follows. Let $S_1 = [2, 5]$ and $q_1 = 5$. For $i \ge 2$, let q_i be the smallest prime such that $q_i > q_{i-1}$ and all prime divisors of $q_i - 1$ are contained in the list S_{i-1}. Define $S_i := [2, 5, \ldots, q_{i-1}, q_i]$. Then

$$\begin{aligned}
S_2 &= [2, 5, 11], & q_2 &= 11,\\
S_3 &= [2, 5, 11, 17], & q_3 &= 17,\\
S_4 &= [2, 5, 11, 17, 23], & q_4 &= 23,\\
S_5 &= [2, 5, 11, 17, 23, 41], & q_5 &= 41,\\
S_6 &= [2, 5, 11, 17, 23, 41, 47], & q_6 &= 47,
\end{aligned}$$

and so on. Define $S := \lim_{i\to\infty} S_i$. Then

$$\frac{n}{f(n)} = \frac{2^t}{2^t - 1} \cdot \prod_{i=1}^{m} \frac{p_i^{t_i}}{p_i^{t_i} - 1} \le \frac{2^m}{2^m - 1} \cdot \prod_{i=1}^{m} \frac{p_i}{p_i - 1} \le \frac{2^m}{2^m - 1} \cdot \prod_{i=1}^{m} \frac{q_i}{q_i - 1}.$$

Using a computer, we have verified for $2 \le m \le 57096$ that

$$\frac{2^m}{2^m - 1} \prod_{i=1}^{m} \frac{q_i}{q_i - 1} < 2 \quad \text{and} \quad \frac{2^{57096}}{2^{57096} - 1} \prod_{i=1}^{57096} q_i > 4.48 \cdot 10^{457008}.$$

Clearly, $n/f(n) > 1$, and since $n/f(n)$ must be an integer by Lemma 3.1, the theorem follows. □

Remark 3.3. Whether or not the list S constructed in the proof of Theorem 1.11 is finite, sieve methods [Halberstam and Richert 1974] can be used to show that the product

$$\frac{2^m}{2^m-1}\prod_{i=1}^{m}\frac{q_i}{q_i-1} \tag{3-2}$$

is bounded above. We conjecture that (3-2) is less than 2 for all $m \geq 2$, but we are unable to provide a proof since a tight explicit bound is both tedious and difficult to compute using sieve methods. The truth of this conjecture would imply that the answer to Question 1.9 is affirmative.

4. The proof of Theorem 1.12

Definition 4.1. Let t be a positive integer, and let q be a prime divisor of 2^t-1. We say that q is a *primitive divisor* of 2^t-1 if q does not divide 2^s-1 for any positive integer $s<t$.

Theorem 4.2 [Bang 1886]. *Let $t \geq 2$ be an integer. Then 2^t-1 has a primitive divisor except when $t=6$.*

Theorem 4.3. *Let G be a minimal abelian POS group, such that*

$$G \simeq (\mathbb{Z}_2)^t \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{k-1}} \times (\mathbb{Z}_{p_k})^2 \times \mathbb{Z}_{p_{k+1}} \times \cdots \times \mathbb{Z}_{p_m},$$

where $p_1 < p_2 < \cdots < p_m$ are odd primes. Then $p_1 = 3$ and $2^t-1 = 2^{p_k}-1 = p_i p_j$, for some $i \neq j$.

Proof. As before, let

$$n = |G| = 2^t p_k^2 \prod_{\substack{i=1\\ i\neq k}}^{m} p_i \quad \text{and} \quad f(n) = (2^t-1)(p_k^2-1)\prod_{\substack{i=1\\ i\neq k}}^{m}(p_i-1).$$

Since G is a POS group, $n/f(n)$ is an integer by Lemma 3.1.

Next, note that $n \equiv 0 \pmod 3$. For if not, then $p_k > 3$ and $p_k^2 - 1 \equiv 0 \pmod 3$. Then, since $f(n) \equiv 0 \pmod{p_k^2-1}$, we have that $f(n) \equiv 0 \pmod 3$, which contradicts the fact that $n/f(n)$ is an integer. This proves that $p_1 = 3$.

Now, suppose that p is an odd prime divisor of t. Then 2^p-1 divides 2^t-1, and so 2^p-1 divides n. Consequently, every prime divisor of 2^p-1 is p_i for some i, and then $p_i - 1 \equiv 0 \pmod p$. Also, for each such p_i, we have that p_i-1 divides n. Thus, since n is not divisible by the cube of any odd prime, it follows that 2^p-1 has at most two distinct odd prime divisors. Therefore, we are led to consider the following five cases:

(1) $2^p - 1 = p_k^2$ for some odd prime divisor p of t.

(2) $2^p - 1 = p_i p_k^2$ for some i, and some odd prime divisor p of t.

(3) There exists an odd prime that divides t, and for every odd prime p that divides t, we have that $2^p - 1 = p_i$ for some i.

(4) There exists at least one odd prime p that divides t such that $2^p - 1 = p_i p_j$ for some $i \neq j$.

(5) No odd prime divides t; that is $t = 2^a$.

Ljunggren [1943] proved that Case (1) is impossible.

In Case (2), we have that $p_i - 1 \equiv 0 \pmod{p}$ and $p_k - 1 \equiv 0 \pmod{p}$. Then $(p_i - 1)(p_k^2 - 1) \equiv 0 \pmod{p^2}$, which says that p^2 divides n. Hence, $p = p_k$. But this contradicts the fact that $p_k - 1 \equiv 0 \pmod{p}$. Hence, Case (2) is impossible as well.

For Case (3), we show first that t has exactly one odd prime divisor. Suppose that p and q are odd prime divisors of t. Then $2^p - 1 = p_i$ and $2^q - 1 = p_j$ for some i and j. Then $p_i - 1 \equiv 0 \pmod{p}$ and $p_j - 1 \equiv 0 \pmod{q}$. By Theorem 4.2, there exists an odd prime $r \neq p_i, p_j$ such that $2^{pq} - 1 \equiv 0 \pmod{r}$. Since $2^{pq} - 1$ divides $2^t - 1$, we have that $f(n) \equiv 0 \pmod{r}$, and so $r = p_v$ for some v. Since p_v is a primitive divisor, it follows that $p_v - 1 \equiv 0 \pmod{pq}$. But then $(p_i - 1)(p_v - 1) \equiv 0 \pmod{p^2}$, and $(p_j - 1)(p_v - 1) \equiv 0 \pmod{q^2}$, which implies that $p = q$.

Thus, t has at most one odd prime divisor. Suppose $t = 2^a p^b$. Let $2^p - 1 = p_i$. Then $p_i - 1 \equiv 0 \pmod{p}$. If $b \geq 2$, we can use Theorem 4.2 to produce a prime divisor $p_j \neq p_i$ of $2^{p^2} - 1$ such that $p_j - 1 \equiv 0 \pmod{p^2}$. But then $(p_i - 1)(p_j - 1) \equiv 0 \pmod{p^3}$, which contradicts the fact that $n/2^t$ is cube-free. Therefore, we only need to consider here the two possibilities $t = 2^a p$ and $t = p$, since the possibility that $t = 2^a$ is handled separately below as Case (5).

Suppose first that $t = 2^a p$. As before, let $2^p - 1 = p_i$. Then $p_i - 1 \equiv 0 \pmod{3}$ and $p_i - 1 \equiv 0 \pmod{p}$. Suppose that $a \geq 1$. Then $2^t - 1 \equiv 0 \pmod{3}$, so that $(2^t - 1)(p_i - 1) \equiv 0 \pmod{9}$, which implies that $p_k = 3$. If $p = 3$, then $2^6 - 1$ divides $2^t - 1$, and so $(2^t - 1)(p_i - 1) \equiv 0 \pmod{27}$, which is a contradiction. On the other hand, if $p \neq 3$, then by Theorem 4.2, there exists a prime $q \neq p_i$ such that $q - 1 \equiv 0 \pmod{2^a p}$. Hence, $(p_i - 1)(q - 1) \equiv 0 \pmod{p^2}$, which implies that $p = p_k = 3$, again a contradiction. Therefore, $a = 0$ and $t = p$, which is the second possibility above. Again, let $2^p - 1 = p_i$. Then $p_i - 1 \equiv 0 \pmod{p}$, so that $p \neq p_i$. Also, $p_i - 1 \equiv 0 \pmod{3}$. If $p_k \neq 3$, then $(p_k^2 - 1)(p_i - 1) \equiv 0 \pmod{9}$, which is impossible since the only square that divides n is $p_k^2 \neq 9$. Hence, $p_k = 3$. If $p = 3 = p_k$, then $n \equiv 0 \pmod{8}$, but $n \not\equiv 0 \pmod{16}$. However, if $p = 3$, then n would be divisible by $(2^{p_k} - 1)(p_k^2 - 1) = (7 - 1)(3^2 - 1)$, which implies that $n \equiv 0 \pmod{16}$. This contradiction shows that $p \neq 3$. Also, since p is odd, we have that $p_i \neq 3$. Thus, all three primes p, p_i and $p_k = 3$ are distinct. If $p \equiv 1 \pmod{3}$, then $2^6 - 1$ divides $2^{p-1} - 1 = p_i - 1$, and so the number of elements of order pp_i is

$$(p-1)(p_i-1) = 2(p-1)(2^{p-1}-1) \equiv 0 \pmod{27},$$

which does not divide n. Thus, $p \equiv 2 \pmod 3$. Now, let q be an odd prime divisor of $p-1$. Then 2^q-1 and $2^{2q}-1$ divide $2^{p-1}-1$, and so both divide n. Let r be a primitive divisor of 2^q-1, and let s be a primitive divisor of $2^{2q}-1$. Since $p \equiv 2 \pmod 3$, we have that $q \neq 3$, and therefore the existence of s is guaranteed by Theorem 4.2. Then

$$r-1 \equiv 0 \equiv s-1 \pmod q.$$

Since $r \neq s$, it follows that either $r \neq p$ or $s \neq p$. Suppose, without loss of generality, that $r \neq p$. Note that $r \neq 3$ so that the number of elements of order pr is $(p-1)(r-1)$. But

$$(p-1)(r-1) \equiv 0 \pmod{q^2},$$

which implies that $q = 3$, a contradiction. Hence, we conclude that no odd primes divide $p-1$. Write $p-1 = 2^a$. Then the number of elements of order p_i is

$$p_i - 1 = 2^p - 2 = 2(2^{2^a}-1) \equiv 0 \pmod 3.$$

If $a \geq 7$, then 6700417 and 274177 divide $2^{2^a}-1$, and the number of elements of order $p_i \cdot 6700417 \cdot 274177$ is

$$2(2^{2^a}-1)(6700416)(274176) \equiv 0 \pmod{27},$$

which does not divide n. Hence, $a \leq 6$, and it is easy to check that 2^a+1 is prime exactly when $a = 1$, 2 or 4. Since $p \equiv 2 \pmod 3$, then $a = 2$ or 4. If $a = 2$, then $p = 5$, and $31 = 2^5-1$ divides n. But then, the number of elements of order $3^2 \cdot 5 \cdot 31$, which is $(3^2-1)(5-1)(31-1) = 2^6 \cdot 3 \cdot 5$, does not divide n. Similarly, if $a = 4$, then $p = 17$, and the power of 2 that divides $f(n)$ is greater than the power of 2 that divides n. Therefore, Case (3) is impossible.

We proceed now to Case (4). Suppose that p is an odd prime dividing t such that $2^p-1 = p_i p_j$, for some $i \neq j$. Then $p_i-1 \equiv p_j-1 \equiv 0 \pmod p$, so that p^2 divides the number of elements of order $p_i p_j$, and thus p^2 divides n. Hence, $p = p_k$. If there exists a prime $q \neq p$ that divides t, then $2^{pq}-1$ divides n. By Theorem 4.2, there is a primitive divisor p_s of $2^{pq}-1$ with $s \notin \{i, j\}$. Then p divides p_s-1, and hence p^3 divides $(p_i-1)(p_j-1)(p_s-1)$, the number of elements of order $p_i p_j p_s$. This contradiction shows that $p = p_k$ is the only odd prime that divides t. An argument similar to the one used in Case (3) shows that p^2 does not divide t. Then, as in Case (3), we only have to consider the two possibilities: $t = 2^a p$ and $t = p$. Suppose that $t = 2^a p$, with $a \geq 1$. Since $2^p-1 = p_i p_j$, with $i \neq j$, it follows that $p \neq 3$. Then, by Theorem 4.2, there exists a primitive divisor p_s

of $2^{2p}-1$. Thus, $s \notin \{i, j\}$ and $p_s - 1 \equiv 0 \pmod{p}$. But then we have that the number of elements in G of order $p_i p_j p_s$ is

$$(p_i - 1)(p_j - 1)(p_s - 1) \equiv 0 \pmod{p^3}.$$

Hence, $a = 0$ and $t = p = p_k$.

This brings us to Case (5). Assume now that $t = 2^a$. As in Case (3), if $a \geq 7$, then 6700417 and 274177 divide $2^{2^a} - 1$, and n is divisible by the number of elements in G of order $2 \cdot 6700417 \cdot 274177$, which is $(2^{2^a} - 1)(6700416)(274176)$. But $(2^{2^a} - 1)(6700416)(274176)$ cannot divide n since

$$(2^{2^a} - 1)(6700416)(274176) \equiv 0 \pmod{27},$$

and $n/2^t$ is cube-free. Thus, $a \leq 6$. It is straightforward to check that each of these cases, in some way, violates the hypotheses of the theorem. For example, if $a = 6$, then n is divisible by

$$2^{64} - 1 = 3 \cdot 5 \cdot 17 \cdot 257 \cdot 641 \cdot 65537 \cdot 6700417.$$

Hence, $(2^{64} - 1) \cdot 640$ and $(2^{64} - 1) \cdot 6700416$ must also divide n. However, $(2^{64}-1) \cdot 640 \equiv 0 \pmod{25}$ and $(2^{64}-1) \cdot 6700416 \equiv 0 \pmod{9}$, which contradicts the fact that n is divisible by exactly one odd square. Checking the remaining cases completes the proof of the theorem. □

Remark 4.4. Without loss of generality, we can assume that $p_i < p_j$ in the statement of the conclusion of Theorem 4.3. Also, this conclusion implies that $3 = p_1 < p_k < p_i < p_j$, with $p_k \geq 11$. Thus, $m \geq 4$.

Proof of Theorem 1.12. Let G be a minimal abelian POS group such that

$$G \simeq (\mathbb{Z}_2)^t \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{k-1}} \times (\mathbb{Z}_{p_k})^2 \times \mathbb{Z}_{p_{k+1}} \times \cdots \times \mathbb{Z}_{p_m},$$

where $p_1 < p_2 < \cdots < p_m$ are odd primes, with $1 \leq m \leq 5$. By Theorem 4.3, we have that $p_1 = 3$ and $2^t - 1 = 2^{p_k} - 1 = p_i p_j$ for some $i \neq j$. By Remark 4.4, we can also assume that $p_k \geq 11$ and that $m = 4$ or $m = 5$.

Consider first the case when $m = 4$. In this case, we have

$$\frac{n}{f(n)} = \frac{2^{p_k} \cdot 3 \cdot p_k^2 \cdot p_i \cdot p_j}{(2^{p_k} - 1) \cdot 2 \cdot (p_k^2 - 1) \cdot (p_i - 1) \cdot (p_j - 1)} = \frac{2^{p_k - 1} \cdot 3 \cdot p_k^2}{(p_k^2 - 1) \cdot (p_i - 1) \cdot (p_j - 1)}.$$

Since $p_i - 1 \equiv p_j - 1 \equiv 0 \pmod{p_k}$, it follows that either

(1) $p_k - 1 = 2^a \cdot 3$ and $p_k + 1 = 2^b$ or

(2) $p_k - 1 = 2^a$ and $p_k + 1 = 2^b \cdot 3$.

