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On the associated primes of the third power of the cover ideal

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An algebraic approach to graph theory involves the study of the edge ideal and the cover ideal of a given graph. While a lot is known for the associated primes of powers of the edge ideal, much less is known for the associated primes of the powers of the cover ideal. The associated primes of the cover ideal and its second power are completely determined. A configuration called a *wheel* is shown to always appear among the associated primes of the third power of the cover ideal.

1. Introduction

We start with some definitions and notation, for which we follow [Harris et al. 2008; Villarreal 2001]. A (finite) *graph* G consists of two finite sets, the vertex set V_G and the edge set E_G, whose elements are unordered pairs of vertices. An edge $\{x_i, x_j\} \in E_G$ is written $x_i x_j$ (or $x_j x_i$). If $x_i x_j$ is an edge, we say that the vertices x_i and x_j are *adjacent* and that the edge is *incident* to x_i and x_j. All our graphs will be *simple*, meaning that the only possible edges are $x_i x_j$ for $i \neq j$.

A subset $C \subseteq V_G$ is a (*vertex*) *cover* of G if each edge in E_G is incident to a vertex in C. A cover C is *minimal* if no proper subset of C is a cover of G.

The results of this paper are in the area of algebraic graph theory, where algebraic methods are used to investigate properties of graphs. Indeed, a graph G with vertex set $V_G = \{x_1, \dots, x_n\}$ can be related to the polynomial ring $R = \mathsf{k}[x_1, \dots, x_n]$, where k is a field. In the following we take the liberty of referring to x_i as a variable in the polynomial ring and as a vertex in the graph G, without any further specification. Given a ring R, we denote by $(f_1, \dots, f_l)$ the ideal of R generated by the elements $f_1, \dots, f_l \in R$.

Two ideals of the polynomial ring $R = \mathsf{k}[x_1, \dots, x_n]$ that have proven most useful in studying the properties of a graph G with vertex set $V_G = \{x_1, \dots, x_n\}$

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and edge set E_G are the *edge ideal*

$$I_G = (x_i x_j \mid x_i x_j \in E_G)$$

and the *cover ideal*

$$J_G = (x_{i_1} \cdots x_{i_k} \mid x_{i_1}, \ldots, x_{i_k} \text{ is a minimal cover of } G).$$

Both are square-free monomial ideals, that is, they are generated by monomials in which each variable appears at most one time.

One of the most basic tools in commutative algebra to study an ideal I of a noetherian ring R is to compute the finite set of *associated prime ideals* of I, which is denoted by $\operatorname{Ass}(R/I)$ (for details, see [Eisenbud 1995]). In the case of a monomial ideal L in a polynomial ring $S = \mathsf{k}[x_1, \ldots, x_n]$, an element in $\operatorname{Ass}(S/L)$ is a *monomial prime ideal*, which is an ideal generated by a subset of the variables. Because of this fact we can record the following definition.

Definition. Let L be a monomial ideal in the polynomial ring $S = \mathsf{k}[x_1, \ldots, x_n]$ and let $P = (x_{i_1}, \ldots, x_{i_s})$ be a monomial prime ideal. If there exists a monomial m such that $x_{i_j} m \in L$ for each $j = 1, \ldots, s$ and $x_i m \notin L$ for every $i \neq i_1, \ldots, i_s$ then P is an *associated* prime to L. We denote by $\operatorname{Ass}(S/L)$ the set of all associated (monomial) primes of L.

Chen et al. [2002] gave a constructive method for determining primes associated to the powers of the edge ideal, but much less is known about cover ideals. It is known that, given a graph G and its cover ideal J_G, a monomial prime ideal P is in $\operatorname{Ass}(S/J_G)$ if and only if $P = (x_i, x_j)$ and $x_i x_j$ is an edge of G (see [Villarreal 2001], for example).

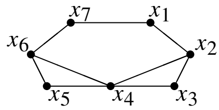
The initial point of our investigation is a result of Francisco, Ha and Van Tuyl (Theorem 1.1 below) describing the associated primes of the ideal $(J_G)^2$.

Let G be a graph. A *path* in G is a sequence of distinct vertices $x_1, x_2, \ldots, x_k$ such that $x_j x_{j+1} \in E_G$ for $j = 1, 2, \ldots, k-1$. The *length* of such a path is $k-1$, one less than the number of vertices. If $x_k x_1$ is also an edge of G, we say that the graph C with vertex set $\{x_1, x_2, \ldots, x_k\}$ and edge set $\{x_1x_2, \ldots, x_{k-1}x_k, x_kx_1\}$ is a *cycle* (in G). A cycle with an odd number of vertices is also called an *odd hole*.

