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The Steiner problem involves finding a shortest path network connecting a specified set of points. In this paper, we examine the Steiner problem for three points on the surface of a regular tetrahedron. We prove several important properties about Steiner minimal trees on a regular tetrahedron. There are infinitely many ways to connect three points on a tetrahedron, so we present a way to eliminate all but a finite number of possible solutions. We provide an algorithm for finding a shortest network connecting three given points on a regular tetrahedron. The solution can be found by direct measurement of the remaining possible Steiner trees.

1. Introduction

The Steiner problem asks to find a shortest path network to connect a given set of points on a surface. In this paper we will study the three point Steiner problem on a regular tetrahedron. We will provide an algorithm in Section 10, Algorithm 10.1, that determines a solution to the three point Steiner problem on the regular tetrahedron.

On the Euclidean plane, the Steiner problem has been studied extensively; see [Gilbert and Pollak 1968; Hwang et al. 1992; Ivanov and Tuzhilin 1994, Chapter 9; Melzak 1961; Zacharias 1914–1921]. The Steiner problem for three points on the Euclidean plane was formally introduced in the seventeenth century by Fermat; see [Hwang et al. 1992; Kuhn 1974; Zacharias 1914–1921]. A general algorithm to find the solution to the Steiner problem for \( n \) points on the Euclidean plane was first developed by Melzak [1961] (see also [Hwang et al. 1992]).

The Steiner problem on the surface of the tetrahedron is not as straightforward as on the plane. In particular, a geodesic segment connecting any two points on the surface of the tetrahedron is not unique (see top part of Figure 1 on next page). Consequently, there are infinitely many locally stable shortest-length trees connecting any three points on the surface of the tetrahedron (see Figure 1, bottom). In this

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paper, we provide an algorithm that eliminates all but a small number of path networks that need be considered as possible minimizers. Amongst these remaining candidates, a shortest path network can be found using direct measurement.

This research contributes to the growing set of strategies for solving Steiner problems on surfaces in general. Algorithms exist to find the solution for the Steiner problem on certain surfaces of constant curvature. The problem was studied in [Weng 2001; Litwhiler and Aly 1980; Brazil et al. 1998] for on curved surfaces, including spheres. March and Halverson [2005] studied Steiner trees in hyperbolic space. Lee et al. [2011] studied the Steiner problem on wide and narrow cones. Penrod [2007] and May and Mitchell [2007] developed algorithms to solve Steiner problems on the flat torus. Caffarelli et al. [2010] studied the Steiner problem on surfaces of revolution. Brune and Sipe [2009] developed an algorithm to find a shortest path between two points on the surface of the regular tetrahedron. This research about the Steiner problem on the regular tetrahedron may provide further insight into the Steiner problem on more general piecewise linear surfaces.

2. Preliminaries

We begin by setting up the basic framework for the Steiner problem on a regular tetrahedron \( T \). Let \( \mathcal{A} = \{a_1, a_2, \ldots, a_n\} \) be a set of given points on \( T \) called terminal points, and let \( L \) be a path network (also on \( T \)) connecting the points in \( \mathcal{A} \). A path network connects a collection of arcs, only possibly meeting at the endpoints such that the network contains a path connecting any two points of \( \mathcal{A} \). If \( L \) is a shortest path network, the edges must be geodesics. \( L \) must also be a tree since if \( L \) contained a cycle, one of the edges could be removed. The goal of the Steiner problem is to find a shortest path network \( L \) connecting the points of \( \mathcal{A} \). A shortest path network may have additional vertices called Steiner points. The
solution to the Steiner problem is called the Steiner minimal tree, which will be denoted by SMT(\mathcal{S}).

As defined in [Hwang and Weng 1986], a tree with \( n \) fixed points is called a Steiner tree on \( n \) fixed points if it satisfies the following conditions:

1. There are at most \( n - 2 \) Steiner points.
2. Each Steiner point has exactly three incident edges.
3. Any pair of edges meeting at any vertex of the tree form an angle with measure at least \( 120^\circ \).

Note that for a tree with no degree-two Steiner points, the number of edges minus the number of vertices is 1, which in fact implies condition 1. A tree that has exactly \( n - 2 \) Steiner points is called a full Steiner tree. A tree that has fewer than \( n - 2 \) Steiner points is called a degenerate Steiner tree.

