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Preimages of quadratic dynamical systems

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For a quadratic polynomial with rational coefficients, we consider the problem of bounding the number of rational points that eventually land at a given constant after iteration, called preimages of the constant. It was shown by Faber, Hutz, Ingram, Jones, Manes, Tucker, and Zieve (2009) that the number of rational preimages is bounded as one varies the polynomial. Explicit bounds on the number of preimages of zero and -1 were addressed in subsequent articles. This article addresses explicit bounds on the number of preimages of any algebraic number for quadratic dynamical systems and provides insight into the geometric surfaces parameterizing such preimages.

1. Introduction

Fix an algebraic number field K and a number $c \in K$ and define an endomorphism of the affine line by

$$f_c : \mathbb{A}^1_K \to \mathbb{A}^1_K, \qquad f_c(x) = x^2 + c.$$

If we define f_c^N to be the N-fold composition of the morphism f_c, and f_c^{-N} to be the inverse image of a in $\mathbb{A}^1_K$ under f_c^N, then for $a \in \mathbb{A}^1(K)$, the set of *rational iterated preimages of a* is given by

$$\bigcup_{N \geq 1} f_c^{-N}(a)(K) = \{x_0 \in \mathbb{A}^1(K) : f_c^N(x_0) = a \text{ for some } N \geq 1\}.$$

Heuristically, finding iterated preimages amounts to solving progressively more complicated polynomial equations, so K-rational solutions should be a rarity. The situation becomes more interesting as we vary c, which has the effect of varying the morphism f_c.

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Definition 1.1. Define

$$\kappa(a) = \sup_{c \in K} \#\left\{ \bigcup_{N \geq 1} f_c^{-N}(a)(K) \right\}.$$

A special case of the main theorem in [Faber et al. 2009] shows that $\kappa(a)$ is finite, but does not give an explicit bound. Note that it is easy to construct a pair (a, c) with arbitrarily many rational preimages simply by fixing c and taking $a = f_c(N)(0)$. The fact that $\kappa(a)$ is finite shows that, for a given a, such c values are rarely defined over the same field.

When needed for clarity, we include the field K in the notation as $\kappa(a, K)$. In this article, we focus on a weaker notion $\bar{\kappa}(a)$ that bounds the "typical" number of rational preimages.

Definition 1.2. Define

$$\bar{\kappa}(a, K) = \limsup_{c \in K} \#\left\{ \bigcup_{N \geq 1} f_c^{-N}(a)(K) \right\}.$$

In essence $\bar{\kappa}(a)$ differs from $\kappa(a)$ by excluding at most finitely many c values from consideration, thus, $\bar{\kappa}(a) \leq \kappa(a)$.

The cases of $a = 0$ and $a = -1$ were studied in [Faber et al. 2011; Hyde 2010], respectively, and it was shown that

$$\bar{\kappa}(0, \mathbb{Q}) = \bar{\kappa}(-1, \mathbb{Q}) = 6.$$

In the first of these papers, a significant amount of effort went into the more difficult task of showing that $\kappa(0, \mathbb{Q}) = 6$, assuming some standard conjectures. This article addresses the situation from the more general setting of allowing a to vary and examining the "preimage surfaces" instead of "preimage curves." We also allow arbitrary number fields K. Our main result is the following theorem.

Theorem 1.3. *For $a \in \overline{\mathbb{Q}}$ and for any fixed algebraic number field K we have*

$$\bar{\kappa}(a, K) = \begin{cases} 10 & \text{if } a = -\frac{1}{4}, \\ 6 \text{ or } 8 & \text{if } a \text{ is one of the three third critical values}, \\ 4 & \text{if } a \in S \cap K, \\ 6 & \text{otherwise}. \end{cases}$$

The set S is the finite set of a values (in $\overline{\mathbb{Q}}$) where the elliptic surface with two rational first preimages and four rational second preimages and the elliptic surface with two rational first preimages, (at least) two rational second preimages, and (at least) two rational third preimages both have specialization with rank zero at a.

The elliptic surface parameterizing values of a and c with two rational first preimages, (at least) two rational second preimages, and (at least) two rational third preimages has generic rank two (Theorem 3.3). Thus, finding the set of a values where the corresponding specialization is an elliptic curve of rank zero is a generalization of the problem studied by Masser and Zannier [2008]. The same authors have shown that such sets are finite [Masser and Zannier 2012], implying the set S is finite. The critical values are defined in Definition 2.1.

The organization of the article is as follows. In Section 3 we examine the lower bound for $\bar{\kappa}(a)$ by finding the generic rank over $\mathbb{Q}$ of the elliptic surfaces corresponding to arrangements of 6 preimages. In Section 4 we examine the upper bound on $\bar{\kappa}(a)$ by showing that all arrangements of $2N$ preimages for some N correspond to curves of genus greater than 1. In Section 5 we prove Theorem 1.3. In Section 6 we prove some additional properties of the preimage surfaces that are tangential to the proof of Theorem 1.3, yet still of interest. First we parameterize the possible torsion subgroups of the elliptic surface corresponding to two rational first preimages and four rational second preimages. Then, starting on page 362, we examine exceptional pairs (a, c) that are excluded by considering $\bar{\kappa}(a)$ instead of $\kappa(a)$.

We present these results for two reasons. First, by working with the "moduli surfaces" parameterizing arrangements of preimages, our problem can be reduced to the classical Diophantine problem of finding rational points on curves and surfaces. Second, our setting provides a nice example in which elliptic surfaces naturally arise and we apply specialization theorems, rank arguments, height functions, and use explicitly that the geometry of a curve has implications for its arithmetic through the use of Falting's theorem.

We make heavy use of the algebra and number theory systems Magma and PARI/gp version 2.3.2.

A similar analysis would almost certainly be possible for the families of maps of the form $x^d + c$, where $d \geq 2$ is a positive integer. In fact, for any family of polynomial maps of fixed degree it seems likely that the same methods would apply. For more general rational maps, at the very least, there would be additional complications for the genus calculations. This problem poses an interesting direction for further study.

2. Preimage curves and surfaces

In this section we summarize the necessary geometric theory of preimage curves developed in [Faber et al. 2011; 2009], and then introduce the preimage surfaces we consider in this article. Let K be a number field. As in the introduction, we define a morphism $f_c : \mathbb{A}^1_K \to \mathbb{A}^1_K$ for any $c \in K$ by the formula $f_c(x) = x^2 + c$. We could view f_c as an endomorphism of $\mathbb{P}^1_K$, but the point at infinity is totally invariant for this type of morphism and, thus, dynamically uninteresting. Fix a point $a \in K$ and a positive integer N. Define an algebraic set

$$Y^{\mathrm{pre}}(N,a) = V(f_c^N(x) - a) \subset \mathbb{A}_K^2 = \operatorname{Spec} K[x,c].$$

If $Y^{\mathrm{pre}}(N,a)$ is geometrically irreducible, we define the *N-th preimage curve*, denoted $X^{\mathrm{pre}}(N,a)$, to be the unique complete curve birational to $Y^{\mathrm{pre}}(N,a)$.