In (1), we get that

$$2 = 2^b - 2^a \cdot 3 = 2^a(2^{b-a} - 3),$$

which implies that $a = 1$ and $b = 3$. Hence, $p_k = 7$, which contradicts the fact that $p_k \geq 11$. In (2), we get two possibilities. The first possibility gives

$$2 = 2^a(2^{b-a} \cdot 3 - 1),$$

which implies that $a = b = 0$. Thus $p_k = 2$, which is impossible. The second possibility yields

$$2 = 2^b(3 - 2^{a-b}),$$

which implies that either $a = 2$ and $b = 1$, in which case $p_k = 5$; or $a = b = 0$, in which case $p_k = 2$. Both situations are impossible. Hence, there are no POS groups satisfying the conditions of the theorem with $m = 4$.

Now suppose that $m = 5$. Then

$$\frac{n}{f(n)} = \frac{2^{p_k} \cdot 3 \cdot p \cdot p_k^2 \cdot p_i \cdot p_j}{(2^{p_k} - 1) \cdot 2 \cdot (p-1) \cdot (p_k^2 - 1) \cdot (p_i - 1) \cdot (p_j - 1)}.$$

Since $p_k < p_i < p_j$, we have $\frac{p_j}{p_j - 1} < \frac{p_i}{p_i - 1} < \frac{p_k}{p_k - 1}$. Thus,

$$\frac{n}{f(n)} \leq \frac{2^{p_k} \cdot 3 \cdot 5 \cdot p_k^4}{(2^{p_k} - 1) \cdot 2 \cdot 4 \cdot (p_k^2 - 1) \cdot (p_k - 1)^2}.$$

It is straightforward to show that

$$g(x) = \frac{15 \cdot 2^x \cdot x^4}{8 \cdot (2^x - 1)(x^2 - 1)(x - 1)^2}$$

is a decreasing function for $x \geq 2$, and that $g(x) < 2$ when $x \geq 32$. It follows that $n/f(n) < 2$ when $p_k \geq 37$. Clearly, $n/f(n) > 1$, and since we are assuming that $n/f(n)$ is an integer, we only have to check p_k with $11 \leq p_k \leq 31$. The fact that $2^{p_k} - 1$ must be the product of two distinct primes rules out all primes in this range except $p_k = 11$ and $p_k = 23$. If $p_k = 23$, then $2^{23} - 1 = 47 \cdot 178481$ divides n. But then $178481 - 1 = 2^4 \cdot 5 \cdot 23 \cdot 97$ also divides n, which contradicts the fact that $m = 5$. Verifying that the case $p_k = 11$ gives the POS group in the statement of the theorem completes the proof. □

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On the associated primes of the third power of the cover ideal

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An algebraic approach to graph theory involves the study of the edge ideal and the cover ideal of a given graph. While a lot is known for the associated primes of powers of the edge ideal, much less is known for the associated primes of the powers of the cover ideal. The associated primes of the cover ideal and its second power are completely determined. A configuration called a *wheel* is shown to always appear among the associated primes of the third power of the cover ideal.

1. Introduction

We start with some definitions and notation, for which we follow [Harris et al. 2008; Villarreal 2001]. A (finite) *graph* G consists of two finite sets, the vertex set V_G and the edge set E_G, whose elements are unordered pairs of vertices. An edge $\{x_i, x_j\} \in E_G$ is written $x_i x_j$ (or $x_j x_i$). If $x_i x_j$ is an edge, we say that the vertices x_i and x_j are *adjacent* and that the edge is *incident* to x_i and x_j. All our graphs will be *simple*, meaning that the only possible edges are $x_i x_j$ for $i \neq j$.

A subset $C \subseteq V_G$ is a (*vertex*) *cover* of G if each edge in E_G is incident to a vertex in C. A cover C is *minimal* if no proper subset of C is a cover of G.

The results of this paper are in the area of algebraic graph theory, where algebraic methods are used to investigate properties of graphs. Indeed, a graph G with vertex set $V_G = \{x_1, \dots, x_n\}$ can be related to the polynomial ring $R = \mathsf{k}[x_1, \dots, x_n]$, where k is a field. In the following we take the liberty of referring to x_i as a variable in the polynomial ring and as a vertex in the graph G, without any further specification. Given a ring R, we denote by $(f_1, \dots, f_l)$ the ideal of R generated by the elements $f_1, \dots, f_l \in R$.

Two ideals of the polynomial ring $R = \mathsf{k}[x_1, \dots, x_n]$ that have proven most useful in studying the properties of a graph G with vertex set $V_G = \{x_1, \dots, x_n\}$

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and edge set E_G are the *edge ideal*

$$I_G = (x_i x_j \mid x_i x_j \in E_G)$$

and the *cover ideal*

$$J_G = (x_{i_1} \cdots x_{i_k} \mid x_{i_1}, \ldots, x_{i_k} \text{ is a minimal cover of } G).$$

Both are square-free monomial ideals, that is, they are generated by monomials in which each variable appears at most one time.

One of the most basic tools in commutative algebra to study an ideal I of a noetherian ring R is to compute the finite set of *associated prime ideals* of I, which is denoted by $\operatorname{Ass}(R/I)$ (for details, see [Eisenbud 1995]). In the case of a monomial ideal L in a polynomial ring $S = \mathsf{k}[x_1, \ldots, x_n]$, an element in $\operatorname{Ass}(S/L)$ is a *monomial prime ideal*, which is an ideal generated by a subset of the variables. Because of this fact we can record the following definition.

Definition. Let L be a monomial ideal in the polynomial ring $S = \mathsf{k}[x_1, \ldots, x_n]$ and let $P = (x_{i_1}, \ldots, x_{i_s})$ be a monomial prime ideal. If there exists a monomial m such that $x_{i_j} m \in L$ for each $j = 1, \ldots, s$ and $x_i m \notin L$ for every $i \neq i_1, \ldots, i_s$ then P is an *associated* prime to L. We denote by $\operatorname{Ass}(S/L)$ the set of all associated (monomial) primes of L.

Chen et al. [2002] gave a constructive method for determining primes associated to the powers of the edge ideal, but much less is known about cover ideals. It is known that, given a graph G and its cover ideal J_G, a monomial prime ideal P is in $\operatorname{Ass}(S/J_G)$ if and only if $P = (x_i, x_j)$ and $x_i x_j$ is an edge of G (see [Villarreal 2001], for example).

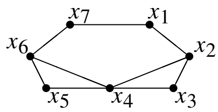
The initial point of our investigation is a result of Francisco, Ha and Van Tuyl (Theorem 1.1 below) describing the associated primes of the ideal $(J_G)^2$.

Let G be a graph. A *path* in G is a sequence of distinct vertices $x_1, x_2, \ldots, x_k$ such that $x_j x_{j+1} \in E_G$ for $j = 1, 2, \ldots, k-1$. The *length* of such a path is $k-1$, one less than the number of vertices. If $x_k x_1$ is also an edge of G, we say that the graph C with vertex set $\{x_1, x_2, \ldots, x_k\}$ and edge set $\{x_1 x_2, \ \ldots, \ x_{k-1} x_k, \ x_k x_1\}$ is a *cycle* (in G). A cycle with an odd number of vertices is also called an *odd hole*.

Given a graph G and a set of vertices $W \subseteq V_G$, the *graph generated by W* has vertex set W and edge set $\{xy \mid xy \in E_G, \ x \in W, \ y \in W\}$.

Theorem 1.1 [Francisco et al. 2010]. *Let G be a graph with vertex set $\{x_1, \ldots, x_n\}$, edge set E_G and cover ideal J_G. A monomial prime ideal $P = (x_{i_1}, \ldots, x_{i_k})$ of the polynomial ring $S = \mathsf{k}[x_1, \ldots, x_n]$ is in the set* $\operatorname{Ass}(S/J_G^2)$ *if and only if either*

- *$k = 2$ and $x_{i_1} x_{i_2} \in E_G$, or*
- *k is odd and the graph generated by $x_{i_1}, \ldots, x_{i_k}$ is an odd hole.*

As an example, if G is the graph 

we will have

$\mathrm{Ass}(J) =$
$$\{(x_1, x_2), (x_1, x_7), (x_2, x_3), (x_2, x_4), (x_3, x_4), (x_4, x_5), (x_4, x_6), (x_5, x_6), (x_6, x_7)\}$$

(the associated prime of J consists of the primes generated by two variables that correspond to the edges of the graph), and

$$\mathrm{Ass}(J^2) = \mathrm{Ass}(J) \cup \{(x_2, x_3, x_4), (x_4, x_5, x_6), (x_1, x_2, x_4, x_6, x_7)\}$$

(the associated prime of J_G^2 contains all the primes that are either generated by two variables corresponding to edges or generated by three variables corresponding to odd cycles of G).

In this paper we study the associated primes of the third power of the cover ideal, the ideal J_G^3. We prove that the primes generated by the variables corresponding to the vertices of a *wheel* (see next definition) always appear among the associated primes of J_G^3. This result is connected with the coloring number of a graph, as discussed at the end of Section 2.

The algebra system Macaulay2 was used for all the computations in this paper, and in particular in finding the pattern that led to the main theorem.

2. Centered odd holes and the main theorem

Definition. A graph C is said to be a *wheel* if $V_C = V_H \cup \{y\}$, where H, called the *rim* of C, is an odd hole such that the graph generated by H in C is H itself, and y, called the *center* of C, is a vertex adjacent in C to at least three vertices of H and belonging to at least two odd cycles in C. (It follows that y belongs to at least three odd cycles in C.) The rim H and center y are part of the data needed to specify a wheel, as they may not be uniquely determined by C.

Let C be a wheel with rim H and center y. A vertex $x \in V_H$ is *radial* if xy is an edge of C. Let there be k radial vertices, labeled consequently $x_1, \dots, x_k$ *in order around the wheel*. We leave it to the reader to specify precisely what this means. For $i = 1, \dots, k-1$, we denote by l_i the length of the path in H joining x_i to x_{i+1} (and not going through any other radial vertex). Similarly l_k denotes the length of the path in H from x_k to x_1.

For the main theorem we will need the following lemma, where we use the notation $|\ |$ for the *size* (that is, the number of vertices) of a graph.

Lemma 2.1. *Let C be a wheel with rim H and center y, and let k be its radial number. If W is a vertex cover for C that contains y, then $|W| \geq |C|/2 + 1$. If W is a vertex cover for C that does not contain y, then*

$$|W| \geq k + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor.$$

Moreover,

$$k+\left\lfloor\frac{l_1-1}{2}\right\rfloor+\cdots+\left\lfloor\frac{l_k-1}{2}\right\rfloor\geq\frac{|C|}{2}+1. \tag{2-1}$$

Proof. Let V_H be the vertex set of H. Assume that W contains the vertex y. The vertex set $W\cap V_H$ has to be a vertex cover for H. Moreover, since H is an odd hole, the cardinality of $W\cap V_H$ has to be at least $(|H|+1)/2$, which is equal to $|C|/2$. Therefore the cardinality of W is $|C|/2+1$.

Assume now that W does not contain the vertex y. Let $x_1,\dots,x_k$ be the radial vertices. Since $y\notin W$, all the radial vertices are in W. As $W\cap V_H$ is a cover of H, in the path from x_i to x_{i+1} we need at least $\lfloor(l_i-1)/2\rfloor$ vertices, for $i=1,\dots,k-1$, and we need $\lfloor(l_k-1)/2\rfloor$ vertices for the path from x_k to x_1.

To prove (2-1) we write

$$\begin{aligned}k+\left\lfloor\frac{l_1-1}{2}\right\rfloor+\cdots+\left\lfloor\frac{l_k-1}{2}\right\rfloor &\geq k+\frac{l_1-1}{2}+\cdots+\frac{l_k-1}{2}\geq\frac{l_1}{2}+\cdots+\frac{l_k}{2}+\frac{k}{2}\\ &\geq\frac{l_1+\cdots+l_k+1}{2}+\frac{k-1}{2}\geq\frac{|C|}{2}+1,\end{aligned}$$

where in the last inequality we used the fact that $k\geq 3$. □

In the following we will make an abuse of notation: if G is a graph with vertices $x_1,\dots,x_n$ and H is a subgraph generated by the vertices $x_{i_1},\dots,x_{i_k}$, by H we also denote the prime monomial ideal $(x_{i_1}\dots,x_{i_k})$ in the polynomial ring $\mathsf{k}[x_1,\dots,x_n]$. Here is our main theorem.

Theorem 2.2. *Let G be a graph with vertex set $V_G=\{x_1,\dots,x_n\}$ and assume that G has a subgraph C which is a wheel. Let $S=\mathsf{k}[x_1,\dots,x_n]$ and let J be the cover ideal of G. Then the set $\operatorname{Ass}(S/J^3)$ is not contained in the set $\operatorname{Ass}(S/J^2)$, and in fact $C\in\operatorname{Ass}(S/J^3)\setminus\operatorname{Ass}(S/J^2)$.*

Proof. By Lemma 2.11 in [Francisco et al. 2011], we may assume that $G=C$. Let y be the center of the wheel C, and let $x_1,x_2,\dots,x_k$ be the radial vertices. Denote by x_{i_j}, for $j=1,\dots,l_i-1$, the vertices between x_i and x_{i+1} if $i<k$ and the vertices between x_k and x_1 if $i=k$.

That C is not in $\operatorname{Ass}(S/J^2)$ follows from Theorem 1.1, since C is neither an odd hole nor an edge.

To show that C is in $\operatorname{Ass}(S/J^3)$ we need to find a monomial c such that $c\notin J^3$ and $xc\in J^3$ for each vertex x of C. Let c be the monomial

$$c=y^2\prod_{i=1,\dots,k}x_i^2\prod_{\substack{i=1,\dots,k\\ j=1,\dots,l_i-1}}x_{ij}^a,\quad\text{where } a=\begin{cases}1 & \text{if } j \text{ is odd},\\ 2 & \text{if } j \text{ is even}.\end{cases}$$

To show that c is the desired monomial, we first prove that

$$\deg c=k+2+n+\left\lfloor\frac{l_1-1}{2}\right\rfloor+\cdots+\left\lfloor\frac{l_k-1}{2}\right\rfloor. \tag{2-2}$$

Let n be the size of H. For a monomial m we denote by $\deg m$ the degree of m. In computing $\deg c$, the contribution from the variables y and x_i, for $i = 1, \dots, k$, is given by $2k + 2$. For $i = 1, \dots, k-1$, between x_i and x_{i+1}, there are $l_i - 1$ vertices, and there are $l_k - 1$ vertices between x_k and x_1. Given an integer s, there are $\lfloor s/2 \rfloor$ even integers and $\lceil s/2 \rceil$ odd integers between 1 and s. Therefore, in computing $\deg c$, the contribution from the variables x_{ij} is given by

$$2\left\lfloor \frac{l_1-1}{2} \right\rfloor + \dots + 2\left\lfloor \frac{l_k-1}{2} \right\rfloor + \left\lceil \frac{l_1-1}{2} \right\rceil + \dots + \left\lceil \frac{l_k-1}{2} \right\rceil.$$

The degree of the monomial c is therefore equal to

$$\begin{aligned}
&2k+2+2\left\lfloor \frac{l_1-1}{2} \right\rfloor + \dots + 2\left\lfloor \frac{l_k-1}{2} \right\rfloor + \left\lceil \frac{l_1-1}{2} \right\rceil + \dots + \left\lceil \frac{l_k-1}{2} \right\rceil \\
&\quad = 2k+2+\left(\left\lfloor \frac{l_1-1}{2} \right\rfloor + \left\lceil \frac{l_1-1}{2} \right\rceil\right) + \dots + \left(\left\lfloor \frac{l_k-1}{2} \right\rfloor + \left\lceil \frac{l_k-1}{2} \right\rceil\right) \\
&\qquad\qquad + \left\lfloor \frac{l_1-1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k-1}{2} \right\rfloor \\
&\quad = k+2+k+(l_1-1)+\dots+(l_k-1)+\left\lfloor \frac{l_1-1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k-1}{2} \right\rfloor \\
&\quad = k+2+l_1+\dots+l_k+\left\lfloor \frac{l_1-1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k-1}{2} \right\rfloor \\
&\quad = k+2+n+\left\lfloor \frac{l_1-1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k-1}{2} \right\rfloor.
\end{aligned}$$

The last line establishes (2-2).