The Steiner problem for \( n \) fixed points on the plane can be solved in finite time using Melzak’s algorithm [1961]. We will utilize these results for the regular tetrahedron since the plane can be viewed as a branched cover of the regular tetrahedron. The Steiner problem on \( \mathcal{T} \) is more complex than on the plane because there are infinitely many geodesics that could connect two points. Thus, the process of solving the Steiner problem on \( \mathcal{T} \) is initially a problem of narrowing down potential path networks.

The algorithm used to solve the 3-point Steiner problem in Euclidean space was developed by Torricelli, Cavalieri, Simpson, Heinen, and Bertrand (see [Hwang et al. 1992]). For convenience, we repeat it here.

Algorithm 2.1. This algorithm provides a shortest network connecting three given points in Euclidean space.

1. Let \( A, B, \) and \( C \) be given. Label \( A, B, \) and \( C \) so that \( m \angle ABC \geq m \angle ACB \) and \( m \angle ABC \geq m \angle BAC \).

2. Determine whether Case 1 or 2 applies.

Case 1. If \( m \angle ABC > 120^\circ \), the Steiner minimal tree is degenerate and it is \( AB \cup BC \). The algorithm is complete (see figure for example).

\[
\text{m} \angle ABC = 121.52^\circ
\]

Case 2. If \( m \angle ABC \leq 120^\circ \), proceed to Steps (3)–(6).
(3) Create an equilateral triangle $\triangle BCE$ where $E$ is on the opposite side of $\overrightarrow{BC}$ from $A$.

(4) Construct $\overline{EA}$. This line segment is called the Simpson line. (The length of the Simpson Line is known to have the same length as the $\text{SMT}(A, B, C)$ [Hwang et al. 1992].)

(5) Next, circumscribe a circle about $\triangle BCE$. The point of intersection of that circle and $\overline{EA}$ is the Steiner point $S$.

(6) Connect each of $A$, $B$, and $C$ to $S$ to form $\text{SMT}(A, B, C)$. By construction, every two edges of the tree which meet at the Steiner point have angle $120^\circ$ [Gilbert and Pollak 1968]. The algorithm is complete.

Another observation relevant to our discussion of the Steiner problem on the regular tetrahedron is that no geodesic passes through the vertices of a narrow cone [Lee et al. 2011]. Since a small neighborhood of a vertex is a narrow cone, no shortest path network will pass through any vertices of $\mathcal{T}$. Hence, a shortest path network can only meet a vertex of $\mathcal{T}$ if a fixed point is placed on that vertex [Ivanov and Tuzhilin 1994, Chapter 9].

3. Tiling the plane

In this section we will show how to construct a branched covering of the plane onto the regular tetrahedron. For further reference, see [Ivanov and Tuzhilin 1994, Chapter 9].

Consider a regular tetrahedron with faces labeled 1, 2, 3, and 4. Cut along the edges common to faces 1 and 2, 1 and 4, and 2 and 4 and lay it on the plane, as
shown in the figure. We will use this configuration to tile the plane.

Notice that face 1 is adjacent to face 2 on the tetrahedron. Thus, in order to
represent that on the plane, we must place a tile corresponding to face 2 so it
becomes adjacent to a tile corresponding to face 1. This is accomplished by placing
a tile corresponding to face 2 that is an $180^\circ$ rotation about a common vertex.
Similarly, we must place a tile corresponding to face 4 so that the tile corresponding
to face 1 and a tile corresponding to face 4 have a common edge in the plane as
they do on the tetrahedron. Since each face on the tetrahedron is adjacent to the
other three faces, then each face should be adjacent to all of the other faces on
the plane. If copies of each face are placed at $180^\circ$ rotations about each of their
respective vertices, this results in a comprehensive tiling of the Euclidean plane.