Definition 2.1. We say a is an *N-th critical value* of f_c if

$$f_{c_0}^N(0) = a \quad \text{and} \quad \frac{df_c^N(0)}{dc}\bigg|_{c=c_0} = 0.$$

Theorem 2.2 [Faber et al. 2009, Corollary 2.4 and Theorem 3.2]. *Suppose N is a positive integer and $a \in K$ is not a critical value of f_c^j for any $2 \le j \le N$. Then $Y^{\mathrm{pre}}(N,a)$ is nonsingular, geometrically irreducible, and the genus of $X^{\mathrm{pre}}(N,a)$ is $(N-3)2^{N-2}+1$.*

For $a \in K$, define a morphism $\psi : Y^{\mathrm{pre}}(N,a) \to \mathbb{A}^N$ by

$$\psi(x,c) = \left(x,\, f_c(x),\, f_c^2(x),\, f_c^3(x), \ldots,\, f_c^{N-1}(x)\right).$$

We recall the following theorem.

Theorem 2.3 [Faber et al. 2011, Proposition 4.2].

(a) *The projective closure of the image of ψ is a complete intersection of quadrics with homogenous ideal*

$$J = (Z_{N-1}^2 + Z_i Z_N - Z_{i-1}^2 - aZ_N^2 : i = 1,2,3,\ldots,N-1).$$

(b) *The points of $V(J)$ on the hyperplane $Z_N = 0$ have homogeneous coordinates*

$$(\epsilon_0 : \cdots : \epsilon_{N-1} : 0), \qquad \epsilon_i = \pm 1.$$

In particular, there are 2^{N-1} of them. Moreover, they are all nonsingular points of $V(J)$.

(c) *If $Y^{\mathrm{pre}}(N,a)$ is nonsingular, then $X^{\mathrm{pre}}(N,a) \cong V(J)$ and the complement of the affine part $X^{\mathrm{pre}}(N,a) \smallsetminus Y^{\mathrm{pre}}(N,a)$ consists of 2^{N-1} points.*

Definition 2.4. We define the *N-th preimage surface* $X^{\mathrm{pre}}(N)$ as the surface fibered over $\mathbb{P}_K^1$ by a. The fiber over a is given by $X^{\mathrm{pre}}(N,a)$ if $Y^{\mathrm{pre}}(N,a)$ is geometrically irreducible and $V(J)$ otherwise. In particular, for each $a \in K$ not a critical value of f_c, we get a nonsingular curve in $\mathbb{P}_K^N$.

$$\begin{array}{ccc} X^{\mathrm{pre}}(N) & \qquad & X^{\mathrm{pre}}(N,a) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \mathbb{P}_K^1 & & a \end{array}$$

Note that for a fixed a_0, the affine points $(x_0, c_0, 1)$ on the curve $X^{\mathrm{pre}}(N, a_0)$ are in bijection with the N-th preimages $x_0 \in f_{c_0}^{-N}(a_0)$.

We will consider the N-th preimage surfaces in the language of function fields. In particular, consider the function field $K(a)$ which is comprised of all rational functions in a with K-rational coefficients. We consider the surfaces defined as

$$Y^{\mathrm{pre}}(N) = V(f_c^N(x) - a) \subset \mathbb{A}^2_{K(a)}$$

and

$$X^{\mathrm{pre}}(N) = V(Z_{N-1}^2 + Z_i Z_N - Z_{i-1}^2 - aZ_N^2 : i = 1, 2, 3, \ldots, N-1) \subset \mathbb{P}^N_{K(a)}.$$

The genus formula (Theorem 2.2) applies to each fiber for which $Y^{\mathrm{pre}}(N, a)$ is nonsingular and geometrically irreducible. In particular, $X^{\mathrm{pre}}(1)$ and $X^{\mathrm{pre}}(2)$ have fibers of genus 0, $X^{\mathrm{pre}}(3)$ has fibers of genus 1, and $X^{\mathrm{pre}}(N)$ for $N \geq 4$ has fibers of genus > 1 (with finitely many exceptional fibers for each N). Therefore, for $N > 3$ and all but finitely many $a \in K$, it follows from Falting's theorem that there are only finitely many points $(x, c) \in X^{\mathrm{pre}}(N, a)$. Thus, except for the finitely many a values, the N-th preimages for $N > 3$ have no contribution to $\bar{\kappa}(a)$. This premise is the content of Corollary 4.2 and the rest of Section 4 addresses the exceptional a values.

Throughout this article we discuss arrangements of preimages. For example, by a 222 arrangement we mean that there are two rational first preimages, (at least) two rational second preimages, and (at least) two rational third preimages. Similarly, a 2424 arrangement has two rational first preimages, four rational second preimages, (at least) 2 rational third preimages, and (at least) four rational fourth preimages. Note that any 226 arrangement would have to be part of a 246 arrangement since the forward image of a rational point is still a rational point.

3. Arrangements of six preimages

By examining the arrangements of six preimages we are able to prove the following lower bound for $\bar{\kappa}(a)$.

Theorem 3.1. *Let K be a number field. There is a finite set S such that*

$$\begin{cases} \bar{\kappa}(a) \geq 6 & \text{if } a \in K \backslash (S \cap K), \\ \bar{\kappa}(a) = 4 & \text{if } a \in S \cap K. \end{cases}$$

Proof. The 22 curve over the function field $K(a)$ is the curve whose points correspond to two rational first preimages and (at least) two rational second preimages. It has fibers of genus 0 [Faber et al. 2009] and at least one $\mathbb{Q}$-rational section for each choice of a, $(1, 1, 0)$. Thus, each fiber has infinitely many rational points and $\bar{\kappa}(a) \geq 4$.