To prove that c does not belong to J^3, we first show the strict inequality

$$\deg c < 2\left(\frac{|C|}{2}+1\right) + k + \left\lfloor \frac{l_1-1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k-1}{2} \right\rfloor. \tag{2-3}$$

For suppose this inequality is not satisfied. Then (2-2) gives

$$k+2+n+\left\lfloor \frac{l_1-1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k-1}{2} \right\rfloor \geq 2\left(\frac{|C|}{2}+1\right) + k + \left\lfloor \frac{l_1-1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k-1}{2} \right\rfloor,$$

which means that

$$2+n \geq 2\left(\frac{|C|}{2}+1\right).$$

But $|C| = |H| + 1 = n + 1$. Thus

$$2+n \geq 2\left(\frac{n+1}{2}+1\right) = n+2+1,$$

which is impossible. Therefore (2-3) holds.

Let us show that (2-3) implies that $c \notin J^3$. Assume otherwise; then $c = hm_1m_2m_3$ with $m_i \in J$ for $i = 1, 2, 3$. Since $m_i \in J$, the variables that appear in each m_i correspond to a minimal cover of C. Lemma 2.1 says that such a cover has at least $|C|/2 + 1$ vertices if it contains y and at least $k + \lfloor (l_1-1)/2 \rfloor + \dots + \lfloor (l_k-1)/2 \rfloor$

— a number at least as large as $|C|/2+1$ — if not. Using the fact that at least one of the three covers must *not* contain y, we thus obtain

$$\begin{aligned}\deg c &= \deg h + \deg m_1 + \deg m_2 + \deg m_3 \\ &\geq \deg h + 2\Big(\frac{|C|}{2}+1\Big) + k + \Big\lfloor \frac{l_1-1}{2} \Big\rfloor + \cdots + \Big\lfloor \frac{l_k-1}{2} \Big\rfloor.\end{aligned}$$

This contradicts (2-3) (since $\deg h \geq 0$) and so shows that $c \notin J^3$.

To finish the proof of Theorem 2.2 we need to show that for every vertex $x \in V_C$ we have $xc \in J^3$.

Let x be any vertex of H and relabel the vertices of H starting from $x = t_1$ clockwise $t_2, \ldots, t_n$, where n is the size of H. We can write $xc = m_1m_2m_3$, where

$$m_1 = y \prod_{i \text{ odd}} t_i, \quad m_2 = yt_1 \prod_{i \text{ even}} t_i, \quad m_3 = \prod_{i=1,\ldots,k} x_i \prod_{\substack{i=1\ldots,k \\ j \text{ even}}} x_{ij}.$$

Note that m_1 and m_2 correspond to covers, as they contain y and every other vertex of H. Also m_3 corresponds to a cover as all the x_i are included, and therefore all the edges connecting y to H are covered, and every other vertex in the path from x_i to x_{i+1} is included.

Finally we need to write $yc = m_1m_2m_3$ with $m_i \in J$ for $i = 1, 2, 3$. For this assume that x_1 is such that the path from x_k to x_1 is odd. Relabel the vertices $x_1 = t_1$ and then clockwise to t_n. Let

$$m_1 = y \prod_{i \text{ odd}} t_i.$$

Note that m_1 will give a cover as we are considering every other vertex in the odd cycle and the vertex y. Now let l the least even number so that t_l corresponds to a radial vertex x_g, for some g. Set

$$m_2 = y \prod_{\substack{l \leq i \leq n \\ i \text{ even}}} t_i \prod_{\substack{1 \leq i \leq l \\ i \text{ odd}}} t_i.$$

Because we are considering every other vertex from t_1 to t_{l-1}, every other vertex from t_l, and the center y, the monomial m_2 corresponds to a cover of the wheel.

Finally

$$m_3 = yx_gx_{g+1}\ldots x_k \prod_{\substack{i=g,\ldots,k \\ j \text{ even}}} x_{ij} \prod_{\substack{i=1,\ldots,l-1 \\ i \text{ even}}} t_i.$$

Also m_3 gives a cover as it contains every other vertex from t_2 to $t_l = x_g$, every other vertex from x_i to x_{i+1}, for $i = g, \ldots, k-1$, every other vertex from x_k to x_1, and the center y. Notice that x_1 is missing from the monomial m_3 but the vertex y is listed in the monomial as for the vertex preceding x_1, because of the assumption that the path $x_k, \dots, x_1$ in H is odd. □

For every ideal I in a polynomial ring S (or a more general ring), one can compute the sequence of sets $\operatorname{Ass}(S/I^n)$ for $n \in \mathbb{N}$. Brodmann [1979] proved, in much greater generality, that there exists a positive integer a such that

$$\bigcup_{i=1}^{a_I} \operatorname{Ass}(S/I^i) = \bigcup_{i=1}^{\infty} \operatorname{Ass}(S/I^i). \tag{2-4}$$

Very little is known about the value of a_I. In [Francisco et al. 2011], the authors give an upper bound for a_I in the case that I is an edge ideal for a graph.

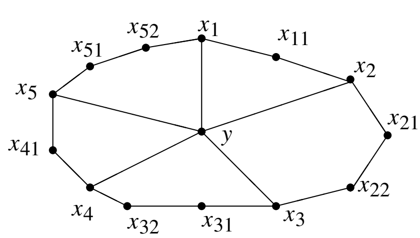
The value of a_J, where J is the cover ideal of a graph G, is related to the *coloring number* of G, that is, the least number of colors that one needs to color the vertices of G so that two adjacent vertices always have different colors. We denote the coloring number of G by $\chi(G)$. It is shown in [Francisco et al. 2011] that, in (2-4), $a_J \geq \chi(G) - 1$ when J is the cover ideal of G. The same paper gives examples for which $a_J > \chi(G) - 1$. Centered odd holes are an infinite family of such examples.

Corollary 2.3. *Let C be a wheel with cover ideal J. If C has a vertex that is not radial, then $a_J \geq \chi(C)$.*

Proof. Because C contains an odd hole, one needs at least three colors for the vertexes of C. We first show that $\chi(C) = 3$. Let $\{a, b, c\}$ be a list of three colors. Assume that x is a vertex of C which is not radial. Color the vertex x and the center y with c, and finally color the remaining vertices alternating a and b.

The main theorem implies that $a_J \geq 3$. □

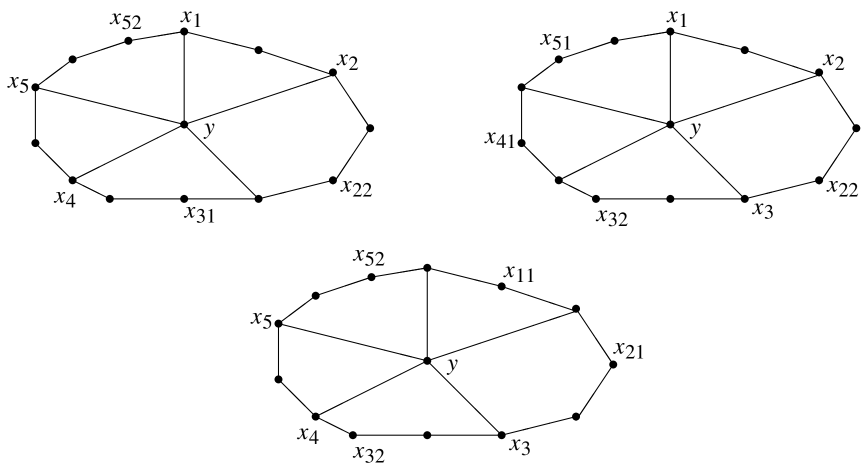
We finish the paper with an example that illustrates the idea behind the proof of the main theorem. Consider this wheel:



The monomial c used in the proof of the main theorem is given by

$$c = x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 x_{22}^2 x_{32}^2 x_{52}^2 x_{11} x_{21} x_{31} x_{41} x_{51}.$$

We can write $yc = m_1 m_2 m_3$, where the monomials m_1, m_2, and m_3 correspond to the following covers:



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Soap film realization of isoperimetric surfaces with boundary

Jacob Ross, Donald Sampson and Neil Steinburg

(Communicated by Frank Morgan)

We examine surfaces of the type proved to be minimizing under a connectivity condition by Dorff et al. We determine which of these surfaces are stable soap films. The connectivity condition is shown to be very restrictive; few of these surfaces are stable (locally minimizing) without it.

1. Introduction

Surface area minimization in soap bubbles and soap films is one of the more fascinating subjects in mathematics today. Metacalibration techniques — a generalization of the calibrations popularized by Harvey and Lawson [1982] (see also [Morgan 1988, Chapter 6]) — were developed to investigate the problems that arise in surface minimization. In particular metacalibration techniques prove very useful in solving a new class of problems with both fixed volume and fixed boundary constraints. We call these problems *equitent problems* after Lawlor et al. Equitent stands for equal content (volume condition) and equal extent (boundary condition) [Dorff et al. 2008].

In this paper we consider a certain class of equitent problems addressed in [Dorff et al. 2011]. It was shown there that certain equitent surfaces are globally minimizing under a connectivity condition that restricts the surfaces' homotopy class. This connectivity condition is not however true for general minimizing surfaces. We examine which of these surfaces are locally minimizing without the connectivity condition. This is equivalent to showing these surfaces are realizable as a soap film. We demonstrate this for those surfaces that are proved to be locally minimal.

2. The surfaces of Dorff et al.

Equitent surfaces are constructed via the union of sections of spheres and planes. Starting with a cone over a wire-frame polyhedron, the center of the cone is then

MSC2010: primary 49Q10; secondary 49Q05, 53A10.

Keywords: bubble, soap film, isoperimetric, equitent, metacalibration, minimization, minimal, surface area.

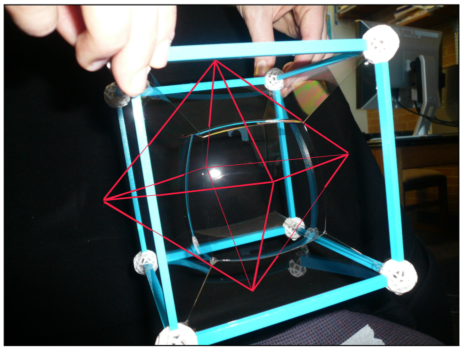


Figure 1. Equitent surface constructed on a cube wireframe.

replaced by a volume (bubble) that is enclosed by spherical caps in the same polyhedral arrangement. See the example soap film in Figure 1. Dorff et al. categorize these figures by the dual figure to the wire frame polyhedron. This dual figure, called the *connectivity graph*, is used to define the planes and spheres used in the construction of these surfaces and describes the adjacency conditions on the resulting surface. The specifics of the construction are not requisite to our results.

In their paper Dorff et al. also define a connectivity condition, which is that exterior regions share boundary only if the corresponding vertices in the connectivity graph are adjacent. They prove that the constructed surfaces are globally area minimizing among all surfaces that enclose the same fixed volume, have the same wire frame polyhedral boundary, and satisfy the connectivity condition.

3. Soap film stability

Theorem. *Among all the minimal surfaces of Dorff et al. in* $\mathbb{R}^3$, *there are only six that are stable as a soap film*: *those whose connectivity graphs are a single point*, *edge*, *equilateral triangle*, *regular tetrahedron*, *regular octahedron*, *or regular icosahedron.*

Proof. We relax the connectivity condition and look at which surfaces are locally area minimizing among surfaces that enclose the same fixed volume and have the same wire frame polyhedral boundary. We reduce conditions for local minimality to conditions on the connectivity graph.

First, in the constuction of the surfaces Dorff et al. require that the connectivity graph to be a uniform polyhedron (polytope) of unit edge length. A uniform polyhedron is one with regular polygon faces and congruent vertices. This guarantees the existence of particular vector fields needed in the minimization proof. They also require the circumradius of the connectivity graph to be strictly less than 1. A circumradius greater than or equal to 1 would create a central bubble of volume zero.

Uniform polyhedra that meet this condition are limited to the tetrahedron, cube, octahedron, icosahedron, triangular prism, pentagonal prism, square antiprism, and pentagonal antiprism.

Minimality conditions come from the work of Jean Taylor [1976]. She proved that Plateau's rules for soap films must hold for locally minimizing surfaces in $\mathbb{R}^3$. These are:

(1) Soap films are made of smooth surfaces of constant mean curvature.

(2) Soap films always meet in threes along a smooth curve, meeting at equal angles of 120^o.

(3) These curves meet in fours at a point, meeting at equal angles of $\cos^{-1}(-\frac{1}{3})$ (approximately 109^o).

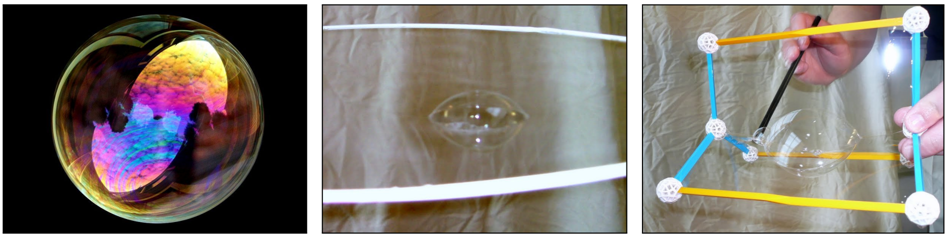
The first and third rules always hold as a result of the surface's construction. The second rule, however, further limits the number of connectivity graphs that can be formed. In the construction, each face of the connectivity graph corresponds to one of these curves (from a vertex of the wire-frame polyhedron) and each edge corresponds to a smooth surface connecting to this curve (from an edge of the wire-frame polyhedron). Thus the second rule implies that connectivity graphs must be constrained to have only triangular faces.

The uniform polyhedra that meet the conditions on the construction and satisfy this second rule are limited to the tetrahedron, octahedron, and icosahedron. For connectivity graphs in lower dimensions that also satisfy these conditions, we have a single point (0 dimensions), a line segment (1 dimension), and an equilateral triangle (2 dimensions). □

These conditions are very restrictive; out of the 18 convex uniform polyhedrons and infinite sets of prisms, antiprisms, and lower dimensional figures, only six equitent surfaces can be created in $\mathbb{R}^3$. In the next section we demonstrate each of these surfaces as a soap film.

4. Realization of the bubbles

Equitent surfaces can be realized as a soap film by dipping a wire-frame in a soap solution and blowing a soap bubble onto the surface. (It may however take several tries to get a surface of a particular homotopy class, and have it last long enough to take a picture!) Each of the six connectivity graphs identified in the last section do generate a stable minimal surface when realized as a soap film this way. Note that the wire-frame polyhedron in each case is the dual figure to the connectivity graph. Also note that the number of vertices in the connectivity graph corresponds to the number of exterior regions separated by the equitent surface.

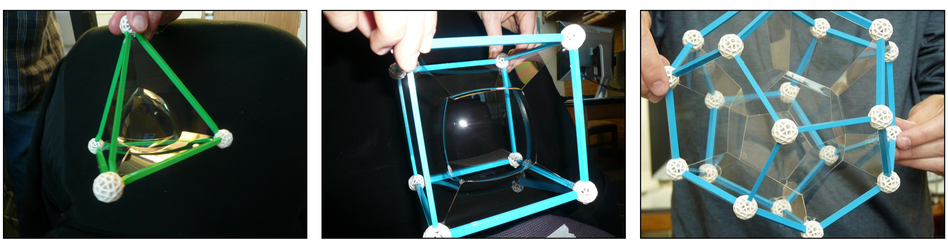


Dimension 0 Dimension 1 Dimension 2

Figure 2. Equitent surfaces with lower dimensional connectivity graph.