Given a graph G and a set of vertices $W \subseteq V_G$, the *graph generated by* W has vertex set W and edge set $\{xy \mid xy \in E_G,\ x \in W,\ y \in W\}$.

Theorem 1.1 [Francisco et al. 2010]. *Let G be a graph with vertex set $\{x_1, \ldots, x_n\}$, edge set E_G and cover ideal J_G. A monomial prime ideal $P = (x_{i_1}, \ldots, x_{i_k})$ of the polynomial ring $S = \mathsf{k}[x_1, \ldots, x_n]$ is in the set $\operatorname{Ass}(S/J_G^2)$ if and only if either*

- *$k = 2$ and $x_{i_1} x_{i_2} \in E_G$, or*
- *k is odd and the graph generated by $x_{i_1}, \ldots, x_{i_k}$ is an odd hole.*

As an example, if G is the graph  we will have

$\mathrm{Ass}(J) =$

$$\{(x_1, x_2), (x_1, x_7), (x_2, x_3), (x_2, x_4), (x_3, x_4), (x_4, x_5), (x_4, x_6), (x_5, x_6), (x_6, x_7)\}$$

(the associated prime of J consists of the primes generated by two variables that correspond to the edges of the graph), and

$$\mathrm{Ass}(J^2) = \mathrm{Ass}(J) \cup \{(x_2, x_3, x_4), (x_4, x_5, x_6), (x_1, x_2, x_4, x_6, x_7)\}$$

(the associated prime of J_G^2 contains all the primes that are either generated by two variables corresponding to edges or generated by three variables corresponding to odd cycles of G).

In this paper we study the associated primes of the third power of the cover ideal, the ideal J_G^3. We prove that the primes generated by the variables corresponding to the vertices of a *wheel* (see next definition) always appear among the associated primes of J_G^3. This result is connected with the coloring number of a graph, as discussed at the end of Section 2.

The algebra system Macaulay2 was used for all the computations in this paper, and in particular in finding the pattern that led to the main theorem.

2. Centered odd holes and the main theorem

Definition. A graph C is said to be a *wheel* if $V_C = V_H \cup \{y\}$, where H, called the *rim* of C, is an odd hole such that the graph generated by H in C is H itself, and y, called the *center* of C, is a vertex adjacent in C to at least three vertices of H and belonging to at least two odd cycles in C. (It follows that y belongs to at least three odd cycles in C.) The rim H and center y are part of the data needed to specify a wheel, as they may not be uniquely determined by C.

Let C be a wheel with rim H and center y. A vertex $x \in V_H$ is *radial* if xy is an edge of C. Let there be k radial vertices, labeled consequently $x_1, \dots, x_k$ *in order around the wheel*. We leave it to the reader to specify precisely what this means. For $i = 1, \dots, k-1$, we denote by l_i the length of the path in H joining x_i to x_{i+1} (and not going through any other radial vertex). Similarly l_k denotes the length of the path in H from x_k to x_1.

For the main theorem we will need the following lemma, where we use the notation $|\ \ |$ for the *size* (that is, the number of vertices) of a graph.

Lemma 2.1. *Let C be a wheel with rim H and center y, and let k be its radial number. If W is a vertex cover for C that contains y, then $|W| \geq |C|/2 + 1$. If W is a vertex cover for C that does not contain y, then*

$$|W| \geq k + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor.$$

Moreover,

$$k+\left\lfloor\frac{l_1-1}{2}\right\rfloor+\cdots+\left\lfloor\frac{l_k-1}{2}\right\rfloor\geq\frac{|C|}{2}+1. \tag{2-1}$$

Proof. Let V_H be the vertex set of H. Assume that W contains the vertex y. The vertex set $W\cap V_H$ has to be a vertex cover for H. Moreover, since H is an odd hole, the cardinality of $W\cap V_H$ has to be at least $(|H|+1)/2$, which is equal to $|C|/2$. Therefore the cardinality of W is $|C|/2+1$.

Assume now that W does not contain the vertex y. Let $x_1,\ldots,x_k$ be the radial vertices. Since $y\notin W$, all the radial vertices are in W. As $W\cap V_H$ is a cover of H, in the path from x_i to x_{i+1} we need at least $\lfloor(l_i-1)/2\rfloor$ vertices, for $i=1,\ldots,k-1$, and we need $\lfloor(l_k-1)/2\rfloor$ vertices for the path from x_k to x_1.