Theorem 3.3 shows that the 222 surface has generic rank at least 2 (exactly 2 over $\mathbb{Q}$). Theorem 3.2 shows that the 24 surface has generic rank 0 over $\mathbb{Q}$. Let S be the (possibly empty) set of a values for which both the 222 and 24 surface specialize to rank 0. By [Masser and Zannier 2012] the set of a values where the 222 surface has rank 0 is finite and thus, S is finite. If $a \in S \cap K$, $\bar{\kappa}(a) = 4$, otherwise $\bar{\kappa}(a) \geq 6$. $\square$

Second preimages. We consider the situation where the preimage tree is full to the second level; that is, there are two rational first preimages and four rational second preimages:

$$\begin{tikzcd}
 & & & a & & & \\
 & t \arrow[urr, "f_c"] & & & & -t \arrow[ull, "f_c"'] & \\
s \arrow[ur, "f_c"] & & -s \arrow[ul, "f_c"'] & & u \arrow[ur, "f_c"] & & -u. \arrow[ul, "f_c"']
\end{tikzcd}$$

We can define this curve over the function field $K(a)$ as

$$X_{24} = V(s^2 - tz - (t^2 - az^2), u^2 + tz - (t^2 - az^2)) \subseteq \mathbb{P}^3_{K(a)}.$$

The fibers (when nonsingular) have genus one with at least one rational section $(1, 1, 1, 0)$ so we can produce a minimal Weierstrass model (using Magma) as an elliptic curve over the function field $K(a)$ as

$$E_{24}(a) : v^2 w = u^3 + (4a - 1)u^2 w + 16auw^2 + (64a^2 - 16a)w^3$$

with j-invariant

$$j(a) = \frac{(16a^2 - 56a + 1)^3}{a(4a + 1)^4}$$

and discriminant

$$\Delta(a) = a(4a + 1)^4.$$

The only fibers which are not elliptic curves are $a = 0$ and $a = -\frac{1}{4}$. This is in fact a rational elliptic surface since it has a Weierstrass model satisfying $\deg(a_i) \leq i$ for a_i the coefficients of an elliptic curve in Weierstrass form [Shioda 1990, page 237].

Theorem 3.2. *$E_{24}(a)(\mathbb{Q}(a))$ has rank 0 and torsion subgroup $\mathbb{Z}/4\mathbb{Z}$ generated by*

$$T(a) = (2, 8a + 2, 1).$$

Proof. We use the main theorem of [Oguiso and Shioda 1991] to see that the rank over $\mathbb{Q}(a)$ is zero. We compute the Kodaira symbols in Magma to get

$$[\langle I4, 1\rangle, \langle I1*, 1\rangle, \langle I1, 1\rangle].$$

From row 72 in the table [Oguiso and Shioda 1991] we have that the rank of $E_{24}(a)(\mathbb{Q}(a))$ is zero. Examining the torsion, we see that the point

$$(2, 8a+2, 1)$$

has order 4 and the specialization $E_{24}(1)(\mathbb{Q})$ has torsion subgroup $\mathbb{Z}/4\mathbb{Z}$. Since the specialization map is injective on torsion on all nonsingular fibers, $E_{24}(a)$ has torsion subgroup exactly $\mathbb{Z}/4\mathbb{Z}$. $\square$

Third preimages. From Theorem 2.3 we see that the elliptic surface parameterizing third preimages of a over the function field $K(a)$ is given by

$$X_{222} = V(z_2^2 + z_1z_3 - z_0^2 - az_3^2,\, z_2^2 + z_2z_3 - z_1^2 - az_3^2) \subseteq \mathbb{P}^3_{K(a)}.$$

Using the cuspidal point $(-1, 1, 1, 0)$ from Theorem 2.3 as the section at infinity we can find a minimal model in Magma as

$$E_{222}(a) : v^2w = u^3 + \left(16a + \tfrac{942}{13}\right)u^2w + \left(\tfrac{10048}{13}a + \tfrac{293084}{169}\right)uw^2 \\ + \left(1024a^2 + \tfrac{1620800}{169}a + \tfrac{30250696}{2197}\right)w^3$$

with j-invariant

$$j(a) = \frac{(16a^2+3)^2}{(4a+1)^2(256a^3+368a^2+104a+23)}$$

and discriminant

$$\Delta(a) = (4a+1)^2(256a^3+368a^2+104a+23).$$

As expected, the only fibers which are not elliptic curves are the fibers over $a = -\frac{1}{4}$ and the three third critical values. This is in fact a rational elliptic surface since it has a Weierstrass model satisfying $\deg(a_i) \le i$ for a_i the coefficients of an elliptic curve in Weierstrass form [Shioda 1990, page 237].

Theorem 3.3. *$E_{222}(a)(\mathbb{Q}(a))$ has rank 2 generated by the two independent sections*

$$P(a) = \left(-\tfrac{262}{13}, 32a+8, 1\right) \quad \textit{and} \quad Q(a) = \left(-\tfrac{366}{13}, 32a+8, 1\right).$$

Proof. We use the main theorem of [Oguiso and Shioda 1991] to see that the rank over $\mathbb{Q}(a)$ is exactly two. We compute the Kodaira symbols in Magma to get

$$[\langle I1, 3\rangle, \langle I2, 1\rangle, \langle I1*, 1\rangle].$$

From row 30 in the table [Oguiso and Shioda 1991] we have that the rank of $E_{222}(a)(\mathbb{Q}(a)) = 2$. Since the specialization map is injective on torsion on all fibers where E_{222} is nonsingular, and the specialization $E_{222}(0)$ has no torsion, there are no rational torsion sections. We can see $P(a)$ and $Q(a)$ are actually the generators by finding a specialization $E_{222}(a_0)$ which is rank 2 with generators $P(a_0)$ and $Q(a_0)$. For $a = 4$ we have

$$E_{222}(4) : v^2 w = u^3 + \tfrac{1774}{13} u^2 w + \tfrac{815580}{169} u w^2 + \tfrac{150527944}{2197} w^3$$

and from Magma the generators are

$$\left(-\tfrac{262}{13}, 136, 1\right) \quad \text{and} \quad \left(-\tfrac{1146}{13}, 136, 1\right).$$

In terms of $P(4)$ and $Q(4)$ these are

$$P(4) \quad \text{and} \quad P(4) + Q(4).$$

Thus, $P(4)$ and $Q(4)$ generate the Mordell-Weil group $E_{222}(4)$ and, hence, $P(a)$ and $Q(a)$ generate the Mordell-Weil group of $E_{222}(a)$. □

4. Arrangements of eight or more preimages

We examine when the genus of the fibers of preimage surfaces of various arrangements of $2N$ preimages is greater than 1 and, thus, by Falting's theorem have a finite number of rational points over an algebraic number field. In particular, if every $2N$ arrangement has genus greater than 1 for some N, then $\bar{\kappa}(a) < 2N$. The difficulty lies in determining the genus when the fiber is singular. We treat the nonsingular case in the following theorem.

Theorem 4.1. *If the curve (fiber) defining an arrangement of* $2N$ *rational preimages of* a *is nonsingular, then it has genus* $(N-3)2^{N-2} + 1$.