For lower dimensional connectivity graphs we see that the surface realized from a single point is a spherical bubble with no wire frame (Figure 2, left). A single edge as a connectivity graph yields a lens shaped bubble on a planar surface. Here we represent the wire-frame as a circle (any polygon in two dimensions will do); see Figure 2, middle. From an equilateral triangle we have a "football" shaped bubble connected to three planar surfaces (Figure 2, right).

For the three dimensional connectivity graphs, a polyhedral shaped bubble with spherical caps will be formed. These figures will also have planar surfaces connecting to each edge of the bubble. For tetrahedral, octahedral, or icosahedral connectivity graphs we get a tetrahedron-, cube-, or dodecahedron-shaped bubble, respectively. See Figure 3.



Tetrahedral graph Octahedral graph Icosahedral graph

Figure 3. Equitent surfaces with dimension-3 connectivity graph.

5. Conclusion

As noted earlier, we have seen that the connectivity condition of Dorff et al. is a very restrictive condition. Each of the locally minimizing surfaces were known prior to their work, though perhaps not yet proven to be minimal. The real impact of their paper comes from the pioneering new method of metacalibration and how we can use it to tackle equitent problems. Their paper gives the first new results

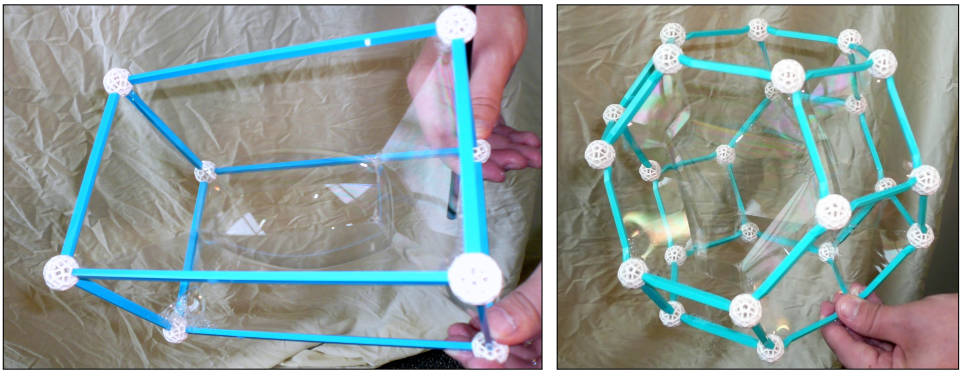


Figure 4. Other examples of equitent surfaces: rectangular prism wire-frame (left) and negative-pressure soap bubbles (right).

proven using this method, though it has also been used to provide new proofs of some multiple bubble problems [Dilts et al. $\geq$ 2011].

We hope to be able to generalize the metacalibration approach to handle further equitent problems. This includes finding an alternate construction of equitent surfaces that relaxes the uniformity condition on the connectivity graphs. This would allow us to investigate surfaces such as those generated on a rectangular prism wire-frame, not just a cube (Figure 4, left).

Another problem to consider are equitent surfaces that would be generated by connectivity graphs of circumradius greater than or equal to 1. Such surfaces are stable in $\mathbb{R}^2$ and $\mathbb{R}^3$, though the central bubble has negative pressure and the faces bow inwards (Figure 4, right).

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Zero forcing number, path cover number, and maximum nullity of cacti

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The zero forcing number of a graph is the minimum size of a zero forcing set. This parameter is useful in the minimum rank/maximum nullity problem, as it gives an upper bound to the maximum nullity. The path cover number of a graph is the minimum size of a path cover. Results for comparing the parameters are presented, with equality of zero forcing number and path cover number shown for all cacti and equality of zero forcing number and maximum nullity for a subset of cacti. (A cactus is a graph where each edge is in at most one cycle.)

1. Introduction

Throughout this paper, a *graph* $G = (V_G, E_G)$ will mean a simple (no loops, no multiple edges) undirected graph. We will assume a finite and non-empty vertex set V_G. The edge set E_G consists of two-element subsets of vertices. If $\{x, y\} \in E_G$, we say x and y are *neighbors* or x and y are *adjacent*, and write $x \sim y$.

The zero forcing number of a graph was introduced in [AIM 2008] and the related terminology was developed in [Barioli et al. 2009], [Barioli et al. 2010], and [Hogben 2010]. Referring to it as the graph infection number, physicists have used this parameter in studying quantum systems control [Burgarth and Giovannetti 2007; Burgarth and Maruyama 2009; Severini 2008]. Consider a black and white vertex coloring of a graph G. From the initial coloring, vertices change color according to the *color-change rule*: If v is the only white neighbor of a black vertex u, then change the color of v to black. Applying the color-change rule to u to change the color of v, we say *u forces v* and write $u \to v$. Given an initial coloring of G, the *derived set* is the set of vertices colored black after the color-change rule is applied until no more changes are possible. If the set Z of vertices initially colored black has derived set that is all the vertices of G, we say Z is a *zero forcing set* for G. A zero forcing set with the minimum number of vertices is called an *optimal zero forcing set*, and this minimum size of a zero forcing set for a graph G is the *zero forcing number* of the graph, denoted $\mathrm{Z}(G)$.

MSC2010: primary 05C50; secondary 15A03.

Keywords: zero forcing number, path cover number, maximum nullity, cut-edge, cut-vertex, cactus.

The *path cover number* $\mathrm{P}(G)$ of a graph G is the smallest positive integer m such that there are m vertex-disjoint induced paths in G such that every vertex of G is a vertex of one of the paths.

An association between graphs and matrices is made in the following way. Denote by $S_n(\mathbb{R})$ the set of $n \times n$ real symmetric matrices. The *graph* of $A \in S_n(\mathbb{R})$, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \dots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Given a graph G, the *set of symmetric matrices described by G* is $\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}$. The *minimum rank of* G is $\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G)\}$ and the *maximum nullity of* G is $\mathrm{M}(G) = \max\{\mathrm{null}\, A : A \in \mathcal{S}(G)\}$. Clearly $\mathrm{mr}(G) + \mathrm{M}(G) = |G|$, where the *order* $|G|$ is the number of vertices in G. Because of this relationship, finding the value of one of these two parameters for a graph is equivalent to finding the value for both.

Following are theorems relating the zero forcing number to path cover number and maximum nullity of a graph. These bounds will be used in later results.

Theorem 1.1 [Hogben 2010]. *For any graph G*, $\mathrm{P}(G) \leq \mathrm{Z}(G)$.

Theorem 1.2 [AIM 2008]. *For any graph G*, $\mathrm{M}(G) \leq \mathrm{Z}(G)$.

It is well known that if G is a tree then $\mathrm{P}(G) = \mathrm{Z}(G)$ [AIM 2008] and $\mathrm{P}(G) = \mathrm{M}(G)$ [Johnson and Duarte 1999], so the three parameters are equal.

In this paper, we compare the graph parameters $\mathrm{Z}(G)$, $\mathrm{P}(G)$, and $\mathrm{M}(G)$. In Section 2, we present the effect on the parameters after the deletion of a single vertex or the deletion of a single edge. These (mostly known) results will be utilized in later sections. Results of similar type for each of the graph parameters are presented in a unified format to emphasize the relationship to each other. The main result of Section 3 is equality of zero forcing number and path cover number for cacti, where a cactus is a graph where each edge is in at most one cycle. In Section 4, we prove zero forcing number is equal to maximum nullity for a restricted family of cacti. Section 5 summarizes our results and suggests further research.

Additional properties and some notation. Here we present additional terminology, notation, and theorems that will be used. For a given zero forcing set Z, a *chronological list of forces* is a listing of the forces used to construct the derived set in the order they are performed. A *forcing chain* for a chronological list of forces is a sequence of vertices $(v_1, v_2, \dots, v_k)$ such that for $i = 1, \dots, k-1$, $v_i \to v_{i+1}$, and a *maximal forcing chain* is a forcing chain that is not a proper subsequence of any other forcing chain. The collection of maximal forcing chains for a chronological list of forces is called the *chain set* of the chronological list of forces, and an *optimal chain set* is a chain set from a chronological list of forces of an optimal zero forcing set. When a chain set contains a chain consisting of a single vertex, we say that the chain set contains the vertex as a *singleton*. For a zero forcing set Z, a *reversal* of Z is the set of vertices which are last in the forcing chains in the chain set of some chronological list of forces [Barioli et al. 2010].

Theorem 1.3 [Barioli et al. 2010]. *If Z is a zero forcing set of G then so is any reversal of Z.*

Observation 1.4. If Z' is a reversal of Z, then $|Z'| = |Z|$. In particular, if Z is an optimal zero forcing set, then a reversal Z' of Z is also an optimal zero forcing set.

A vertex v is called *terminal* if it is the endpoint of a path in some minimum path cover. It is called *doubly terminal* if it is in a path by itself in some minimum path cover, and is called *simply terminal* if it is terminal but not doubly terminal.

For a graph $G = (V_G, E_G)$ and $W \subseteq V_G$, the *induced subgraph* $G[W]$ is the graph with vertex set W and edge set $\{\{v, w\} \in E_G : v, w \in W\}$. The subgraph induced by $\overline{W} = V_G \setminus W$ will be denoted by $G - W$, or in the case W is a single vertex $\{v\}$, by $G - v$. For $e \in E_G$, the subgraph $(V_G, E_G \setminus \{e\})$ will be denoted by $G - e$.

A graph is called *connected* if any two vertices are linked by a path. If a graph is not connected, we say it is *disconnected*. The maximal connected subgraphs of a graph are called the *components* of the graph. If the graph $G - v$ has more connected components than G, then v is called a *cut-vertex* of G. Similarly, a *cut-edge* of a graph is one such that its deletion increases the number of connected components.

2. Edge spread and vertex spread

We present a number of (mostly known) results which will be used in later sections. They are grouped and formatted in such a way as to emphasize commonality between the types of results for the different parameters.

Edge spread. In this subsection, we consider the effects on zero forcing number, path cover number, and maximum nullity when deleting a single edge from a graph. For a graph G and an edge e of G, the *rank edge spread* of e in G is $\mathrm{r}_e(G) = \mathrm{mr}(G) - \mathrm{mr}(G - e)$, the *null edge spread* of e in G is $\mathrm{n}_e(G) = \mathrm{M}(G) - \mathrm{M}(G - e)$, and the *zero edge spread* of e in G is $\mathrm{z}_e(G) = \mathrm{Z}(G) - \mathrm{Z}(G - e)$ [Edholm et al. 2010]. Here we make an analogous definition concerning change in path cover number when deleting an edge.

Definition 2.1. The *path edge spread* of e in G is $\mathrm{p}_e(G) = \mathrm{P}(G) - \mathrm{P}(G - e)$.

First we present the bounds on the zero edge spread and path edge spread and attempt to characterize edges with a given edge spread value.

Theorem 2.2 [Edholm et al. 2010]. *For every graph G and every edge $e = \{v, w\}$ of G, $-1 \le \mathrm{z}_e(G) \le 1$. If $\mathrm{z}_e(G) = 1$, then there exists an optimal chain set such that e is not an edge in any chain.*

Theorem 2.3. *For every graph G and every edge $e = \{v, w\}$ of G, $-1 \le \mathrm{p}_e(G) \le 1$. If $\mathrm{p}_e(G) = 1$, then there exists a minimum path cover such that v and w are not in the same path.*

Proof. Let G be a graph and $e = \{v, w\}$ be an edge in G. Consider a minimum path cover of G. If v and w are not covered by the same path, then this path cover of G is also a path cover of $G - e$. If v and w are covered by the same path in the path cover of G, then splitting the path into two paths will create a path cover of $G - e$. Either way, $\mathrm{P}(G - e) \le \mathrm{P}(G) + 1$ so $\mathrm{p}_e(G) \ge -1$.

Consider a minimum path cover of $G - e$. If v and w are not covered by the same path, then this path cover of $G - e$ is also a path cover of G (observe that this case cannot occur if $\mathrm{p}_e(G) = 1$). If v and w are covered by the same path in the path cover of $G - e$, there must be a vertex on the path between them. Let x be the vertex that is between v and w on the path and adjacent to v. Split the path between v and x. This is a path cover of G, but with one more than $\mathrm{P}(G-e)$ paths. In the case $\mathrm{p}_e(G) = 1$, this is a minimum path cover of G with v and w in different paths. Regardless of the path edge spread, $\mathrm{P}(G) \le \mathrm{P}(G - e) + 1$ so $\mathrm{p}_e(G) \le 1$. □

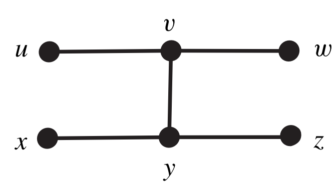
Theorem 2.4 [Edholm et al. 2010]. *Let $e = \{v, w\}$ be an edge of G. If $\mathrm{z}_e(G) = -1$, then for every optimal zero forcing chain set of G, e is an edge in a chain.*

Theorem 2.5. *Let $e = \{v, w\}$ be an edge of G. If $\mathrm{p}_e(G) = -1$, then for every minimum path cover of G, v and w are in the same path.*

Proof. The contrapositive will be proved. Let G be a graph and $e = \{v, w\}$ be an edge of G. Suppose there is a minimum path cover of G in which v and w are not in the same path. This path cover of G is also a path cover of $G - e$, so $\mathrm{P}(G - e) \le \mathrm{P}(G)$. Hence $\mathrm{p}_e(G) \ge 0$. □

Theorem 2.5 can be viewed as a partial converse to the second statement in Theorem 2.3. Here we provide an example showing that the converse of the second statement in Theorem 2.3 is not true. This example also shows the converse of the second statement in Theorem 2.2 is false.

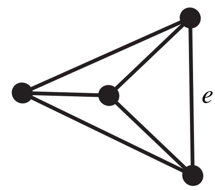
Example 2.6. Let G be this graph:



For $e = \{v, y\}$ we have $\mathrm{p}_e(G) = 0$, but v and y are not in the same path in the minimum path cover.

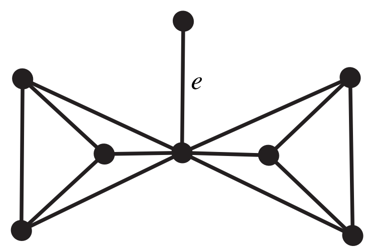
Although the bounds on $\mathrm{z}_e(G)$ and $\mathrm{p}_e(G)$ are the same, the parameters are not generally comparable, as can be seen in Examples 2.7 and 2.8 below. Null edge spread has the same bounds as well, and [Edholm et al. 2010] gives examples showing the incomparability of $\mathrm{z}_e(G)$ with $\mathrm{n}_e(G)$.

Example 2.7. Let G be this graph:



Here $\mathrm{Z}(G) = 3$ and $\mathrm{Z}(G - e) = \mathrm{P}(G) = \mathrm{P}(G - e) = 2$. Therefore, $\mathrm{z}_e(G) = 1 > 0 = \mathrm{p}_e(G)$.

Example 2.8. Let G be this graph:



Here $\mathrm{Z}(G) = 5$, $\mathrm{Z}(G - e) = 6$, and $\mathrm{P}(G) = \mathrm{P}(G - e) = 4$. Therefore, $\mathrm{z}_e(G) = -1 < 0 = \mathrm{p}_e(G)$.

Under the conditions of Observation 2.9 we can use one of parameters $\mathrm{z}_e(G)$ or $\mathrm{p}_e(G)$ to determine the other.

Observation 2.9. Let G be a graph such that $\mathrm{P}(G) = \mathrm{Z}(G)$ and let e be an edge of G. Then:

(1) $\mathrm{p}_e(G) \geq \mathrm{z}_e(G)$.

(2) If $\mathrm{z}_e(G) = 1$, then $\mathrm{p}_e(G) = 1$.

(3) If $\mathrm{p}_e(G) = -1$, then $\mathrm{z}_e(G) = -1$.

Next we consider edge spreads when the edge is a cut-edge.