To prove (2-1) we write

$$\begin{aligned}k+\left\lfloor\frac{l_1-1}{2}\right\rfloor+\cdots+\left\lfloor\frac{l_k-1}{2}\right\rfloor&\geq k+\frac{l_1-1}{2}+\cdots+\frac{l_k-1}{2}\geq\frac{l_1}{2}+\cdots+\frac{l_k}{2}+\frac{k}{2}\\&\geq\frac{l_1+\cdots+l_k+1}{2}+\frac{k-1}{2}\geq\frac{|C|}{2}+1,\end{aligned}$$

where in the last inequality we used the fact that $k\geq 3$. □

In the following we will make an abuse of notation: if G is a graph with vertices $x_1,\ldots,x_n$ and H is a subgraph generated by the vertices $x_{i_1},\ldots,x_{i_k}$, by H we also denote the prime monomial ideal $(x_{i_1}\ldots,x_{i_k})$ in the polynomial ring $\mathsf{k}[x_1,\ldots,x_n]$. Here is our main theorem.

Theorem 2.2. *Let G be a graph with vertex set $V_G=\{x_1,\ldots,x_n\}$ and assume that G has a subgraph C which is a wheel. Let $S=\mathsf{k}[x_1,\ldots,x_n]$ and let J be the cover ideal of G. Then the set* $\operatorname{Ass}(S/J^3)$ *is not contained in the set* $\operatorname{Ass}(S/J^2)$, *and in fact $C\in\operatorname{Ass}(S/J^3)\setminus\operatorname{Ass}(S/J^2)$.*

Proof. By Lemma 2.11 in [Francisco et al. 2011], we may assume that $G=C$. Let y be the center of the wheel C, and let $x_1,x_2,\ldots,x_k$ be the radial vertices. Denote by x_{i_j}, for $j=1,\ldots,l_i-1$, the vertices between x_i and x_{i+1} if $i<k$ and the vertices between x_k and x_1 if $i=k$.

That C is not in $\operatorname{Ass}(S/J^2)$ follows from Theorem 1.1, since C is neither an odd hole nor an edge.

To show that C is in $\operatorname{Ass}(S/J^3)$ we need to find a monomial c such that $c\notin J^3$ and $xc\in J^3$ for each vertex x of C. Let c be the monomial

$$c=y^2\prod_{i=1,\ldots,k}x_i^2\prod_{\substack{i=1,\ldots,k\\ j=1,\ldots,l_i-1}}x_{ij}^a,\quad\text{where } a=\begin{cases}1&\text{if } j \text{ is odd},\\2&\text{if } j \text{ is even}.\end{cases}$$

To show that c is the desired monomial, we first prove that

$$\deg c=k+2+n+\left\lfloor\frac{l_1-1}{2}\right\rfloor+\cdots+\left\lfloor\frac{l_k-1}{2}\right\rfloor. \tag{2-2}$$

Let n be the size of H. For a monomial m we denote by $\deg m$ the degree of m. In computing $\deg c$, the contribution from the variables y and x_i, for $i = 1, \dots, k$, is given by $2k+2$. For $i = 1, \dots, k-1$, between x_i and x_{i+1}, there are $l_i - 1$ vertices, and there are $l_k - 1$ vertices between x_k and x_1. Given an integer s, there are $\lfloor s/2 \rfloor$ even integers and $\lceil s/2 \rceil$ odd integers between 1 and s. Therefore, in computing $\deg c$, the contribution from the variables x_{ij} is given by

$$2\left\lfloor \frac{l_1-1}{2} \right\rfloor + \cdots + 2\left\lfloor \frac{l_k-1}{2} \right\rfloor + \left\lceil \frac{l_1-1}{2} \right\rceil + \cdots + \left\lceil \frac{l_k-1}{2} \right\rceil.$$