Proof. A complete intersection in $\mathbb{P}^m$ is defined as a subscheme Y of $\mathbb{P}^m$ whose homogeneous ideal I can be generated by $r = \operatorname{codim}(Y, \mathbb{P}^m)$ elements [Hartshorne 1977, Exercise II.8.4]. Each surface arranging $2N$ points can be described by the equations

$$f_c(z_1) = a \quad \text{and} \quad f_c(z_i) = (-1)^{\epsilon} z_j \text{ for } 2 \le i \le N$$

where $1 \le j < N$ and $\epsilon = \pm 1$ depending on the arrangment of points. After homogenization and elimination of c from this system of equations we obtain a description of each fiber as a curve defined by $N-1$ degree two hypersurfaces in $\mathbb{P}^N$ and, hence, a complete intersection. From [Hirzebruch 1966, §22] or [Arslan and Sertöz 1998, Corollary 2] we get a formula for the arithmetic genus of a complete intersection of $N-1$ degree two hypersurfaces in $\mathbb{P}^N$ as

$$p_a = \sum_{m=1}^{N-1} (-1)^{m+1} \binom{N-1}{m} \phi_N(-2m)$$

where $\phi_N(z)$ comes from the Hilbert polynomial of the $2N$ curve and is given by

$$\phi_N(z) = \frac{(z+1)(z+2)\cdots(z+N)}{N!} = \binom{z+N}{N}.$$

Since the arithmetic genus is equal to the geometric genus for nonsingular curves [Hartshorne 1977, Proposition IV.1.1], the genus is independent of the arrangement of the preimages and from [Faber et al. 2009, Theorem 1.5] we get the simpler formula

$$g = (N-3)2^{N-2} + 1. \qquad \square$$

Corollary 4.2. *If the curve (fiber) defining an arrangement of $2N$ rational preimages of a is nonsingular, then the genus is greater than 1 for $2N \geq 8$.*

We have thus reduced the computation of $\bar{\kappa}(a, K)$ to checking a values where the fiber is singular for arrangements with 8 (or more) rational preimages (224, 242, 2222). The method is as follows.

(a) Using the Jacobian criterion, determine all of the singular fibers (a values).

(b) Determine the δ-invariants of each singular point to determine the genus of each singular fiber.

Recall that the δ-invariant of a singularity P is defined as

$$\delta_P = \sum_Q \tfrac{1}{2} m_Q(m_Q - 1),$$

where the sum ranges over the infinitely near points of P and m_Q are their multiplicities. See [Sendra et al. 2008, Section 3.2] for the basic definitions and the case of plane curves and [Brieskorn and Knörrer 1986, Section 9.2, Theorem 7] for a more general discussion. As the singularity analysis computations are identical in form for all of the singularities, we outline the method, include the first such computation, and omit the details for the other singularities. The singularity analysis proceeds as follows.

(a) Let $C \subseteq \mathbb{P}^N$ be a singular curve with singular point P. We move P to $(0, \ldots, 0, 1)$ and dehomogenize.

(b) Project onto a singular plane curve with isomorphic tangent space at the singular point.

(c) Analyze the singularity of the plane curve with blow-ups and compute the δ-invariant.

Examining the 224 surface. One possible 224 arrangement of 8 preimages is this:

$$\begin{tikzcd}
 & & & & a & \\
 & & & t \arrow[ur, "f_c"] & & -t \arrow[ul, "f_c"'] \\
 & s \arrow[urr, "f_c"] & & & -s \arrow[ul, "f_c"'] & \\
q \arrow[ur, "f_c"] & & -q \arrow[ul, "f_c"'] & r \arrow[ur, "f_c"] & & -r \arrow[ul, "f_c"']
\end{tikzcd}$$

Every other 224 arrangement differs only by renaming, so this is the only distinct 224 arrangement. The curve is defined by three degree two equations in $\mathbb{P}^4$ as

$$C_{224} = V(az^2 - t^2 - (tz - s^2), az^2 - t^2 - (sz - q^2), az^2 - t^2 - (-sz - r^2)) \subseteq \mathbb{P}^4_{K(a)}.$$

Theorem 4.3. *The a values for which the fiber of the* 224 *surface is singular are given by*

$$a \in \left\{-\tfrac{1}{4}, 0, a_1, a_2, a_3\right\},$$

where a_1, a_2, a_3 are the three third critical values of f_c

Proof. We apply the Jacobian criterion to determine the singular points. For each singular point, we can determine the associated a value(s). Examining the hyperplane at infinity $z = 0$ we have the 8 cuspidal points $(\pm 1, \pm 1, \pm 1, 1, 0) \in \mathbb{P}^4$. To check the singularity of these points, we use the Jacobian criterion on the affine chart $\mathbb{A}^4_{q\neq 0}$ with generators

$$\{az^2 - t^2 - (tz - s^2), az^2 - t^2 - (sz - 1), az^2 - t^2 - (-sz - r^2)\}$$

to have the Jacobian matrix at $z = 0$

$$\begin{pmatrix} 0 & 2s & -2t & -t \\ 0 & 0 & -2t & -s \\ 2r & 0 & -2t & s \end{pmatrix}.$$

The determinant of one such maximal minor is $-8rst$, and since $r, s, t \neq 0$, this is nonzero, so the cuspidal points are all nonsingular.

Now we consider the points in the affine chart $\mathbb{A}^4_{z\neq 0}$ which has generators

$$\{a - t^2 - (t - s^2), a - t^2 - (s - q^2), a - t^2 - (-s - r^2)\}.$$

The Jacobian matrix is given by

$$\begin{pmatrix} 0 & 0 & 2s & -2t-1 \\ 2q & 0 & -1 & -2t \\ 0 & 2r & 1 & -2t \end{pmatrix}$$

and the determinants of the maximal minors are

$$\{8qrs, 4qr(-2t-1), 2q(4st-2t-1), -2r(4st+2t+1)\}.$$

The combinations that result in all 4 determinants vanishing are the following.

(a) If $q = r = 0$, then we have $c = \pm s$ and so $c = 0$ and so $a = 0$.

(b) If $q = 0$ and $(4st + 2t + 1) = 0$, then we must have $s \neq -\frac{1}{2}$ so we can solve $t = -\frac{1}{4s+2} = -\frac{1}{4c+2}$. Then we have $s^2 + c = c^2 + c = t$ and the roots of $4c^3 + 6c^2 + 2c + 1 = \frac{df_c^3(0)}{dc}$ combined with $a = f_c(f_c(f_c(0)))$ to get the three third critical values.

(c) If $q \neq 0$, $r = 0$, and $(4st - 2t - 1) = 0$, then we must have $t \neq 0$ and we can solve $s = \frac{2t+1}{4t} = -c$. Then we have $s^2 - s = t$ and the roots of $16t^3 + 4t^2 - 1$ which give the three third critical values.

(d) If $q, r \neq 0$, $s = 0$, and $t = -\frac{1}{2}$, then we have $c = -\frac{1}{2}$ and so $a = -\frac{1}{4}$.

□

We will treat $a = -\frac{1}{4}$ on page 358.

Theorem 4.4. *The genus of C_{224} is*

$$g = \begin{cases} 4 & \text{if } a = 0, \\ 1 & \text{if } a \in \{a_1, a_2, a_3\}, \end{cases}$$

where a_1, a_2, a_3 are the three third critical values of f_c.