Theorem 2.10 [Barioli et al. 2004]. *Let $e = \{v_1, v_2\}$ be a cut-edge of a connected graph G. Let G_1 and G_2 be the connected components of $G - e$ with $v_1 \in G_1$ and $v_2 \in G_2$. Then*

$$\mathrm{r}_e(G) = \begin{cases} 0 & \textit{if } \max_{i=1,2}\{r_{v_i}(G_i)\} = 2, \\ 1 & \textit{otherwise}. \end{cases}$$

Corollary 2.11. *Let $e = \{v_1, v_2\}$ be a cut-edge of a connected graph G. Let G_1 and G_2 be the connected components of $G - e$ with $v_1 \in G_1$ and $v_2 \in G_2$. Then*

$$\mathrm{n}_e(G) = \begin{cases} 0 & \text{if } \min_{i=1,2}\{n_{v_i}(G_i)\} = -1, \\ -1 & \text{otherwise}. \end{cases}$$

Proof. This follows from Theorem 2.10 and the fact that $\mathrm{r}_e(G) + \mathrm{n}_e(G) = 0$ for any graph G and any edge e of G. □

Theorem 2.12. *Let $e = \{v_1, v_2\}$ be a cut-edge of a connected graph G. Let G_1 and G_2 be the connected components of $G - e$ with $v_1 \in G_1$ and $v_2 \in G_2$. Then*

$$\mathrm{z}_e(G) = \begin{cases} -1 & \text{if } v_i \text{ is in an optimal zero forcing set in } G_i \text{ for } i = 1, 2, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Let Z_1 and Z_2 be optimal zero forcing sets for G_1 and G_2, respectively. Let $Z = Z_1 \cup Z_2$. Color the vertices of Z black and the remaining vertices white. Forces can be performed in G_1 until v_1 is black. Forces can be performed in G_2 until v_2 is black. Now the remaining forces can take place in G_1 and in G_2. Therefore Z is a zero forcing set for G and $\mathrm{Z}(G) \leq |Z| = \mathrm{Z}(G_1) + \mathrm{Z}(G_2) = \mathrm{Z}(G - e)$. Hence $\mathrm{z}_e(G) \leq 0$.

Suppose v_1 is an optimal zero forcing set Z_1 for G_1 and v_2 is in an optimal zero forcing set Z_2 in G_2. Let Z_1' be a reversal of Z_1. Then by Observation 1.4, Z_1' is an optimal zero forcing set for G_1 and there is a chronological list of forces in which v_1 does not perform a force (i.e., v_1 is last in the maximal forcing chain which contains it). Let $Z = Z_1' \cup Z_2 \setminus \{v_2\}$. Color the vertices of Z black and the remaining vertices white. Forces can be performed in G_1 until all vertices of G_1 are black and v_1 has not performed a force. Now v_1 is black and v_2 is the only white neighbor of v_1, so $v_1 \to v_2$. Now all the vertices of Z_2 are black and none has performed a force, so all other vertices of G_2 can be forced black. Therefore Z is a zero forcing set for G and $\mathrm{Z}(G) \leq |Z| = \mathrm{Z}(G_1) + \mathrm{Z}(G_2) - 1 = \mathrm{Z}(G - e) - 1$. Theorem 2.2 gives $\mathrm{z}_e(G) \geq -1$, so $\mathrm{z}_e(G) = -1$.

Suppose now that at least one of v_1 or v_2 is not in any optimal zero forcing set for the respective component. Without loss of generality, say v_1 is not in any optimal zero forcing set for G_1. Let Z be an optimal zero forcing set for G and consider the chronological list of forces. Examine the following cases.

Case 1: Suppose $v_1 \to v_2$. Then v_1 cannot force any vertex of G_1. Since v_1 is not in any optimal zero forcing set for G_1, it is not at the end of a forcing chain for any optimal zero forcing set of G_1. Thus v_1 forcing v_2 requires $|Z \cap V_{G_1}| \geq \mathrm{Z}(G_1) + 1$. It must also be that $|Z \cap V_{G_2}| \geq \mathrm{Z}(G_2) - 1$. Then $\mathrm{Z}(G) = |Z| = |Z \cap V_{G_1}| + |Z \cap V_{G_2}| \geq \mathrm{Z}(G_1) + \mathrm{Z}(G_2) = \mathrm{Z}(G - e)$, so $\mathrm{z}_e(G) \geq 0$.

Case 2: Suppose $v_1 \not\to v_2$. Then $|Z \cap V_{G_2}| \geq \mathrm{Z}(G_2)$. Since v_1 is not in any optimal zero forcing set for G_1, it must be that $|Z \cap V_{G_1}| \geq \mathrm{Z}(G_1)$. Then $\mathrm{Z}(G) = |Z| = |Z \cap V_{G_1}| + |Z \cap V_{G_2}| \geq \mathrm{Z}(G_1) + \mathrm{Z}(G_2) = \mathrm{Z}(G - e)$, so $\mathrm{z}_e(G) \geq 0$. □

Theorem 2.13 [Barioli et al. 2004]. *Let $e = \{v_1, v_2\}$ be a cut-edge of a connected graph G. Let G_1 and G_2 be the connected components of $G - e$ with $v_1 \in G_1$ and $v_2 \in G_2$. Then*

$$\mathrm{p}_e(G) = \begin{cases} -1 & \textit{if } v_i \textit{ is terminal in } G_i \textit{ for } i = 1, 2, \\ 0 & \textit{otherwise}. \end{cases}$$

The converse of Theorem 2.4 is open from [Edholm et al. 2010], and the converse of Theorem 2.5 is left open in this paper. We will show that the converses of these theorems are true for a cut-edge.

Theorem 2.14. *Let $e = \{v, w\}$ be a cut-edge of G. If e is an edge in a chain for every optimal zero forcing chain set of G, then $\mathrm{z}_e(G) = -1$.*

Proof. The contrapositive will be proved. Suppose $\mathrm{z}_e(G) \neq -1$. By Theorem 2.12, $\mathrm{z}_e(G) = 0$. Let G_1 and G_2 be the connected components of $G - e$ with $v \in G_1$ and $w \in G_2$. Let Z_1 and Z_2 be optimal zero forcing sets for G_1 and G_2, respectively. Let $Z = Z_1 \cup Z_2$. Color the vertices of Z black and the remaining vertices white. Forces can be performed in G_1 until v is black. Forces can be performed in G_2 until w is black. Now the remaining forces can take place in G_1 and in G_2. Therefore Z is a zero forcing set for G and $e = \{v, w\}$ is not an edge in any chain. Also, $|Z| = \mathrm{Z}(G_1) + \mathrm{Z}(G_2) = \mathrm{Z}(G - e) = \mathrm{Z}(G) - \mathrm{z}_e(G) = \mathrm{Z}(G)$, so Z is an optimal zero forcing set for G. □

Theorem 2.15. *Let $e = \{v, w\}$ be a cut-edge of G. If v and w are in the same path for every minimum path cover of G, then $\mathrm{p}_e(G) = -1$.*

Proof. The contrapositive will be proved. Suppose $\mathrm{p}_e(G) \neq -1$. By Theorem 2.13, $\mathrm{p}_e(G) = 0$. Let G_1 and G_2 be the connected components of $G - e$ with $v \in G_1$ and $w \in G_2$. Consider a path cover of G consisting of minimum path covers of G_1 and G_2. Then v and w are not in the same path of this path cover of G. Also, since $\mathrm{p}_e(G) = 0$, this path cover of G is minimum. □

Vertex spread. In this section, we consider the effects on minimum rank, maximum nullity, zero forcing number, and path cover number when deleting a single vertex from a graph. For a graph G and a vertex v of G, the *rank spread* of v in G is $\mathrm{r}_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v)$ [Barioli et al. 2004], the *null spread* of v in G is $\mathrm{n}_v(G) = \mathrm{M}(G) - \mathrm{M}(G - v)$ [Edholm et al. 2010], the *zero spread* of v in G is $\mathrm{z}_v(G) = \mathrm{Z}(G) - \mathrm{Z}(G - v)$ [Edholm et al. 2010], and the *path spread* of v in G is $\mathrm{p}_v(G) = \mathrm{P}(G) - \mathrm{P}(G - v)$ [Barioli et al. 2005].

Theorem 2.16 [Edholm et al. 2010; Huang et al. 2010]. *For every graph G and vertex v of G, $-1 \leq \mathrm{z}_v(G) \leq 1$.*

Theorem 2.17 [Barioli et al. 2004; Barioli et al. 2005]. *For every graph G and vertex v of G, $-1 \leq \mathrm{p}_v(G) \leq 1$.*

Recall that v being contained as a singleton means it is in a forcing chain by itself in an optimal chain set, and v being doubly terminal means it is in a path by itself in a minimum path cover.

Theorem 2.18 [Edholm et al. 2010]. *Let v be a vertex of G. Then $\mathrm{z}_v(G) = 1$ if and only if there exists an optimal chain set of G that contains v as a singleton.*

Theorem 2.19 [Barioli et al. 2005]. *Let v be a vertex of G. Then $\mathrm{p}_v(G) = 1$ if and only if v is doubly terminal.*

Theorem 2.20 [Edholm et al. 2010]. *Let v be a vertex of G. If $\mathrm{z}_v(G) = -1$, then v is never in an optimal zero forcing set for G.*

Theorem 2.21 [Barioli et al. 2005]. *Let v be a vertex of G. If $\mathrm{p}_v(G) = -1$, then v is not terminal.*

The next theorems give the parameter spreads for a cut-vertex. Recall that v being simply terminal means that v is terminal but not doubly terminal. By Theorems 2.19 and 2.21, this is equivalent to the path spread being zero and v being an endpoint in some minimal path cover.

Theorem 2.22 [Barioli et al. 2004]. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \dots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Then*

$$\mathrm{r}_v(G) = \min \left\{ \sum_{i=1}^{k} \mathrm{r}_v(G_i), 2 \right\}$$

Corollary 2.23. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \dots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[W_i \cup \{v\}]$. Let m denote $\min_{1 \leq j \leq k}\{\mathrm{n}_v(G_j)\}$, and t denote the number of the G_i's in which $\mathrm{n}_v(G_i) = 0$. Then*

$$\mathrm{n}_v(G) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = 0 \text{ and } t = 1, \\ -1 & \text{if } m = 0 \text{ and } t \geq 2, \text{ or if } m = -1. \end{cases}$$

Proof. This follows from Theorem 2.22 and the fact that $\mathrm{r}_v(G) + \mathrm{n}_v(G) = 1$ for any graph G and any vertex v of G. $\square$

Theorem 2.24 [Row 2011]. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \dots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \le i \le k$, let $G_i = G[W_i \cup \{v\}]$. Let m denote $\min_{1 \le j \le k}\{\mathrm{z}_v(G_j)\}$, and t denote the number of the G_i's in which $\mathrm{z}_v(G_i) = 0$ and v is in an optimal zero forcing set. Then*

$$\mathrm{z}_v(G) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = 0 \text{ and } t \le 1, \\ -1 & \text{if } m = 0 \text{ and } t \ge 2, \text{ or if } m = -1. \end{cases}$$

Theorem 2.25 [Barioli et al. 2005]. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \dots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \le i \le k$, let $G_i = G[W_i \cup \{v\}]$. Let m denote $\min_{1 \le j \le k}\{\mathrm{p}_v(G_j)\}$, and t denote the number of the G_i's in which v is simply terminal. Then*

$$\mathrm{p}_v(G) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = 0 \text{ and } t \le 1, \\ -1 & \text{if } m = 0 \text{ and } t \ge 2, \text{ or if } m = -1. \end{cases}$$

3. Comparing Z(G) and P(G) for cacti

A block of a graph is a maximal connected subgraph without a cut-vertex. A *cactus* is a graph in which each block is either a cycle or an edge. In other words, a cactus is a graph in which any two cycles share at most one vertex. An example of a cactus is shown in Figure 1. In this section, we prove $\mathrm{Z}(G) = \mathrm{P}(G)$ for any cactus G. We begin with a few preliminaries.

Theorem 3.1 [Row 2011]. *Let G be a unicyclic graph. Then* $\mathrm{Z}(G) = \mathrm{P}(G)$.

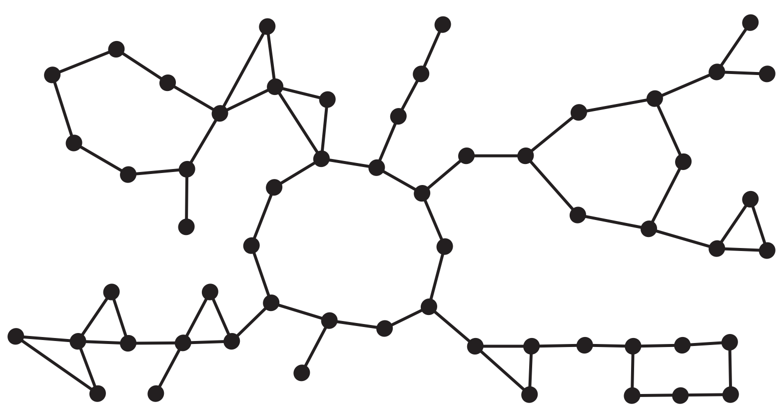


Figure 1. A cactus. No edge is in more than one cycle.

Lemma 3.2. *Let G be a graph, v a vertex in G, and H the graph constructed by appending a leaf w to v in G. Suppose* $\mathrm{Z}(G) = \mathrm{P}(G)$ *and* $\mathrm{Z}(H) = \mathrm{P}(H)$. *The vertex v is in an optimal zero forcing set for G if and only if v is terminal in G.*

Proof. Suppose v is in an optimal zero forcing set for G. An optimal chain set from this optimal zero forcing set determines a path cover of G with $\mathrm{Z}(G) = \mathrm{P}(G)$ paths and v as an endpoint of a path. Hence v is terminal.

Suppose v is terminal in G. Then $e = \{v, w\}$ is a cut edge and the graph $H' = (\{w\}, \varnothing)$ is a single isolated vertex. Therefore, w is terminal in H'. By Theorem 2.13, $\mathrm{p}_e(H) = -1$. By Observation 2.9, $\mathrm{z}_e(H) = -1$. By Theorem 2.12, v is in an optimal zero forcing set for G. □

Theorem 3.3. *Let G be a cactus. Then* $\mathrm{Z}(G) = \mathrm{P}(G)$.

Proof. The theorem will be proved by induction on the number of cycles in the cactus. If there is one cycle, G is a unicyclic graph and by Theorem 3.1, $\mathrm{Z}(G) = \mathrm{P}(G)$. Suppose now that for some $m \geq 2$ any cactus G with less than m cycles satisfies $\mathrm{Z}(G) = \mathrm{P}(G)$. Let G be a cactus with m cycles. Since the cycles are edge disjoint, there is a cut-vertex v such that $G - v$ has connected components with vertex sets $W_1, \ldots, W_k$ and each graph $G_i = G[W_i \cup \{v\}], \forall i = 1, \ldots k$ is a cactus with fewer than m cycles. By the inductive hypothesis, $\mathrm{Z}(G_i) = \mathrm{P}(G_i), \forall i = 1, \ldots, k$ and $\mathrm{Z}(G_i - v) = \mathrm{P}(G_i - v), \forall i = 1, \ldots, k$, so $\mathrm{z}_v(G_i) = \mathrm{p}_v(G_i), \forall i = 1, \ldots, k$. Therefore, $\min_{1 \leq j \leq k}\{\mathrm{z}_v(G_j)\} = \min_{1 \leq j \leq k}\{\mathrm{p}_v(G_j)\}$. For all $i = 1, \ldots k$, consider the graphs H_i constructed by appending a leaf w_i to v in G_i. By the inductive hypothesis, $\mathrm{Z}(G_i) = \mathrm{P}(G_i), \forall i = 1, \ldots k$ and $\mathrm{Z}(H_i) = \mathrm{P}(H_i), \forall i = 1, \ldots k$. By Lemma 3.2, v is in an optimal zero forcing set for G_j if and only if v is terminal in G_j. Then $\mathrm{z}_v(G_j) = 0$ and v is in an optimal zero forcing set for G_j if and only if $\mathrm{p}_v(G_j) = 0$ and v is terminal in G_j if and only if v is simply terminal in G_j by the contrapositive of Theorem 2.19. Then by Theorems 2.24 and 2.25, $\mathrm{z}_v(G) = \mathrm{p}_v(G)$. Hence $\mathrm{Z}(G) = \sum_{i=1}^{k} \mathrm{Z}(G_i - v) + \mathrm{z}_v(G) = \sum_{i=1}^{k} \mathrm{P}(G_i - v) + \mathrm{p}_v(G) = \mathrm{P}(G)$. □

4. Comparing Z(G) and M(G) for cacti

In Section 3 we showed equality of $\mathrm{Z}(G)$ and $\mathrm{P}(G)$ for all cacti G by utilizing Theorem 3.1 for the base case in the induction proof. Since it is not true that $\mathrm{Z}(G) = \mathrm{M}(G)$ for all unicyclic graphs, in this section we focus on a subset of cacti and prove $\mathrm{Z}(G) = \mathrm{M}(G)$ for each graph in this subset.