Points on $T$ will be represented by lower case letters. The corresponding points
in the tiling will be represented by corresponding capital letters. Assume $a$ is on
face 1 on $T$. Then for each tile corresponding to face 1, there is a copy of $A$ on
the tile. Two adjacent tiles contain copies of $A$ which are $180^\circ$ rotations about the
common vertex of the tile containing $A$. A small section of the tiling can be seen
here:

We introduce a coordinate system to notate the different faces of the tiling. In
the tiling, the horizontal lines that separate the triangles will be known as $m_i$, for
$i = \ldots, -2, -1, 0, 1, 2, \ldots$. Similarly define $n_i$ as the lines with the slope equal
to $-\sqrt{3}$. Finally define $p_i$ as the lines with slope $\sqrt{3}$. We thus obtain the following
Using this coordinate system, we can identify individual tiles. For any tile that is bounded by \( m_x, n_y, \) and \( p_z \), we will denote it as \( T_{(x,y,z)} \). Without loss of generality, we will assume that \( T_{(0,0,0)} \) corresponds to face 1, \( T_{(1,-1,0)} \) corresponds to face 2, \( T_{(0,-1,1)} \) corresponds to face 3, and \( T_{(1,0,1)} \) corresponds to face 4. Though each face of the tetrahedron is replicated infinitely many times, each tile in the tiling has a unique labeling according to the lines that bound it.

We now show that this tiling is a branched covering of the plane onto the regular tetrahedron. Let \( \Pi : \mathbb{R}^2 \to \mathcal{T} \) be the natural continuous map that takes each tile of the plane to its corresponding face in \( \mathcal{T} \) homeomorphically. Let \( \mathcal{V} \) be the vertex set of \( \mathcal{T} \). Note that \( \Pi \) is a branched covering map with branch set \( \mathcal{V} \). Then the map

\[
\pi : \mathbb{R}^2 - \Pi^{-1}(\mathcal{V}) \to \mathcal{T} - \mathcal{V}
\]

(which is a restriction of \( \Pi \)) is a covering map of \( \mathcal{T} - \mathcal{V} \). Since \( \pi \) is a covering map, it has the following lifting property: Suppose \( a \in \mathcal{T} - \mathcal{V} \) and \( A \in \Pi^{-1}(a) \). Then any path \( \alpha : [0, 1] \to \mathcal{T} - \mathcal{V} \) so that \( \alpha(0) = a \) has a unique lift to a path \( \tilde{\alpha} : [0, 1] \to \mathbb{R}^2 - \Pi^{-1}(\mathcal{V}) \) with \( \tilde{\alpha}(a) = A \). The map \( \tilde{\alpha} \) is a lift in the sense that \( \pi \circ \tilde{\alpha} = \alpha \). It follows that any embedded path network in \( \mathcal{T} - \mathcal{V} \) containing \( a \) can be uniquely lifted to a path network containing \( A \).

Note that in the case that \( a \in \mathcal{V} \) and \( \Pi(A) = a \), for any embedded path network containing \( a \) in \( \mathcal{T} \), there are two lifts of the path network containing \( A \). These lifts are \( 180^\circ \) rotations of each other about \( A \).

4. The two point problem

This section will briefly describe an algorithm used to construct a shortest path between any two points on a regular tetrahedron. For further details on this process, refer to [Brune and Sipe 2009]. The algorithm detailed here will depend heavily on the following basic geometric property:
Property 4.1. Given any two points $A$ and $B$ on the plane, construct the perpendicular bisector of $AB$ and call it $P_{AB}$. If $X$ is on the $A$ side of $P_{AB}$, then $X$ is closer to $A$. If $X$ is on the $B$ side of $P_{AB}$, then $X$ is closer to $B$.

Definition 4.2. Given two points $P$ and $Q$ on the plane, define $\tilde{H}_{PQ}$ to be the half-plane cut by the perpendicular bisector of $P$ and $Q$ on the $P$ side; that is,

$$\tilde{H}_{PQ} = \{X | PX \leq QX\}.$$ 

The algorithm: a brief synopsis. Suppose there are two points $p$ and $q$ on distinct faces of the tetrahedron. Suppose $\mathbb{R}^2$ is tiled as in Section 3. Recall that $\Pi : \mathbb{R}^2 \to \mathcal{T}$ is the covering map and $\mathbb{R}^2$ is tiled as in Section 3. Then $\Pi^{-1}(p)$ and $\Pi^{-1}(q)$ contain infinitely many points. Let $P \in \Pi^{-1}(p)$. We want to find a point $Q \in \Pi^{-1}(q)$ that realizes a shortest path from $p$ to $q$. The points of $\Pi^{-1}(q)$ that could realize a shortest path to $P$ can be restricted to a star-shaped region. The region consists of an interior hexagon which contains the point $P$, outlined by six tiles which contains points of $\Pi^{-1}(q)$. This region is called an $i$-star for $i = 1, 2, 3,$ or 4, where $i$ is the face of the tetrahedron containing $q$. We illustrate a 4-star when $p$ is on face 1 and $q$ is on face 4:

It was proved in [Brune and Sipe 2009] that this $i$-star always contains a shortest path between two points.
There is a cutting technique that has been shown to reduce the number of possible points of \( \Pi^{-1}(q) \) that could realize a shortest path. Begin by constructing the line segment from point \( P \) to the point \( P' \in \Pi^{-1}(p) \), also located within the 4-star. Then, construct the perpendicular bisector of \( PP' \), denoted \( PP' \) (see Figure 2, left). Every point of \( \Pi^{-1}(q) \) within the star that falls on the same side of \( l \) as \( P \) will now be the only copies of \( \Pi^{-1}(q) \) considered for the shortest path. The portion of the star-shaped region which is on the \( P \) side of \( PP' \) is called \( \tau \) (see Figure 2, right).

There are three points of \( \Pi^{-1}(q) \) in \( \tau \) which we will label as \( Q_1, Q_2, \) and \( Q_3 \), as shown in Figure 2, right. (If \( PP' \) contains a point of \( \Pi^{-1}(q) \) in \( \tau \), then it contains another point of \( \Pi^{-1}(q) \) and either point in \( \Pi^{-1}(q) \) in \( \tau \) can be discarded.) To find \( \min\{P Q_i\} \) where \( i = 1, 2, 3 \), we construct \( \tilde{H}_{Q_i, Q_j} \) for \( i = 1, 2, 3 \) and \( j \neq i \).

Note that the boundary of \( \tilde{H}_{Q_i, Q_j} \) is \( P Q_i Q_j \). If \( Q_i \) is closest to \( P \), then \( P \) must lie in \( \tilde{H}_{Q_i, Q_j} \cap \tilde{H}_{Q_i, Q_k} \). Note that if \( P \) is equally close to \( Q_i \) and \( Q_j \), then \( P \) lies in both \( \tilde{H}_{Q_i, Q_j} \cap \tilde{H}_{Q_i, Q_k} \) and \( \tilde{H}_{Q_j, Q_i} \cap \tilde{H}_{Q_j, Q_k} \). In the figure below, a shortest path is realized by \( \tilde{P} \tilde{Q}_3 \). Hence, \( P \) lies in \( \tilde{H}_{Q_3, Q_1} \cap \tilde{H}_{Q_3, Q_2} \). In particular, \( \Pi(\tilde{P} \tilde{Q}_3) \) is the minimal geodesic connecting \( p \) and \( q \) and will traverse faces 1, 2, and 4.
5. Overview

Suppose \( \{x, y, z\} \in \mathcal{T} \). Recall that \( \Pi \) is the branched covering map described in Section 3. Thus \( \Pi^{-1}(x), \Pi^{-1}(y), \) and \( \Pi^{-1}(z) \) contain infinitely many points. Hence, there are also infinitely many distinct Steiner trees connecting points \( x, y \) and \( z \). Our goal in this paper is to narrow down the number of combinations in the tiled plane which may realize the solution.

As stated earlier, we will divide our discussion of this problem into three cases:

**Case 1:** Three points that can be considered to be on one face of \( \mathcal{T} \).

**Case 2:** Three points that can be considered to be on three distinct faces of \( \mathcal{T} \).

**Case 3:** Any configuration of three points that does not fit into the first two cases (i.e., three points that can only be considered to be on two distinct faces).

Section 6 will address the simplest case where all three points are on a common face of the tetrahedron. Section 7 will introduce the strategies needed for Sections 8 and 9. In Section 8, we will discuss case 2, and in Section 9 we will discuss case 3. We will discuss how to solve the problem for any specific positioning of the points in Section 10.