Proof. There is one singular point for $a = 0$ and four singular points for each a_i. In all cases $\delta_P = 1$ so the genus drops by 1 for each singular point.

We now compute the δ-invariant of one of the singular points for a_1. The 224 curve for a_1 is defined as

$$V(a_1z^2 - t^2 - (tz - s^2), a_1z^2 - t^2 - (sz - q^2), a_1z^2 - t^2 - (-sz - r^2))$$

and if α is a root of

$$4x^3 + 6x^2 + 2x + 1$$

then

$$a_1 = \alpha^4 + 2\alpha^3 + \alpha^2 + \alpha = -\tfrac{1}{4}\alpha^2 + \tfrac{1}{2}\alpha - \tfrac{1}{8}.$$

We label the coordinates as (q, r, s, t, z) and the singular point is

$$P = (0, -\beta, \alpha, \alpha^2 + \alpha, 1)$$

where $\beta^2 = -2\alpha$. We move P to $(0, 0, 0, 0, 1)$ with a translation

$$(q, r, s, t, z) \mapsto (q, r - \beta z, s + \alpha z, t + (\alpha^2 + \alpha)z)$$

to get a new curve $\widetilde{C}$ and singular point $\widetilde{P} = (0, 0, 0, 0, 1)$. We dehomogenize to affine coordinates $(Q, R, S, T) = (q/z, r/z, s/z, t/z)$ and compute the tangent space at $\widetilde{P}$ as

$$\begin{cases} -2T\alpha^2 - 2T\alpha - T + 2S\alpha = 0, \\ -2T\alpha^2 - 2T\alpha - S = 0, \\ -2T\alpha^2 - 2T\alpha + S - 2\beta R = 0. \end{cases} \tag{1}$$

Notice that the second equation of (1) implies the first using the degree 4 polynomial satisfied by α. Thus, the tangent space is given by

$$-2T\alpha^2 - 2T\alpha - S = 0, \quad -2T\alpha^2 - 2T\alpha + S - 2\beta R = 0.$$

Since we want to project $\widetilde{C}$ to a plane curve preserving the tangent space at $\widetilde{P}$ we define

$$u = -2T\alpha^2 - 2T\alpha - S, \quad v = -2T\alpha^2 - 2T\alpha + S - 2\beta R,$$

with inverse

$$S = \beta R - \frac{u}{2} + \frac{v}{2}, \quad T = \frac{\beta R}{-2\alpha^2 - 2\alpha} + \frac{u}{-4\alpha^2 - 4\alpha} + \frac{v}{-4\alpha^2 - 4\alpha},$$

and make the change of variables $(Q, R, S, T) \mapsto (Q, R, u, v)$ to get a new curve $\widetilde{C}'$ and point $\widetilde{P}'$. The tangent space at $\widetilde{P}'$ is given by $u = v = 0$. We now project $\widetilde{C}'$ onto a plane curve in the QR-plane. To project we eliminate the variables u, v from the three defining equations of $\widetilde{C}'$ to get the single equation

$$\begin{aligned}
&(2\alpha+1)Q^8 + \big((-8\alpha-4)R^2 + (16\beta\alpha+8\beta)R + (16\alpha^2-4)\big)Q^6 \\
&\quad + \big((12\alpha+6)R^4 + (-48\beta\alpha-24\beta)R^3 + (-144\alpha^2-64\alpha+4)R^2 \\
&\qquad\qquad + (96\beta\alpha^2+32\beta\alpha-8\beta)R + (-64\alpha^2-24\alpha-8)\big)Q^4 \\
&\quad + \big((-8\alpha-4)R^6 + (48\beta\alpha+24\beta)R^5 + (240\alpha^2+128\alpha+4)R^4 \\
&\qquad + (-320\beta\alpha^2-192\beta\alpha-16\beta)R^3 + (384\alpha^2+208\alpha+128)R^2 \\
&\qquad\qquad + (-128\beta\alpha^2-96\beta\alpha-64\beta)R - 32\alpha\big)Q^2 \\
&\quad + (2\alpha+1)R^8 + (-16\beta\alpha-8\beta)R^7 + (-112\alpha^2-64\alpha-4)R^6 \\
&\qquad + (224\beta\alpha^2+160\beta\alpha+24\beta)R^5 + (-320\alpha^2-152\alpha-136)R^4 \\
&\qquad\qquad + (128\beta\alpha^2+32\beta\alpha+96\beta)R^3 + (-64\alpha^2+64\alpha)R^2 = 0,
\end{aligned}$$

defining a plane curve in $\mathbb{A}^2$ with variables (Q, R). Notice that the only points of the form $(0, 0, u, v)$ on $\widetilde{C}'$ is the point $(0, 0, 0, 0)$ (in the other words, the singular point is the only point that projects onto $(0, 0)$), so we proceed with analyzing the plane curve singularity $(0, 0)$. Blowing-up once resolves the singularity and we see that it has multiplicity 2. So we compute

$$\delta_P = \frac{1}{2}(2 \cdot 1) = 1.$$

A similar analysis is done on all of the other singularities to get $\delta_P = 1$ for all P for all $a \in \{0, a_1, a_2, a_3\}$. Hence, we have

$$\begin{cases} g = 5 - 1 = 4 & \text{if } a = 0, \\ g = 5 - (1 + 1 + 1 + 1) = 1 & \text{if } a = a_1, a_2, a_3. \end{cases} \qquad \square$$

Examining the 242 surface. One possible 242 arrangement of 8 preimages is this:

$$\begin{tikzcd}
 & & & & a & & \\
 & & t \arrow[urr, "f_c"] & & & -t \arrow[ul, "f_c"'] & \\
 & s \arrow[ur, "f_c"] & & -s \arrow[ul, "f_c"'] & u \arrow[ur, "f_c"] & & -u \arrow[ul, "f_c"'] \\
q \arrow[ur, "f_c"] & & -q \arrow[ul, "f_c"'] & & & &
\end{tikzcd}$$

Every other 242 arrangement differs only by renaming, so this is the only distinct 242 arrangement. The surface is defined by 3 degree two equations in $\mathbb{P}^4$ as

$$C_{242} = V\left(az^2 - t^2 - (tz - s^2), az^2 - t^2 - (-tz - u^2), az^2 - t^2 - (sz - q^2)\right) \subseteq \mathbb{P}^4_{K(a)}.$$

Theorem 4.5. *The a values for which the fiber of the 242 surface is singular are given by*

$$a \in \left\{-\tfrac{1}{4}, 0, 2, a_1, a_2, a_3\right\}$$

where a_1, a_2, a_3 are the three third critical values of f_c.