Let C_n be an n-cycle and let $U \subseteq V_{C_n}$. The graph H obtained from C_n by appending a leaf to each vertex in U is called a *partial n-sun*. If $U = V_{C_n}$, then H is called an n-sun. It was shown in [Barioli et al. 2005] that $\mathrm{M}(H) = \mathrm{P}(H)$ for partial n-suns except for n-suns with $n > 3$ odd.

If there are at least two components of the graph $G - v$ which are paths, each joined to v in G at only one endpoint, then vertex v is called *appropriate*. A vertex v is called a *peripheral leaf* if v is adjacent to only one other vertex u, and u is adjacent to no more than two vertices. The *trimmed form* of a graph G is an induced subgraph obtained by a sequence of deletions of appropriate vertices, isolated paths, and peripheral leaves until no more such deletions are possible.

Theorem 4.1 [Row 2011]. *If the trimmed form of G, $\breve{G}$, can be obtained by performing n_1 deletions of appropriate vertices, n_2 deletions of isolated paths, and n_3 deletions of peripheral leaves, then $\mathrm{Z}(G) = \mathrm{Z}(\breve{G}) + n_2 - n_1$.*

Theorem 4.2 [Barioli et al. 2005]. *If the trimmed form of G, $\breve{G}$, can be obtained by performing n_1 deletions of appropriate vertices, n_2 deletions of isolated paths, and n_3 deletions of peripheral leaves, then $\mathrm{M}(G) = \mathrm{M}(\breve{G}) + n_2 - n_1$.*

Theorem 4.3 [Barioli et al. 2005]. *The trimmed form of a unicyclic graph G is either the empty graph or a partial n-sun.*

Observation 4.4. The trimmed form of a unicyclic graph G in which at least one of the cycle vertices has only two neighbors is not an n-sun.

The following theorem and lemma will be used in the proof of Theorem 4.7, the main result of this section.

Theorem 4.5. *Let G be a unicyclic graph in which the cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors. Then $\mathrm{Z}(G) = \mathrm{M}(G)$.*

Proof. Let $\breve{G}$ be the trimmed form of G. By Theorem 4.3 and Observation 4.4, $\breve{G}$ is either the empty graph or a partial n-sun, but not an n-sun with n odd and greater than three. The formulas from [Barioli et al. 2005] give $\mathrm{M}(\breve{G}) = \mathrm{P}(\breve{G})$. Theorem 3.1 gives $\mathrm{Z}(\breve{G}) = \mathrm{P}(\breve{G})$, so $\mathrm{Z}(\breve{G}) = \mathrm{M}(\breve{G})$. Then $\mathrm{Z}(G) = \mathrm{M}(G)$ by Theorems 4.1 and 4.2. □

Lemma 4.6. *Let G be a graph, v a vertex in G, and H the graph constructed from G by appending a leaf w to v, then appending a leaf x to w. Suppose $\mathrm{Z}(G) = \mathrm{M}(G)$ and $\mathrm{Z}(H) = \mathrm{M}(H)$. The vertex v is in an optimal zero forcing set for G if and only if $\mathrm{n}_v(G) = 0$.*

Proof. By construction, $e = \{v, w\}$ is a cut edge and the graph

$$H' = \{\{w, x\}, \{\{w, x\}\}\}$$

is a path on two vertices. Since $\mathrm{Z}(H') = \mathrm{M}(H')$, $\mathrm{z}_e(H) = \mathrm{n}_e(H)$. Also, w is in an optimal zero forcing set for H' and $n_w(H') = 0$. Then $\mathrm{n}_v(G) = 0 \Leftrightarrow \mathrm{n}_e(H) = -1 \Leftrightarrow \mathrm{z}_e(H) = -1 \Leftrightarrow v$ is in an optimal zero forcing set for G by Corollary 2.11 and Theorem 2.12. □

Here we present the main result of the section.

Theorem 4.7. *Let G be a cactus in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors. Then* $\mathrm{Z}(G) = \mathrm{M}(G)$.

Proof. Let G be a cactus in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors. The theorem will be proved by induction on the number of cycles in the cactus. If there is one cycle, G is a unicyclic graph in which the cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors, and by Theorem 4.5, $\mathrm{Z}(G) = \mathrm{M}(G)$. Suppose now that for some $m \geq 2$ any cactus G in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors with less than m cycles satisfies $\mathrm{Z}(G) = \mathrm{M}(G)$. Let G be a cactus in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors with m cycles. Since the cycles are edge disjoint, there is a cut-vertex v such that $G - v$ has connected components with vertex sets $W_1, \ldots, W_k$ and each graph $G_i = G[W_i \cup \{v\}], \forall i = 1, \ldots k$ is a cactus in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors with fewer than m cycles. By the inductive hypothesis, $\mathrm{Z}(G_i) = \mathrm{M}(G_i), \forall i = 1, \ldots, k$ and $\mathrm{Z}(G_i - v) = \mathrm{M}(G_i - v), \forall i = 1, \ldots, k$, so $\mathrm{z}_v(G_i) = \mathrm{n}_v(G_i), \forall i = 1, \ldots, k$. Therefore, $\min_{1 \leq j \leq k}\{\mathrm{z}_v(G_j)\} = \min_{1 \leq j \leq k}\{\mathrm{n}_v(G_j)\}$. For all $i = 1, \ldots k$, consider the graphs H_i constructed by appending a leaf w_i to v in G_i then appending a leaf x_i to w_i. By the inductive hypothesis, $\mathrm{Z}(G_i) = \mathrm{M}(G_i), \forall i = 1, \ldots k$ and $\mathrm{Z}(H_i) = \mathrm{M}(H_i), \forall i = 1, \ldots k$. By Lemma 4.6, v is in an optimal zero forcing set for G_j if and only if $\mathrm{n}_v(G_j) = 0$. Then $\mathrm{z}_v(G_j) = 0$ and v is in an optimal zero forcing set for G_j if and only if $\mathrm{n}_v(G_j) = 0$. Then by Theorem 2.24 and Corollary 2.23, $\mathrm{z}_v(G) = \mathrm{n}_v(G)$. Hence

$$\mathrm{Z}(G) = \sum_{i=1}^{k} \mathrm{Z}(G_i - v) + \mathrm{z}_v(G) = \sum_{i=1}^{k} \mathrm{M}(G_i - v) + \mathrm{n}_v(G) = \mathrm{M}(G). \qquad \square$$

The restrictions imposed on the cacti in this section are sufficient for $\mathrm{Z}(G) = \mathrm{M}(G)$, but are not necessary, as can be seen in the following example.

Example 4.8. The graph G shown in Figure 2 does not satisfy the property that each odd cycle of size five or more has at least one vertex with only two neighbors, but does satisfy $\mathrm{Z}(G) = \mathrm{M}(G)$.

5. Conclusions and open questions

We utilized cut-vertex and cut-edge results for zero forcing number, path cover number, and maximum nullity to build graphs having equality of parameters from smaller graphs having equality of the same parameters. Specifically, from knowing $\mathrm{Z}(G) = \mathrm{P}(G)$ for unicyclic graphs we showed $\mathrm{Z}(G) = \mathrm{P}(G)$ for cacti, and from

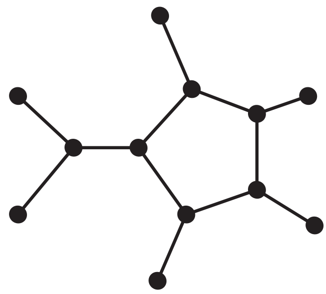


Figure 2. A cactus G that is not in the restricted family but which satisfies $\mathrm{Z}(G) = \mathrm{M}(G)$.

$\mathrm{Z}(G) = \mathrm{M}(G)$ for a restricted family of unicyclic graphs we showed $\mathrm{Z}(G) = \mathrm{M}(G)$ for a restricted family of cacti.

Question 5.1. *What other graphs with equality of some parameters have additional properties that would allow cut-vertex and cut-edge results to be utilized to "build" larger graphs having equality of the parameters*?

Question 5.2. *What are necessary conditions for a cactus to satisfy* $\mathrm{Z}(G) = \mathrm{M}(G)$?

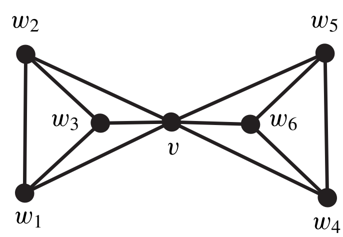
The converse of Theorem 2.4 is open from [Edholm et al. 2010]. We proved the converse holds if e is a cut-edge. We also proved the converse of Theorem 2.5 holds for a cut-edge.

Question 5.3. *Is the converse of Theorem 2.5 true*? *That is*, *if v and w are in the same path for every minimum path cover of G*, *does* $\mathrm{p}_e(G) = -1$ *where* $e = \{v, w\}$?

In general, v being in an optimal zero forcing set does not imply it being terminal, nor does v being terminal imply it being in an optimal zero forcing set, as evidenced by Examples 5.5 and 5.6 below. With the hypothesis that $\mathrm{Z}(G) = \mathrm{P}(G)$, we do get v in an optimal zero forcing set implying v terminal, as can be seen in the first part of the proof for Lemma 3.2 where the graph H is not used. The hypothesis about H is needed in Lemma 4.6 (see Example 5.7).

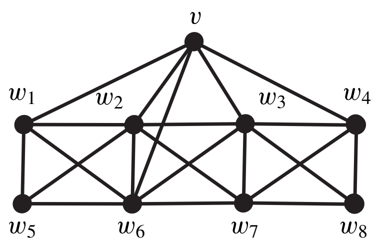
Question 5.4. *Is the graph H from the hypothesis of Lemma 3.2 necessary for the conclusion*? *For a graph G with* $\mathrm{Z}(G) = \mathrm{P}(G)$, *does vertex v being terminal imply v is in an optimal zero forcing set*?

Example 5.5. The vertex v is a cut-vertex for this graph G:



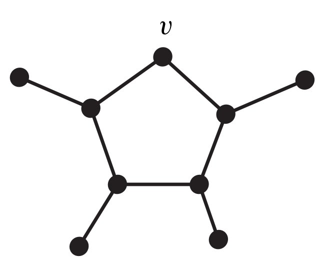
Now both $G[\{v, w_1, w_2, w_3\}]$ and $G[\{v, w_4, w_5, w_6\}]$ are K_4, so we can write $\mathrm{z}_v(G[\{v, w_1, w_2, w_3\}]) = \mathrm{z}_v(G[\{v, w_4, w_5, w_6\}]) = 1$ and v is simply terminal in $G[\{v, w_1, w_2, w_3\}]$ and $G[\{v, w_4, w_5, w_6\}]$. Hence $\mathrm{z}_v(G) = 1$ and $\mathrm{p}_v(G) = -1$ by Theorems 2.24 and 2.25. Therefore, v is in an optimal zero forcing set but not terminal by Theorems 2.18 and 2.21.

Example 5.6. Let G be this graph:



Then $\mathrm{Z}(G - v) = 5$ by [AIM 2008]. By Theorem 2.16, $\mathrm{Z}(G) \geq 4$ and moreover $\{w_2, w_3, w_5, w_6\}$ is a zero forcing set, so $\mathrm{Z}(G) = 4$. The graph $G - v$ is not a path, so $\mathrm{P}(G - v) \geq 2$ and $\{(w_1, w_2, w_3, w_4, w_5), (w_6, w_7, w_8, w_9, w_{10})\}$ is a path cover for $G - v$. Therefore, $\mathrm{P}(G - v) = 2$. By Theorem 2.17, and considering G is not a path, $2 \leq \mathrm{P}(G) \leq 3$. To show $\mathrm{P}(G) \neq 2$, attempt to cover G with two induced paths and consider w_5. If w_5 was in a path by itself, the other eight vertices cannot be covered with a single induced path, so w_5 has to be in a path with other vertices. Since the three neighbors of w_5 are all neighbors of each other, w_5 has to be an endpoint of an induced path. Consider which neighbor is in the path with w_5. If w_1 is with w_5, then w_2 and w_6 have to be in the other path, then v, w_3, and w_7 have to be with w_5 and w_1, then w_4 and w_8 have to be with w_2 and w_6, but $G[\{w_2, w_4, w_6, w_8\}]$ is not a path. If w_2 is with w_5, then w_1 and w_6 have to be in the other path, then v has to be with w_5 and w_2, then w_3 has to be with w_1 and w_6, then w_7 has to be with w_5, w_2, and v, but $G[\{v, w_2, w_5, w_7\}]$ is not a path. If w_6 is with w_5, then w_1 and w_2 have to be in the other path, then v has to be with w_5 and w_6, then w_3 has to be with w_1 and w_2, then w_7 has to be with w_5, w_6, and v, but $G[\{v, w_5, w_6, w_7\}]$ is not a path. So $\mathrm{P}(G) \geq 3$. Hence $\mathrm{z}_v(G) = -1$ and $\mathrm{p}_v(G) = 1$. Hence, v is terminal but never in an optimal zero forcing set by Theorems 2.19 and 2.20.

Example 5.7. Let G be this graph:



Then $\mathrm{Z}(G) = \mathrm{M}(G)$ and $\mathrm{n}_v(G) = 0$, but v is not in an optimal zero forcing set for G.

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Jacobson's refinement of Engel's theorem for Leibniz algebras

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We develop Jacobson's refinement of Engel's Theorem for Leibniz algebras. We then note some consequences of the result.

Since Leibniz algebras were introduced in [Loday 1993] as a noncommutative generalization of Lie algebras, one theme has been to extend Lie algebra results to Leibniz algebras. In particular, Engel's theorem has been extended in [Ayupov and Omirov 1998; Barnes 2011; Patsourakos 2007]. In the second of these works, the classical Engel's theorem is used to give a short proof of the result for Leibniz algebras. The proofs in the other two papers do not use the classical theorem and, therefore, the Lie algebra result is included in the result. In this note, we give two proofs of the generalization to Leibniz algebras of Jacobson's refinement to Engel's theorem, a short proof which uses Jacobson's theorem and a second proof which does not use it. It is interesting to note that the technique of reducing the problem to the special Lie algebra case significantly shortens the proof for the general Leibniz algebras case. This approach has been used in a number of situations [Barnes 2011]. We also note some standard consequences of this theorem. The proofs of the corollaries are exactly as in Lie algebras (see [Kaplansky 1971]). Our result can be used to directly show that the sum of nilpotent ideals is nilpotent, and hence one has a nilpotent radical. In this paper, we consider only finite dimensional algebras and modules over a field $\mathbb{F}$.

An algebra A is called Leibniz if it satisfies $x(yz) = (xy)z + y(xz)$. Denote by R_a and L_a, respectively, right and left multiplication by $a \in A$. Then

$$R_{bc} = R_c R_b + L_b R_c, \tag{1}$$

$$L_b R_c = R_c L_b + R_{bc}, \tag{2}$$

$$L_c L_b = L_{cb} + L_b L_c. \tag{3}$$

MSC2010: primary 17A32; secondary 17B30.