The degree of the monomial c is therefore equal to

$$\begin{aligned}
&2k+2+2\left\lfloor \frac{l_1-1}{2} \right\rfloor + \cdots + 2\left\lfloor \frac{l_k-1}{2} \right\rfloor + \left\lceil \frac{l_1-1}{2} \right\rceil + \cdots + \left\lceil \frac{l_k-1}{2} \right\rceil \\
&\quad = 2k+2+\left(\left\lfloor \frac{l_1-1}{2} \right\rfloor + \left\lceil \frac{l_1-1}{2} \right\rceil\right) + \cdots + \left(\left\lfloor \frac{l_k-1}{2} \right\rfloor + \left\lceil \frac{l_k-1}{2} \right\rceil\right) \\
&\qquad\qquad + \left\lfloor \frac{l_1-1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_k-1}{2} \right\rfloor \\
&\quad = k+2+k+(l_1-1)+\cdots+(l_k-1)+\left\lfloor \frac{l_1-1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_k-1}{2} \right\rfloor \\
&\quad = k+2+l_1+\cdots+l_k+\left\lfloor \frac{l_1-1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_k-1}{2} \right\rfloor \\
&\quad = k+2+n+\left\lfloor \frac{l_1-1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_k-1}{2} \right\rfloor.
\end{aligned}$$

The last line establishes (2-2).

To prove that c does not belong to J^3, we first show the strict inequality

$$\deg c < 2\left(\frac{|C|}{2}+1\right) + k + \left\lfloor \frac{l_1-1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_k-1}{2} \right\rfloor. \tag{2-3}$$

For suppose this inequality is not satisfied. Then (2-2) gives

$$k+2+n+\left\lfloor \frac{l_1-1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_k-1}{2} \right\rfloor \geq 2\left(\frac{|C|}{2}+1\right)+k+\left\lfloor \frac{l_1-1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_k-1}{2} \right\rfloor,$$

which means that

$$2+n \geq 2\left(\frac{|C|}{2}+1\right).$$

But $|C| = |H|+1 = n+1$. Thus

$$2+n \geq 2\left(\frac{n+1}{2}+1\right) = n+2+1,$$

which is impossible. Therefore (2-3) holds.

Let us show that (2-3) implies that $c \notin J^3$. Assume otherwise; then $c = hm_1m_2m_3$ with $m_i \in J$ for $i = 1, 2, 3$. Since $m_i \in J$, the variables that appear in each m_i correspond to a minimal cover of C. Lemma 2.1 says that such a cover has at least $|C|/2+1$ vertices if it contains y and at least $k + \lfloor (l_1-1)/2 \rfloor + \cdots + \lfloor (l_k-1)/2 \rfloor$

— a number at least as large as $|C|/2+1$ — if not. Using the fact that at least one of the three covers must *not* contain y, we thus obtain

$$\begin{aligned}\deg c &= \deg h + \deg m_1 + \deg m_2 + \deg m_3 \\ &\geq \deg h + 2\Big(\frac{|C|}{2}+1\Big) + k + \Big\lfloor \frac{l_1-1}{2} \Big\rfloor + \cdots + \Big\lfloor \frac{l_k-1}{2} \Big\rfloor.\end{aligned}$$

This contradicts (2-3) (since $\deg h \geq 0$) and so shows that $c \notin J^3$.

To finish the proof of Theorem 2.2 we need to show that for every vertex $x \in V_C$ we have $xc \in J^3$.

Let x be any vertex of H and relabel the vertices of H starting from $x = t_1$ clockwise $t_2, \ldots, t_n$, where n is the size of H. We can write $xc = m_1 m_2 m_3$, where

$$m_1 = y \prod_{i \text{ odd}} t_i, \quad m_2 = y t_1 \prod_{i \text{ even}} t_i, \quad m_3 = \prod_{i=1,\ldots,k} x_i \prod_{\substack{i=1\ldots,k \\ j \text{ even}}} x_{ij}.$$

Note that m_1 and m_2 correspond to covers, as they contain y and every other vertex of H. Also m_3 corresponds to a cover as all the x_i are included, and therefore all the edges connecting y to H are covered, and every other vertex in the path from x_i to x_{i+1} is included.

Finally we need to write $yc = m_1 m_2 m_3$ with $m_i \in J$ for $i = 1, 2, 3$. For this assume that x_1 is such that the path from x_k to x_1 is odd. Relabel the vertices $x_1 = t_1$ and then clockwise to t_n. Let

$$m_1 = y \prod_{i \text{ odd}} t_i.$$

Note that m_1 will give a cover as we are considering every other vertex in the odd cycle and the vertex y. Now let l the least even number so that t_l corresponds to a radial vertex x_g, for some g. Set

$$m_2 = y \prod_{\substack{l \leq i \leq n \\ i \text{ even}}} t_i \prod_{\substack{1 \leq i \leq l \\ i \text{ odd}}} t_i.$$

Because we are considering every other vertex from t_1 to t_{l-1}, every other vertex from t_l, and the center y, the monomial m_2 corresponds to a cover of the wheel.