Proof. We apply the Jacobian criterion to determine the singular points. For each singular point, we can determine the associated a value(s). Examining the hyperplane at infinity, $z = 0$, we have the 8 cuspidal points $(\pm 1, \pm 1, \pm 1, 1, 0) \in \mathbb{P}^4$. To check the singularity of these points, we use the Jacobian criterion on the affine chart $\mathbb{A}^4_{q \neq 0}$ with generators

$$\left\{az^2 - t^2 - (tz - s^2), az^2 - t^2 - (-tz - u^2), az^2 - t^2 - (sz - 1)\right\}.$$

The Jacobian matrix at $z=0$ is given by

$$\begin{pmatrix} 2s & 0 & -2t & -t \\ 0 & 2u & -2t & t \\ 0 & 0 & -2t & -s \end{pmatrix}.$$

The determinant of one maximal minor is $-8sut$, and since $s,u,t \neq 0$, this is nonzero, so the cuspidal points are all nonsingular.

Now we consider the points in the affine chart $\mathbb{A}^4_{z\neq 0}$ which has generators

$$\{a-t^2-(t-s^2),\, a-t^2-(-t-u^2),\, a-t^2-(s-q^2)\}.$$

The Jacobian matrix is given by

$$\begin{pmatrix} 0 & 2s & -2t-1 & 0 \\ 0 & 0 & -2t+1 & 2u \\ 2q & -1 & -2t & 0 \end{pmatrix}.$$

The determinants of the maximal minors are

$$\big\{2u(4st+2t+1),\, 4qu(-2t-1),\, 8qus,\, 4qs(-2t+1)\big\}.$$

The combinations that result in all 4 vanishing are as follows:

(a) If $q=0$ and $u=0$, then $f_c^2(0)=a$ and $f_c^3(0)=a$ which is the polynomial equation

$$f_c(f_c(f_c(0)))-f_c(f_c(0))=c^4+2c^3=c^3(c+2)=0$$

so $c=0$ or $c=-2$. So we have $a=0$ or $a=2$.

(b) If $q=0$ and $(4st+2t+1)=0$, then we must have $s\neq -\frac{1}{2}$ so we can solve $t=-1/(4s+2)=-1/(4c+2)$. Then we have $s^2+c=c^2+c=t$ and the roots of

$$4c^3+6c^2+2c+1=\frac{df_c^3(0)}{dc}$$

combined with $a=f_c(f_c(f_c(0)))$ to get the three third critical values.

(c) If $u=0$ and $s=0$, then $c=\pm t$ and so $t=c=0$ and so $a=0$.

(d) If $u=0$ and $t=\frac{1}{2}$, then $c=-\frac{1}{2}$ and so $a=-\frac{1}{4}$.

(e) If $s=0$ and $t=-\frac{1}{2}$, then $c=-\frac{1}{2}$ and so $a=-\frac{1}{4}$. □

We will treat $a=-\frac{1}{4}$ on page 358.

Theorem 4.6. *The genus of C_{242} is* $g=\begin{cases} 3 & \text{if } a=0, \\ 4 & \text{if } a=2, \\ 3 & \text{if } a\in\{a_1,a_2,a_3\}. \end{cases}$

Proof. We proceed as in the proof of Theorem 4.4 for analyzing the singularities.

For $a = 0$ there is one singularity that required two blow-ups to resolve and we get multiplicity 2 for both of the infinitely near points and, hence, $\delta_P = \frac{1}{2}(2 \cdot 1) + \frac{1}{2}(2 \cdot 1) = 2$ and $g = 5 - 2 = 3$.

For $a = 2$ there is one singular point with $\delta_P = 1$ and, hence, $g = 5 - 1 = 4$.

For $a \in \{a_1, a_2, a_3\}$ each curve has two singular points both with $\delta_P = 1$ and, hence, $g = 5 - (1 + 1) = 3$. □

Examining the 2222 surface. One possible 2222 arrangement of 8 preimages is this:

$$\begin{tikzcd}
& & & & a & \\
& & & t \arrow[ur, "f_c"] & & -t \arrow[ul, "f_c"'] \\
& & s \arrow[ur, "f_c"] & & -s \arrow[ul, "f_c"'] & \\
& q \arrow[ur, "f_c"] & & -q \arrow[ul, "f_c"'] & & \\
u \arrow[ur, "f_c"] & & -u \arrow[ul, "f_c"'] & & &
\end{tikzcd}$$

Every other 2222 arrangement differs only by renaming, so this is the only distinct 2222 arrangement. The surface is defined by 4 degree two equations in $\mathbb{P}^5$ as

$$C_{2222} = V(az^2 - t^2 - (tz - s^2),\, az^2 - t^2 - (sz - q^2),\, az^2 - t^2 - (qz - u^2)) \subseteq \mathbb{P}^5_{K(a)}.$$

From [Faber et al. 2009, Theorem 1.3] the only singular fibers are for a the N-th critical values for $2 \le N \le 4$. For $N = 2$ we get $a = -\frac{1}{4}$, which will be treated on page 358. For $N = 3$ we get the three third critical values which we label $a_{3,1}, a_{3,2}, a_{3,3}$. For $N = 4$ we get the seven 4-th critical values, which we label $a_{4,i}$ for $1 \le i \le 7$, and which satisfy

$$a = f_c(f_c(f_c(f_c(0)))) \quad \text{for } 8c^7 + 28c^6 + 36c^5 + 30c^4 + 20c^3 + 6c^2 + 2c + 1 = 0.$$

Theorem 4.7. *The genus of C_{2222} is*

$$g = \begin{cases} 3 & \text{if } a \in \{a_{3,1}, a_{3,2}, a_{3,3}\}, \\ 4 & \text{if } a \in \{a_{4,i} : 1 \le i \le 7\}. \end{cases}$$

Proof. A fiber of the 2222 surface is isomorphic [Faber et al. 2011, Proposition 4.2] to the degree 16 plain curve defined by the equation

$$f_c^4(x) = a.$$

For $a \in \{a_{3,i}\}$ there are three singular points, one of which is $(0, 1, 0)$ and the other two depend on a. The $(0, 1, 0)$ point requires several blow-ups and has $\delta_P = 100$ and each of the other two points have $\delta_P = 1$ for a final genus of $g = \frac{1}{2}(15 \cdot 14) - 102 = 105 - 102 = 3$.

For $a \in \{a_{4,i}\}$ there are two singular points, one of which is $(0, 1, 0)$ and the other depends on a. The $(0, 1, 0)$ point has $\delta_P = 100$ and the point has $\delta_P = 1$ for a final genus of $g = \frac{1}{2}(15 \cdot 14) - 101 = 105 - 101 = 4$. □

Corollary 4.8. *For any $a \in \overline{\mathbb{Q}} \backslash \{-\frac{1}{4}\}$ and any algebraic number field K there are only finitely many $c \in K$ for which there are at least two K-rational 4-th preimages of a.*

The bound $\bar{\kappa}(-\frac{1}{4})$. For $a = -\frac{1}{4}$ the preimages curves are in fact reducible since we have an equation in the generators of the form

$$s^2 + \left(t - \tfrac{1}{2}z\right)^2 = \left(s - \left(t - \tfrac{1}{2}z\right)\right)\left(s + \left(t - \tfrac{1}{2}z\right)\right),$$

where s is a second preimage of a for which $s^2 + c = t$ and $t^2 + c = a$, and an equation of the form

$$u^2 - \left(t + \tfrac{1}{2}z\right)^2 = \left(u - \left(t + \tfrac{1}{2}z\right)\left(u - \left(t + \tfrac{1}{2}z\right)\right),$$

where u is a second preimage of a for which $u^2 + c = -t$. After splitting the preimage curves into their distinct irreducible components we can again proceed with genus calculations.