Keywords: Jacobson's refinement, Engel's Theorem, Leibniz algebras, Lie algebras, nilpotent, bimodule.

Using (1) and (2) we obtain

$$R_cR_b = -R_cL_b. \tag{4}$$

It is known that $L_b = 0$ if $b = a^i$, $i \geq 2$, where $a^1 = a$ and a^n is defined inductively as $a^{n+1} = aa^n$. Furthermore, for $n > 1$, $R_a^n = (-1)^{n-1}R_aL_a^{n-1}$. Therefore R_a is nilpotent if L_a is nilpotent.

For any set X in an algebra, we let $\langle X \rangle$ denote the algebra generated by X. Using (1), $R_{a^2} = (R_a)^2 + L_aR_a$. Furthermore, the associative algebra generated by all R_b, L_b, $b \in \langle a \rangle$ is equal to $\langle R_a, L_a \rangle$. Suppose that $L_a^{n-1} = 0$. Then $R_a^n = 0$. For any $s \in \langle R_a, L_a \rangle$, s^{2n-1} is a combination of terms with each term having at least $2n-1$ factors. Moreover, each of these factors is either L_a or R_a. Any L_a to the right of the first R_a can be turned into an R_a using (4). Hence, any term with $2n-1$ factors can be converted into a term with either L_a in the first $n-1$ leading positions or R_a in the last n postitions. In either case, the term is 0 and $s^{2n-1} = 0$. Thus $\langle R_a, L_a \rangle$ is nil and hence nilpotent.

Let M be an A-bimodule and let $T_a(m) = am$ and $S_a(m) = ma$, $a \in A$, $m \in M$. The analogues of (1)–(4) hold:

$$S_{bc} = S_cS_b + T_bS_c, \tag{5}$$

$$T_bS_c = S_cT_b + S_{bc}, \tag{6}$$

$$T_cT_b = T_{cb} + T_bT_c, \tag{7}$$

$$S_cS_b = -S_cT_b. \tag{8}$$

These operations have the same properties as L_a and R_a, and the associative algebra $\langle T_a, S_a \rangle$ generated by all T_b, S_b, $b \in \langle a \rangle$ is nilpotent if T_a is nilpotent. We record this as

Lemma. *Let A be a finite dimensional Leibniz algebra and let $a \in A$. Let M be a finite dimensional A-bimodule such that T_a is nilpotent on M. Then S_a is nilpotent, and $\langle S_a, T_a \rangle$, the algebra generated by all S_b, T_b, $b \in \langle a \rangle$, is nilpotent.*

A subset of A which is closed under multiplication is called a Lie set.

Theorem (Jacobson's refinement of Engel's theorem for Leibniz algebras). *Let A be a finite dimensional Leibniz algebra and M be a finite dimensional A-bimodule. Let C be a Lie set in A such that $A = \langle C \rangle$. Suppose that T_c is nilpotent for each $c \in C$. Then, for all $a \in A$, the associative algebra $B = \langle S_a, T_a \rangle$ is nilpotent. Consequently B acts nilpotently on M, and there exists $m \in M$, $m \neq 0$, such that $am = ma = 0$ for all $a \in A$.*

Proof 1 (*using the Lie result*). If M is irreducible, then either $MA = 0$ or $ma = -am$ for all a in A and all m in M from [Barnes 2011, Lemma 1.9]. Since left multiplication of A on M gives a Lie module, the Jacobson refinement to Engel's theorem yields that A acts nilpotently on M on the left and hence on M as a bimodule. If M is not irreducible, then A acts nilpotently on the irreducible factors in a composition series of M and hence on M. $\square$

Proof 2 (*independent of the Lie result*). Let $x \in C$. Then T_x is nilpotent and the associative algebra generated by T_b and S_b for all $b \in \langle x \rangle$ is nilpotent by the lemma. Since $\{a \mid aM = 0 = Ma\}$ is an ideal in A, we may assume that A acts faithfully on M.

Let D be a Lie subset of C such that $\langle D \rangle$ acts nilpotently on M, and $\langle D \rangle$ is maximal with these properties. If $C \subseteq \langle D \rangle$, then $A = \langle C \rangle = \langle D \rangle$, and we are done. Thus suppose that $C \nsubseteq \langle D \rangle$, and we will obtain a contradiction.

Let $E = \langle D \rangle \cap C$. E is a Lie set since both $\langle D \rangle$ and C are Lie sets. Since $D \subseteq \langle D \rangle$ and $D \subseteq C$, it follows that $D \subseteq E$ and $\langle D \rangle \subseteq \langle E \rangle$. Since $E \subseteq \langle D \rangle$, $\langle E \rangle \subseteq \langle D \rangle$ and $\langle D \rangle = \langle E \rangle$.

Let $\dim(M) = n$. Since $\langle D \rangle = \langle E \rangle$ acts nilpotently on M, $\sigma_1 \cdots \sigma_n = 0$ where $\sigma_i = S_{d_i}$ or T_{d_i} for $d_i \in E$. Then:
$\sigma_1 \cdots \sigma_i \tau \sigma_{i+1} \cdots \sigma_{2n-1} = 0$ where $\tau = S_a$ or T_a, $a \in A$, for all i.

If x is any product in A with $2n$ terms, of which $2n-1$ come from E, then S_x and T_x are linear combinations of elements as in the last paragraph. Hence $S_x = T_x = 0$, which implies that $x = 0$, since the representation is faithful.

There exists a smallest positive integer j such that $\tau_1 \cdots \tau_j C \subseteq \langle E \rangle$ for all $\tau_1, \ldots, \tau_j$ with $\tau_i = R_{d_i}$ or L_{d_i} where $d_i \in E$. Then there exists an expression $z = \tau_{d_1} \cdots \tau_{d_{j-1}} x \notin \langle E \rangle$ for some $x \in C$ and $d_i \in E$. Note that $z \in C$ since C is a Lie set. Consider zE. Now, zC, $Cz \subseteq C$ and $z\langle E \rangle$, $\langle E \rangle z \subseteq \langle E \rangle$. Therefore zE, $Ez \subseteq E$. Then $z^n E$, $Ez^n \subseteq E$ for all positive integers n, using induction and the defining identity for Leibniz algebras. Then $F = \{z^n, n \geq 1\} \cup E$ is a Lie set contained in C, and since $z \notin \langle E \rangle$, it follows that $\langle E \rangle \subsetneq \langle F \rangle$.

It remains to show that $\langle F \rangle$ acts nilpotently on M. Define $M_0 = 0$ and

$$M_i = \{m \in M \mid Em, mE \subseteq M_{i-1}\}.$$

Since E acts nilpotently on M, $M_k = M$ for some k. We show zM_i, $M_i z \subseteq M_i$. Clearly $zM_0 = M_0 z = 0$. Suppose that z acts invariantly on M_i for all $i < t$. For $m \in M_t$, $d \in E$, $(zm)d = z(md) - m(zd) \in zM_{t-1} + mE \subseteq M_{t-1}$ with similar expressions for $(mz)d$, $d(mz)$ and $d(zm)$. Thus z acts invariantly on each M_i, and hence z^2 does also. Thus $\langle z \rangle$ acts invariantly on each M_i. But $\langle z \rangle$ acts nilpotently on M by the lemma. Hence F acts nilpotently on M, which is a contradiction. $\square$

We obtain the abstract version of the theorem.

Corollary 1. *Let C be a Lie set in a Leibniz algebra A such that $\langle C \rangle = A$ and L_c is nilpotent for all $c \in C$. Then A is nilpotent.*

The following are extensions of results from [Jacobson 1955], whose proofs are the same as in the Lie algebra case.

Corollary 2. *If T is an automorphism of A of order p and has no nonzero fixed points, then A is nilpotent.*

Corollary 3. *If D is a nonsingular derivation of A over a field of characteristic* 0, *then A is nilpotent.*

Corollary 4. *If B and C are nilpotent ideals of A, then $B + C$ is a nilpotent ideal of A.*

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The rank gradient and the lamplighter group

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We introduce the notion of the rank gradient function of a descending chain of subgroups of finite index and show that the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ has uncountably many 2-chains (that is, chains in which each subsequent group has index 2 in the previous group) with pairwise different rank gradient functions. In doing so, we obtain some information on subgroups of finite index in the lamplighter group.

1. Introduction

The lamplighter group, by which we mean the wreath product of the group of order 2 with the infinite cyclic group, denoted $\mathscr{L} = \mathbb{Z}_2 \wr \mathbb{Z}$, is a popular object in group theory and its applications. Just two illustrations of this are Chapter 6 in [Meier 2008] and some select sections in [Lubotzky and Segal 2003]. It is a 2-step solvable group (i.e., metabelian) of exponential growth, infinitely presented and scale invariant [Grigorchuk and Żuk 2001; Nekrashevych and Pete 2011], which is the cornerstone in all known results about the range of L^2-Betti numbers of groups on compact manifolds. In particular, Atiyah's problem about the existence of closed manifolds with noninteger and even irrational L^2-Betti numbers was completely solved on a base of considerations related to $\mathscr{L}$ [Grigorchuk and Żuk 2001; Grabowski 2010; Grigorchuk et al. 2000].

Lackenby [2005] introduced an interesting group-theoretical notion, the rank gradient, which happens to be useful in topology, the theory of countable equivalence relations, the study of amenable groups and other areas. Given a group G and a descending sequence $\{H_n\}_{n=1}^{\infty}$ of subgroups of finite index one can define

$$RG(G, \{H_n\}) = \lim_{n\to\infty} \frac{d(H_n) - 1}{[G : H_n]}$$

to be the rank gradient of the sequence $\{H_n\}$ with respect to G where $d(H)$ denotes the minimal number of generators of a group H.

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Amenable groups were introduced by J. von Neumann in 1929 and play an important role in many areas of mathematics [Nekrashevych and Pete 2011]. There are a number of results due to Lackenby, M. Abért, A. Jaikin-Zapirain and N. Nikolov showing that amenability of G or of certain normal subgroups of G usually implies vanishing of the rank gradient. For instance, finitely generated infinite amenable groups have $RG = 0$ with respect to any normal chain with trivial intersection; see [Abért et al. 2011, Theorem 5].

It is reasonable to study the rank gradient for sequences $\{H_n\}$ with trivial core (i.e., no nontrivial normal subgroups in the intersection $\bigcap_n H_n$). Indeed,

$$RG(G, \{H_n\}) = RG(G/N, \{H_n/N\})$$

if $N \triangleleft G$, $N < \bigcap_n H_n$. The most attention is given to the case when $\bigcap_n H_n = \{1\}$. One of the remaining open questions is this:

Question 1.1 [Abért et al. 2011]. *Let G be a finitely generated infinite amenable group. Is it true that $RG(G, \{H_n\}) = 0$ for any chain with trivial intersection*?

If $\bigcap_{n=1}^{\infty} H_n = H$ then H is a closed subgroup with respect to the profinite topology and $RG(G, \{H_n\})$ is a characteristic of the pair (G, H) which in some situations may characterize the pair (G, H) up to isomorphism. We say two pairs (G, H), (P, Q) are isomorphic if there is an isomorphism $\phi : G \to P$ such that $\phi(H) = Q$.

If $RG(G, \{H_n\}) = 0$ then one may be interested in the decay of the function of the natural argument $n \in \mathbb{N}$ given by

$$\mathrm{rg}(n) = \mathrm{rg}_{(G,\{H_n\})}(n) = \frac{d(H_n) - 1}{[G : H_n]}$$

which we call the *rank gradient function*. We may omit $(G, \{H_n\})$ if the group and chain in consideration are understood. Again, the rate of decay of $\mathrm{rg}(n)$ may be an invariant of the pair (G, H) and may characterize the way H lies in G as a subgroup. Note that the same subgroup can be obtained as the intersection of distinct chains: one can delete certain elements in H_n thereby allowing $\mathrm{rg}(n)$ to decay as fast as one would like and indeed this is not the only way to get different chains with the same intersection. Thus, we restrict our definition to the case when for some prime p, we have $[H_{n+1} : H_n] = p$ and in this case we say the chain is a *p-chain*. Our main result shows that $\mathrm{rg}(n)$ may be used to show that the lamplighter group contains 2-chains with distinct rates of decay of the rank gradient function.

Theorem 1.2. *The group $\mathcal{L}$ has uncountably many 2-chains with pairwise distinct rank gradient functions.*

This result is obtained by explicitly describing subgroups of index 2 in the "higher rank" lamplighter groups $\mathcal{L}_n = \mathbb{Z}_2^n \wr \mathbb{Z}$.

Theorem 1.3. *For any 2-chain $\{H_n\}$ in $\mathscr{L}$ each member H_n is isomorphic to $\mathscr{L}_i = \mathbb{Z}_2^i \wr \mathbb{Z}$ for some $i \le n$.*

This is a corollary of Theorem 2.1 below.

2. Subgroups of index 2 in $\mathscr{L}_n$

Let $\mathscr{L}_n = \mathbb{Z}_2^n \wr \mathbb{Z} = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2^n \rtimes \mathbb{Z}$ (by $\mathbb{Z}_2$ we mean the group of order 2 and the generator of $\mathbb{Z}$ acts by shifting in the direct sum) and let $\mathscr{A}_n = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2^n$ be the base group of $\mathscr{L}_n$. Observe that $\mathscr{L}_n$ is generated by the elements a_i, $i = 1, 2, \dots, n$ and t where t is a generator of the infinite multiplicative cyclic group which we nevertheless denote in the additive way $\mathbb{Z}$, and $a_i \in \mathscr{A}_n$, $i = 1, 2, \dots, n$ are elements given by an $n \times \infty$ matrix with all entries zero except one located in the i-th row and column at position 0 (we assume that the columns are enumerated by the elements of $\mathbb{Z}$). So $\mathscr{L}_n = \langle a_1, \dots, a_n, t\rangle$. We will use similar notation for generation in the remainder of the paper. Observe that if we identify elements of the base group $\mathscr{A}_n$ with two sided infinite (bi-infinite) sequences of columns of dimension n over $\mathbb{Z}_2$ then conjugation by t acts on them as a shift τ in the set of sequences. We will use this fact later.

Theorem 2.1. *Let $H < \mathscr{L}_n$ be a subgroup of index 2. Then either $H \simeq \mathscr{L}_n$ or $H \simeq \mathscr{L}_{2n}$. There are $2^{n+1} - 2$ subgroups of the first type and 1 subgroup of the second type.*

In the proof, we use the following well known result.

Lemma 2.2. *Let $M = \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p \oplus \cdots$ be a finite or infinite direct sum of cyclic groups $\mathbb{Z}_p$ with p a prime. Then every subgroup $P < M$ is a direct summand: $M = P \oplus Q$ for some Q.* (See [Kargapolov and Merzljakov 1977, Chapter 10].)

We will often interpret $\mathbb{Z}_p^n$ as a vector space of dimension n over the prime field $\mathbb{F}_p \simeq \mathbb{Z}_p$. Before we present a proof of Theorem 2.1, we will need the following lemma.