Finally

$$m_3 = y x_g x_{g+1} \ldots x_k \prod_{\substack{i=g,\ldots,k \\ j \text{ even}}} x_{ij} \prod_{\substack{i=1,\ldots,l-1 \\ i \text{ even}}} t_i.$$

Also m_3 gives a cover as it contains every other vertex from t_2 to $t_l = x_g$, every other vertex from x_i to x_{i+1}, for $i = g, \ldots, k-1$, every other vertex from x_k to x_1, and the center y. Notice that x_1 is missing from the monomial m_3 but the vertex y is listed in the monomial as for the vertex preceding x_1, because of the assumption that the path $x_k, \dots, x_1$ in H is odd. □

For every ideal I in a polynomial ring S (or a more general ring), one can compute the sequence of sets $\mathrm{Ass}(S/I^n)$ for $n \in \mathbb{N}$. Brodmann [1979] proved, in much greater generality, that there exists a positive integer a such that

$$\bigcup_{i=1}^{a_I} \mathrm{Ass}(S/I^i) = \bigcup_{i=1}^{\infty} \mathrm{Ass}(S/I^i). \tag{2-4}$$

Very little is known about the value of a_I. In [Francisco et al. 2011], the authors give an upper bound for a_I in the case that I is an edge ideal for a graph.

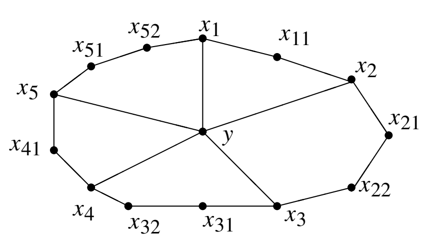
The value of a_J, where J is the cover ideal of a graph G, is related to the *coloring number* of G, that is, the least number of colors that one needs to color the vertices of G so that two adjacent vertices always have different colors. We denote the coloring number of G by $\chi(G)$. It is shown in [Francisco et al. 2011] that, in (2-4), $a_J \geq \chi(G) - 1$ when J is the cover ideal of G. The same paper gives examples for which $a_J > \chi(G) - 1$. Centered odd holes are an infinite family of such examples.

Corollary 2.3. *Let C be a wheel with cover ideal J. If C has a vertex that is not radial, then $a_J \geq \chi(C)$.*

Proof. Because C contains an odd hole, one needs at least three colors for the vertexes of C. We first show that $\chi(C) = 3$. Let $\{a, b, c\}$ be a list of three colors. Assume that x is a vertex of C which is not radial. Color the vertex x and the center y with c, and finally color the remaining vertices alternating a and b.

The main theorem implies that $a_J \geq 3$. □

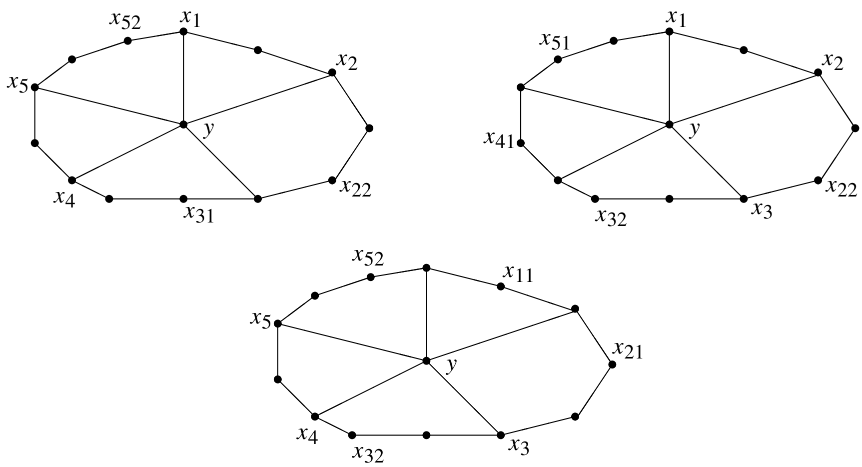
We finish the paper with an example that illustrates the idea behind the proof of the main theorem. Consider this wheel:



The monomial c used in the proof of the main theorem is given by

$$c = x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 x_{22}^2 x_{32}^2 x_{52}^2 x_{11} x_{21} x_{31} x_{41} x_{51}.$$

We can write $yc = m_1 m_2 m_3$, where the monomials m_1, m_2, and m_3 correspond to the following covers:



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