Theorem 4.9. *For any fixed number field K, $\bar{\kappa}\left(-\frac{1}{4}\right) = 10$.*

Proof. Using the Jacobian criterion we compute that the following curves are all nonsingular, and we apply the genus formula from [Hirzebruch 1966, §22] or [Arslan and Sertöz 1998, Corollary 2] to compute the following genera.

$$g = \begin{cases} 1 & \text{in the cases } 224, 2222, 244, 2422 \\ 5 & \text{in the cases } 22222, 2224, 2242, 246, 2442, 2424, 24222. \end{cases}$$

Using Magma, we see that the 244 curve is a rank 1 elliptic curve over $\mathbb{Q}$ isomorphic to

$$v^2 w = u^3 + u^2 w - 9uw^2 + 7w^3$$

so has infinitely many rational points. Therefore, there are infinitely many c with 10 rational preimages of $-\frac{1}{4}$ and only finitely many c values with 12 (or more) rational preimages of $-\frac{1}{4}$. □

5. Proof of Theorem 1.3

Proof. The case $a = -\frac{1}{4}$ was covered in Theorem 4.9.

For a a third critical value we have genus 1 for the 224 curve and, hence, for a large enough extension of $\mathbb{Q}$ it has positive rank and infinitely many rational points. Also, it has no $\mathbb{Q}$-rational points. The 242 curve has genus greater than 1 and, hence, has only finitely many rational points. Thus, for $\bar{\kappa}(a, K)$ to be at least 10 there must be infinitely many rational points on a curve corresponding to an arrangement with rational 4-th preimages, which is not possible by Corollary 4.8. So it is possible for $\bar{\kappa}(a, K)$ to be either 6 or 8 depending on the field.

For all other values of a we have the genus of the 224 and 242 curves are greater than 1 and, hence, have only finitely many rational points. Any arrangement with more points must contain one of these two arrangements, hence $\bar{\kappa}(a, K) \leq 6$. Theorem 3.3 shows that the 222 surface has generic rank 2 and [Masser and Zannier 2012] shows that the set of a where the rank is 0 is finite. Every a value for which both E_{222} and E_{24} specialize to rank 0 has $\bar{\kappa}(a) = 4$, otherwise $\bar{\kappa}(a) = 6$. □

6. Other properties of preimage surfaces

In this section we collect some additional properties of the preimages surfaces that are tangential to the proof of Theorem 1.3, yet still of interest.

Parametrization of torsion subgroups of E_{24}. Recall that Mazur's theorem [1977] gives a description of the possible torsion subgroups of elliptic curves over $\mathbb{Q}$ and that the specialization map is injective on nonsingular fibers. These facts combined with Theorem 3.2 implies that the possible torsion subgroups for a nonsingular specialization of $E_{24}(a)$ must be isomorphic to one of the following groups:

$$\{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/8\mathbb{Z},\ \mathbb{Z}/12\mathbb{Z}\}.$$

We characterize the a values giving rise to a specialization with each of these possible torsion subgroups in the following theorem.

Theorem 6.1. (a) *$E_{24}(a)(\mathbb{Q})$ contains a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ if and only if*

$$a = -t^2 \quad \textit{for} \quad t \in \mathbb{Q}\backslash\left\{0, \pm\tfrac{1}{2}\right\}.$$

(b) *$E_{24}(a)(\mathbb{Q})$ contains a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z}$ if and only if*

$$a = \tfrac{1}{4}t^2(t^2 - 2) \quad \textit{for} \quad t \in \mathbb{Q}\backslash\{0, \pm 1\}.$$

(c) *$E_{24}(a)(\mathbb{Q})$ contains a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ if and only if*

$$a = -\frac{(4t^2 - 4t - 1)^2(4t^2 + 4t - 1)^2}{4(4t^2 + 1)^4} \quad \textit{for} \quad t \in \mathbb{Q}\backslash\left\{0, \pm\tfrac{1}{2}\right\}.$$

(d) *$E_{24}(a)(\mathbb{Q})$ contains a subgroup isomorphic to $\mathbb{Z}/12\mathbb{Z}$ if and only if*

$$a = \frac{(13691470144t^2 - 235376t + 1)(13903463744t^2 - 235376t + 1)^3}{95272651012502978560000000t^6(117688t - 1)^2}$$

for $t \in \mathbb{Q}\backslash\{0, \frac{1}{117688}\}$.

Proof. (a) First suppose $a = -t^2$ for some $t \in \mathbb{Q}\backslash\{0, \pm\frac{1}{2}\}$. Then

$$\{\mathbb{O}, (4t^2 + 1, 0, 1), (4t, 0, 1), (-4t, 0, 1)\}$$

is a subgroup of $E_{24}(-t^2)(\mathbb{Q})$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since there is also a generic torsion point of order 4 (Theorem 3.2), $E_{24}(-t^2)(\mathbb{Q})$ contains a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Next, suppose $E_{24}(a)(\mathbb{Q})$ contains a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and, hence, also a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Thus, $E_{24}(a)(\mathbb{Q})$ has three points of order two. Points of order two must be rational roots of the Weierstrass equation

$$x^3 + (4a - 1)x^2 + (16a)x + 16a(4a - 1) = (x + 4a - 1)(x^2 + 16a). \qquad (2)$$

So, $x^2 + 16a$ must have 2 rational roots, or equivalently, $a = -(x/4)^2 = -t^2$. Hence, there are three rational roots of (2) if and only if $a = -t^2$ for $t \in \mathbb{Q}$. However, if $t = \pm\frac{1}{2}$ then the roots will not be distinct, so we must have $a = -t^2$ for $t \in \mathbb{Q}\backslash\{\pm\frac{1}{2}\}$. For $t = 0$ we get $a = 0$ which is a degenerate case (a singular fiber of $X^{\mathrm{pre}}(2)$).

(b) Suppose $a = t^2(t^2 - 2)/4$ for some $t \in \mathbb{Q}\backslash\{0, \pm 1\}$. Then it can be verified directly that the point $P = (2t(t^2 + t - 1), 2(t - 1)t(t + 1)^3, 1)$ is in $E_{24}(a)(\mathbb{Q})$ and $[2]P = (2, 2(4a + 1), 1)$ is the generator of the cyclic subgroup of order four. So, P generates a cyclic group of order eight.