Lemma 2.3. *Let $M = \mathbb{Z}_p^n$. Every subgroup $P < M$ of index p has a unique "orthogonal" complement $Q < M$ such that $M = P \oplus Q$. The group Q is generated by the element $\bar{a} = (a_1, \dots, a_n)$ which is determined by P. Then P consists of elements $\bar{x} = (x_1, \dots, x_n)$ whose coordinates satisfy the "orthogonality" condition*

$$a_1 x_1 + \cdots + a_n x_n \equiv 0 \pmod{p}.$$

Proof. Let $[M : P] = p$. Consider the subgroup P as a subspace of the vector space $M = \mathbb{Z}_p^n$. Choose a basis of P consisting of elements $\bar{b}_1, \dots, \bar{b}_{n-1}$

$$\bar{b}_1 = (b_{1,1}, \dots, b_{1,n}), \quad \dots, \quad \bar{b}_{n-1} = (b_{n-1,1}, \dots, b_{n-1,n}),$$

with $b_{i,j} \in \mathbb{Z}_p$. Now define the $(n-1) \times n$ matrix $B = (b_{ij})$, which has rank $n-1$, and consider the system of equations

$$\begin{aligned} b_{1,1}x_1 + \cdots + b_{1,n}x_n &= 0 \\ &\vdots \\ b_{n-1,1}x_1 + \cdots + b_{n-1,n}x_n &= 0. \end{aligned}$$

This system has the nontrivial solution $\bar{a} = (a_1, \ldots, a_n)$ and every other solution is some constant multiple of $\bar{a}$. It is then easy to see that $M = P \oplus \langle \bar{a} \rangle$. It is also clear that given some $\bar{a} \in M$ with $\bar{a} \neq 0$, the set of solutions of $a_1x_1 + \cdots + a_nx_n \equiv 0 \pmod{p}$ yields a subgroup P of index p in M. □

Although we do not this, observe that by using tools from linear algebra, the notion of orthogonal complement can be defined in a similar way as we did for a subgroup of index p in an elementary p-group of finite rank. We will use the notation $H^\perp$ to denote the orthogonal complement of a subgroup $H < M$ in M.

Corollary 2.4. *There is a bijection between subgroups of index p in $M = \mathbb{Z}_p^n$ and subgroups of order p given by*

$$H \to H^\perp.$$

We now restrict our attention to the case when $p = 2$.

Proof of Theorem 2.1. Observe that the abelianization $A := (\mathscr{L}_n)_{ab}$ is isomorphic to $\mathbb{Z}_2^n \times \mathbb{Z}$. Define $A^2 < A$ to be the subgroup generated by the squares of elements in A. Then, $A/A^2 \simeq \mathbb{Z}_2^{n+1} = \langle \bar{a}_1, \ldots, \bar{a}_n, \bar{t} \rangle$ where as before $\mathbb{Z} = \langle t \rangle$ denotes the multiplicative infinite cyclic group generated by t, and a bar over some generator, $\bar{a}_i$ or $\bar{t}$ for example, denotes that we are considering the element corresponding to a_i or t of $\mathscr{L}_n$ as an element of the quotient group $\mathscr{L}_n/[\mathscr{L}_n, \mathscr{L}_n]\mathscr{L}_n^2 \simeq \mathbb{Z}_2^{n+1}$. If we consider a_i as an $n \times \infty$ matrix, then it is of the form

$$\begin{pmatrix} \cdots & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & 0 & 1 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & \cdots \end{pmatrix},$$

where the 1 is in the i-th row and the 0-th column. Recall that each a_i is the i-th generator of $\mathscr{A}_n^0$, where we define

$$\mathscr{A}_n = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2^n = \bigoplus_{j \in \mathbb{Z}} \mathscr{A}_n^j.$$

The number of subgroups of index 2 in $\mathscr{L}_n$ is equal to the number of epimorphisms $\mathscr{L}_n \to \mathbb{Z}_2$ which is equal to the number of subgroups of index 2 in $\mathbb{Z}_2^{n+1}$ which is equal to $2^{n+1}-1$ since the kernel of any such epimorphism is an orthogonal complement to a subgroup of order 2 generated by some nonidentity element. We have a short exact sequence

$$1 \to \mathscr{A}_n \to \mathscr{L}_n \xrightarrow{\phi} \langle t\rangle \to 1$$

where ϕ is the natural projection onto $\mathbb{Z}=\langle t\rangle$. Let $H<\mathscr{L}_n$ be of index 2. Then H is normal in $\mathscr{L}_n$ and therefore shift invariant.

There are two cases: either $\phi[H]=\langle t^2\rangle$ or $\phi[H]=\langle t\rangle$.

Case 1. Assume $\phi[H]=\langle t^2\rangle$. In this case $H\cap\mathscr{A}_n=\mathscr{A}_n$, since otherwise we would have $[\mathscr{L}_n : H]\geq 4$ and there is only one subgroup H of index 2 in $\mathscr{L}_n$ with this property. Furthermore, $t^2\in H$ and $H=\mathscr{A}_n\rtimes\langle t^2\rangle$.

Let $D_0<\mathscr{A}_n$, $D_0\simeq\mathbb{Z}_2^{2n}$ be a subgroup of $n\times\infty$ matrices where the only nonzero entries belong to columns with position 0 and 1. Define $D_j=t^{-2j}D_0t^{2j}$. Then notice $D_i\cap D_j=0$ for $i\neq j$ and $\mathscr{A}_n=\bigoplus_{j\in\mathbb{Z}}D_j$. The element t^2 acts by conjugation on $\bigoplus_{j\in\mathbb{Z}}D_j$ as a one-step shift. This implies $H\simeq\mathscr{L}_{2n}$.

Case 2. Now we assume $\phi[H]=\langle t\rangle$. We have $2^{n+1}-2$ such subgroups H. In this case, $H\cap\mathscr{A}_n=P$ is a shift invariant subgroup of index 2 in $\mathscr{A}_n$. Because P is shift invariant, there must be some $x\in\mathscr{A}_n$ whose matrix representation has only one nonzero column, namely the column with position 0, such that $x\notin P$. Let $q\in\mathbb{Z}_2^n$ be the vector with coordinates the same as x. That is, we consider x as an $n\times 1$ vector and relabel it q for clarity. Then let Q^0 be the orthogonal complement to $\langle q\rangle$:

$$\mathscr{A}_n^0=\langle q\rangle\oplus Q^0,$$

where as before we have $\mathscr{A}_n=\bigoplus_{i\in\mathbb{Z}}\mathscr{A}_n^i$. Note that we are considering Q^0 and $\langle q\rangle$ as subgroups of $\mathscr{A}_n^0$ and so Q^0 is a subgroup of H since otherwise we would have $[\mathscr{L}:H]\geq 4$. Define

$$Q=\bigoplus_{i\in\mathbb{Z}}Q^i,\quad \text{where } Q^i=t^{-i}Q^0t^i.$$

Let $\mathscr{R}=\mathbb{Z}_2[t,t^{-1}]$ be the ring of Laurent polynomials in $\mathbb{Z}_2$. It is isomorphic to the group ring $\mathbb{Z}_2[\mathbb{Z}]$ where as before $\mathbb{Z}$ is the additive notation for the multiplicative infinite cyclic group generated by t. The group $\mathscr{A}_n$ can be converted into an $\mathscr{R}$-module M_n by agreeing that the generator t acts on $\mathscr{A}_n$ as the previously defined right-shifting element τ (remember that elements of $\mathscr{A}_n$ can be considered as bi-infinite sequences of columns representing the elements of $\mathbb{Z}_2^n$). Moreover, $\mathscr{A}_n$ is the additive group of this module, M_n is a free $\mathscr{R}$-module of rank n and is isomorphic to $\mathscr{R}^n$.

Observe that Q is a shift invariant subgroup of H. Because of Lemma 2.2 there is a subgroup $S<P$ such that the decomposition $P=Q\oplus S$ holds. Note that

S is also a shift invariant subgroup of P and therefore can be interpreted as an $\mathcal{R}$-module. Therefore P, Q and S can be considered as submodules of M_n and the decomposition of modules $P = Q \oplus S$ holds (we will not change the notation for P, Q, S when considering them as modules or vise versa since it will be clear by the context if we are considering these objects as abelian groups or as $\mathcal{R}$-modules).

We will need the following lemma. Any graduate level textbook in Algebra will contain the fact that a ring of polynomials with coefficients in some field is a principal ideal domain. The ring $\mathcal{R}$ is the localization of the polynomial ring in the multiplicative set consisting of the nonnegative powers of t [Reid 1988]. Many properties of the Laurent polynomial ring follow from the general properties of localization as well as the next one which is a well known fact. However, we were unable to find a suitable reference for this so we add a proof of it below.

Lemma 2.5. *The ring $\mathcal{R}$ is a principal ideal domain.*

Proof. Let I be an ideal in $\mathcal{R}$. Then $I \cap \mathbb{Z}_2[t]$ is an ideal in $\mathbb{Z}_2[t]$ and since the ring of polynomials over a field is a principal ideal domain, $I \cap \mathbb{Z}_2[t] = (f)$ for some $f \in \mathbb{Z}_2[t]$. Then $\mathcal{R}f \subset I$. For $h \in I$, $h = t^{-k}g$ for some $k \in \mathbb{N}$ and $g \in \mathbb{Z}_2[t]$. Thus $g \in I \cap \mathbb{Z}_2[t] = (f)$, and so $h = t^{-k}fa \in \mathcal{R}f$ for some a in $\mathcal{R}$. Therefore $\mathcal{R}f = I$. □

Since they are submodules of a finitely generated free module $M_n \simeq \mathcal{R}^n$ over a principal ideal domain $\mathcal{R}$, the modules P, Q and S are also free. As P is a subgroup of index 2 in $\mathcal{A}_n$, the module P is free of rank n, Q is free of rank $n-1$ and S is free of rank 1. Thus the $\mathcal{R}^n$-module P, when considered as a group generated by the additive group P and the element t which acts by conjugation on P as the shift element τ, becomes isomorphic to $\mathcal{R}^n \rtimes \mathbb{Z} \simeq \mathcal{L}_n$.

We have $2^{n+1}-2$ subgroups H which can be obtained in the second case. Indeed, there are $2^n - 1$ choices for the vector q and therefore the subgroup Q. And to each choice of Q we have two choices to construct H: either to assume that $t \in H$ or that $t \notin H$. In this way, we get $2(2^n - 1) = 2^{n+1} - 2$ subgroups corresponding to *Case 2*. This finishes the proof of the first theorem. □

3. Construction of chains

Since $\mathbb{Z}_2[x, x^{-1}]$ is a principal ideal domain by Lemma 2.5, a shift invariant subgroup T of $\mathcal{A}_1 = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2$ corresponds to a principal ideal $\mathcal{J}$ such that

$$\mathbb{Z}_2[x, x^{-1}]/\mathcal{J} \simeq \mathbb{Z}_2$$

which is a field. This implies that $\mathcal{J}$ is a maximal ideal generated by some irreducible polynomial of degree 1. Thus, $\mathcal{J} = \langle f \rangle$ with $\deg(f) = 1$ so $f = x + 1$. The corresponding element of T is then $\xi = (\ldots, 0, 1, 1, 0, \ldots)$ where the 1's are in the 0 and 1 place respectively. Additionally, ξ is a generator of T as an $\mathcal{R}$-module. One then concludes that T consists of sequences

$$(\ldots, a_{-1}, a_0, a_1, \ldots),$$

where

$$\sum_n a_n \equiv 0 \pmod 2. \tag{1}$$

This observation gives an effective way to construct a subgroup H of index 2 in $\mathcal{L}_n$ with $H \simeq \mathcal{L}_n$. Choose a basis E of $\mathbb{Z}_2^n$ and write elements of $\mathcal{A}_n$ as $n \times \infty$ matrices (where the columns are indexed by $\mathbb{Z}$ as usual) with respect to this basis at position $i \in \mathbb{Z}$. Then take a subgroup of $\mathcal{A}_n$ consisting of elements which satisfy the relation (1) in the first row. After this, choose $t \in H$ or $t \notin H$.

We know that $\mathcal{L}$ contains 3 subgroups of index 2, where 2 are isomorphic to $\mathcal{L}$ and the other is isomorphic to $\mathcal{L}_2$. Furthermore, $\mathcal{L}_2$ has 7 subgroups of index 2, where 1 is isomorphic to $\mathcal{L}_4$ and 6 are isomorphic to $\mathcal{L}_2$, etc. If we take a subgroup $H < \mathcal{L}$ of index 2^k obtained from $\mathcal{L}$ by taking a descending chain of subgroups of index 2 in the previous member of the chain then we have $H \simeq \mathcal{L}_{2^i}$ for some $i \le k$. We can then take a subgroup of index 2 isomorphic to $\mathcal{L}_{2^i}$ (call this choice type 0) or to $\mathcal{L}_{2^{i+1}}$ (call this choice type 1). It is clear that $d(\mathcal{L}_n) = n + 1$ (obviously $\mathcal{L}_n$ is generated by $n + 1$ elements and the abelianization of $\mathcal{L}_n$ is $\mathbb{Z}_2^n \times \mathbb{Z}$ and is $(n + 1)$-generated). Now let $\omega \in \{0, 1\}^{\mathbb{N}}$ be a sequence. Then these two types of choices for subgroups of index 2 allow us to construct a chain $\{H_n^\omega\}$ such that the subgroup H_n^ω is obtained from the previous subgroup H_{n-1}^ω by looking at the n-th term in our sequence ω. That is, a 0 dictates we make a choice of type 0 and a 1 dictates we make a choice of type 1. It is clear that in such a way we obtain uncountably many different chains $\{H_n^\omega\}$ such that each of the functions $\mathrm{rg}^\omega(n)$ are distinct. This provides the proof of Theorem 1.2.

Remark. If $r^\omega = \lim_{n\to\infty} \mathrm{rg}^\omega(n) > 0$ then $r^\omega = 2^{-k}$ for some k and the rank gradient of the chain $\{H_n^\omega\}$ is positive where the number of 0's in the sequence ω is k. In this case, $H^\omega = \bigcap_n H_n^\omega$ contains a nontrivial normal subgroup. In all other cases the rank gradient of the 2-chain is 0.

4. Conclusion

It is clear that the same method used to construct uncountably many rank gradient functions of 2-chains in $\mathcal{L}$ allows one to construct uncountably many 2-chains with pairwise distinct types of decay of the rank gradient function. For instance, one can consider a family of functions $\delta_\alpha(n) = 2^{-n^\alpha}$ with $0 < \alpha < 1$ where to each such function we have a corresponding sequence ω with the property that the rank gradient function $\mathrm{rg}^\omega(n)$ is the best approximation of the function 2^{-n^α}. By “best approximation", we mean the following. Starting with any subgroup $H_1 \simeq \mathscr{L}_2$ of index 2 in $\mathscr{L}$ (which corresponds to the value $\omega_1 = 1$ of the sequence ω and the value $\mathrm{rg}^{\omega}(1) = 1 > \frac{1}{2} = \delta_\alpha(1)$), one can make a choice of type 0 until the rank gradient function becomes less than the value of the function $\delta_\alpha(n)$ for the corresponding value of the argument n. Then make the choice of type 1 until the rank gradient function becomes greater or equal to $\delta_\alpha(n)$ for the corresponding value of n. Then again make the choice of type 0, etc. By continuing this process, we construct a 2-chain that best approximates $\delta_\alpha(n)$. Since the rates of decay of the functions $\delta_\alpha(n)$ are clearly different for different values of α, the (rates of decay of the) corresponding rank gradient functions are also distinct.

Our study is the first step in understanding what types of decay of the rank gradient function may arise in the case of finitely generated residually finite amenable groups.

If $\{H_n\}_{n=1}^{\infty}$ is a descending chain of subgroups of finite index in a residually finite group G, then the intersection $H_* = \bigcap_{n=1}^{\infty} H_n$ is a subgroup of G closed with respect to the profinite topology and indeed any closed subgroup can be obtained in this way. The rank gradient function of the chain $\{H_n\}_{n=1}^{\infty}$ introduced by us may serve as a certain characteristic of the subgroup H_*. Right now it is unclear how $\mathrm{rg}(n)$ depends on the chain $\{H_n\}_{n=1}^{\infty}$ with fixed intersection H_*. Even in the case when $H_* = \{1\}$, it may be that different p-chains with trivial intersection have different rates of decay of $\mathrm{rg}(n)$ but we do not have any examples of this. Of course, it is reasonable to only consider chains with the property that H_{n+1} is a maximal subgroup in $\{H_n\}$. While we have considered the case of the lamplighter group, it will also be interesting to study the decay of the rank gradient function with respect to other amenable groups such as with respect to the 3-generated infinite torsion 2-group of intermediate growth constructed in [Grigorchuk 1980; 1984].

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