Now suppose that $E_{24}(a)(\mathbb{Q})$ has a cyclic subgroup of order eight. If we let $P = (x, y, 1)$ be the generator of the subgroup, then $[2]P$ generates a cyclic group of order four (the generic torsion subgroup). So, we must have $x([2]P) = 2$. This gives us the equation

$$x^4 - 8x^3 - 64ax^2 + 8x^2 - 512a^2x - 1024a^3 + 256a^2 + 64a = 0.$$

Then using the solution to the quartic we have the solutions

$$x = 2 \pm 2\sqrt{4a + 1} + \frac{1}{2}\sqrt{24 + (8a - 1) \pm \frac{512 + 4096a^2 + 256(8a - 1)}{16\sqrt{4a + 1}}}$$

$$x = 2 \pm 2\sqrt{4a + 1} - \frac{1}{2}\sqrt{24 + (8a - 1) \pm \frac{512 + 4096a^2 + 256(8a - 1)}{16\sqrt{4a + 1}}}.$$

In order to have $x \in \mathbb{Q}$, and since x is clearly not 2, we must have $\sqrt{4a+1} \in \mathbb{Q}$. So $a = \frac{b^2-1}{4}$ for some $b \in \mathbb{Q}$. The above roots become

$$x = 2(1 \pm b + b\sqrt{1 \pm b})$$
$$x = 2(1 \pm b - b\sqrt{1 \pm b})$$

from which it follows that $b = \pm(t^2-1)$. Thus, $a = \frac{t^2(t^2-2)}{4}$. Note that for $t = \pm 1$ we get $a = -\frac{1}{4}$ and for $t = 0$ we get $a = 0$ which are all singular fibers.

(c) Clearly, $E_{24}(a)(\mathbb{Q})$ has a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ if and only if $E_2(a)$ has a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and a subgroup isomorphic to $\mathbb{Z}/8\mathbb{Z}$. From the two previous parts, it follows that $a = -t_1^2$ and $a = \frac{1}{4}t_2^2(t_2^2-2)$. These two equations define a curve of genus zero which can be parameterized with Magma and substituted into $a = -t_1^2$ to get the stated form. For $t = 0, \pm\frac{1}{2}$ we get $a = -\frac{1}{4}$, which is a singular fiber.

(d) Since specialization is injective on torsion for nonsingular fibers , $E_{24}(a)(\mathbb{Q})$ has a subgroup isomorphic to $\mathbb{Z}/12\mathbb{Z}$ if and only if there is a point $Q = [x, y] \in E_{24}(a)(\mathbb{Q})$ for which $[3]Q$ generates the generic $\mathbb{Z}/4\mathbb{Z}$ torsion subgroup. In particular, we must have $x([3]Q) = 2$. So we need to find solutions to

$$\frac{x([3]Q)-2}{x-2} = 0$$

where we divide out by $x-2$ since we only wish to exclude the a values which have purely $\mathbb{Z}/4\mathbb{Z}$ torsion. From the *algcurve* package in Maple we get the parametrization given. The two excluded t values correspond to the two singular fibers $a = 0$ and $a = -\frac{1}{4}$. □

Corollary 6.2. *The $a \in \mathbb{Q}$ for which $E_{24}(a)(\mathbb{Q})$ has torsion subgroup exactly $\mathbb{Z}/4\mathbb{Z}$, in other words, the $a \in \mathbb{Q}$ for which the specialization map is an isomorphism on torsion, is a Zariski dense set.*

Proof. From Mazur's theorem and the injectivity of the specialization map, the possible torsion groups of $E_{24}(a)(\mathbb{Q})$ are

$$\{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/8\mathbb{Z},\ \mathbb{Z}/12\mathbb{Z}\}.$$

The condition on a for $E_{24}(a)(\mathbb{Q})_{\text{tors}}$ to not be $\mathbb{Z}/4\mathbb{Z}$ is a closed condition from Theorem 6.1 and the j-invariant. Therefore, every $a \in \mathbb{Q}$ outside of this Zariski closed set satisfies $E_{24}(a)(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z}$ and there is at least one such a,

$$E_{23}(1)(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/4\mathbb{Z}.$$ □

Exceptional (c, a) values over $\mathbb{Q}$.

Rank zero. The methods of [Masser and Zannier 2008; 2012], in principle, can compute the full set S, but in practice such computations are difficult. However, computing the set $S \cap K$ for $[K : \mathbb{Q}] \leq 2$ from Theorem 1.3 is feasible since we have an explicit (small) bound on the order of a torsion point.

We must have both $P(a)$ and $Q(a)$ are torsion on the 222 surface. We have a bound of 18 for the order of a torsion point over a quadratic number field K [Kamienny 1992; Kenku and Momose 1988]. Finding the a for which $P(a)$ or $Q(a)$ is torsion of a given order is solving polynomials equation in a. If there are any a values for which they are both torsion, we compute the rank of $E_{24}(a)$.

Theorem 6.3. *Let S be the set of a values from Theorem 1.3 for which $\bar{\kappa}(a) = 4$. Let K be a quadratic number field. Then, $S \cap K = \varnothing$.*

Proof. Direction computation. □

Full trees of preimages. We can find an a value with arbitrarily many $\mathbb{Q}$-rational preimages by taking a to be the n-th forward image of any wandering $\mathbb{Q}$-rational point. This gives a very deep but potentially sparse preimage tree. Consequently, one may ask if you can find an a and c which gives a full tree to some level. Clearly, if you allow $K/\mathbb{Q}$ to be of large degree, the answer is any level, so we address this question over $\mathbb{Q}$. For example, here is a list of (c, a) with a 246 preimage arrangement.

$$\left(-\tfrac{5248}{2025}, \tfrac{726745984}{284765625}\right), \quad \left(-\tfrac{17536}{5625}, \tfrac{878382976}{244140625}\right), \quad \left(-\tfrac{9153}{6400}, -\tfrac{437896611}{400000000}\right), \quad \left(-\tfrac{24361}{14400}, -\tfrac{42}{25}\right),$$
$$\left(-\tfrac{20817}{25600}, -\tfrac{1078371711}{6400000000}\right), \quad \left(-\tfrac{180625}{97344}, \tfrac{2845625}{5483712}\right), \quad \left(-\tfrac{158848}{99225}, \tfrac{20844352384}{683722265625}\right).$$

Remark 6.4. We were unable to find any pairs (c, a) over $\mathbb{Q}$ with the full 248 arrangement, but it seems reasonable to expect that such an arrangement exists. We searched by choosing the smallest third preimage having height at most log 30,000, since choosing two third preimages which map to same second preimage (up to sign) fixes a unique c value and, hence, a unique a value.

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