6. Case 1: Three points on one face

We know by a theorem proved in [Brune and Sipe 2009] that a shortest path network connecting \( n \) points contained on the same face of a regular tetrahedron is contained within that face. Thus, the Steiner minimal tree for three points on the same face of a tetrahedron can be constructed in that face using the algorithm described in Algorithm 2.1.

7. Geometric properties of Steiner minimal trees

Given \( a, b, c \in \mathcal{T} \) and the corresponding point sets on the tiled plane, there are many ways that points can be selected, each corresponding to a Steiner tree on \( \mathcal{T} \). However, only certain of the combinations realize the Steiner minimal tree on the tetrahedron. The next several results represent strategies that help eliminate fruitless combinations. At this point the reader is encouraged to reread Property 4.1, describing the situation illustrated here:
Lemma 7.1 (perpendicular bisector rule I). Suppose $A, A' \in \Pi^{-1}(a)$ such that $A$ is on tile $T$ and $A'$ is on tile $T'$. Then for any point $B$ on $T$, $AB \leq A'B$. If neither $A$ nor $B$ are a common vertex of $T$ and $T'$, then $AB < A'B$.

Proof. Let $b = \Pi(B)$. Note that $a$ and $b$ are on the same face. We know from a theorem proved in [Brune and Sipe 2009] that a shortest path network connecting $n$ points on the same face is in that same face and here is $ab$, which is realized by $AB$ in $T$. Since $AB$ is a minimum of all paths $A'B$ where $A' \in \Pi^{-1}(a)$, then for all $A' \neq A$, $AB \leq A'B$. If $A$ is not a common vertex of $T$ and $T'$, then $A \neq A'$, so $P_{AA'}$ is defined. If $B$ is not a common vertex of $T$, then $B \in P_{AA'}$. Thus $AB < A'B$. □

Next, let $A$, $B$, and $C$ be points in the tiled plane such that

$$
\Pi(\text{SMT}(A, B, C)) = \text{SMT}(a, b, c).
$$

We will show that the convex hull of the triangular region formed from $A$, $B$, and $C$ cannot contain a vertex of the tiled plane unless that vertex is one of $A$, $B$, or $C$. However, before we prove this, we introduce a definition and a property of triangular regions in general.

Definition 7.2. Given two points $X$ and $V$, let $\Gamma_{XV}$ be the line perpendicular to $XV$ through $V$.

Lemma 7.3. Suppose there is a triangular region with vertices $A$, $B$, and $C$ that contains the point $V$ in the interior. Then there is an $X \in \{A, B, C\}$ such that $\Gamma_{XV}$ separates $X$ from $\{A, B, C\} - \{X\}$.

Proof. If $\Gamma_{AV}$ separates $A$ from $BC$, the proof is done (left figure):

Otherwise, one of $B$ or $C$ is on the same side of $\Gamma_{AV}$ as $A$.

Without loss of generality, suppose $B$ is on the same side of $\Gamma_{AV}$ as $A$ (right figure). Then $m \angle AVB \leq 90^\circ$. Then if $\Gamma_{CV}$ separates $C$ from $A$ and $B$, the proof is done.

If not, one of $A$ or $B$ is on the same side of $\Gamma_{CV}$ as $C$. In the former case we have $m \angle CV A \leq 90^\circ$, while in the latter we have $m \angle CV B \leq 90^\circ$. Here is an illustration
of the second possibility:

Thus, either $m \angle AVC + m \angle AVB \leq 180^\circ$ or $m \angle CVB + m \angle AVB \leq 180^\circ$. In either case, we are in contradiction with the hypothesis that $V$ is in the interior of $\triangle ABC$. Thus, there exists an $X \in \{A, B, C\}$ such that $\Gamma_{XV}$ separates $X$ from \{A, B, C\} – {X}. □

**Theorem 7.4** (vertex rule). Suppose $a$, $b$, and $c \in \mathcal{T}$ and

$\Pi(SMT(A, B, C)) = SMT(a, b, c)$.

Then the image of the convex hull of $\triangle ABC$ under $\Pi$ cannot contain a vertex $v$, unless $v$ is one of $a$, $b$, or $c$.

**Proof.** By way of contradiction, suppose a vertex $V$ of the tiling is contained in the interior of the convex hull of $\triangle ABC$. Construct $SMT(A, B, C)$, and label the Steiner point $S$ (the Steiner tree may possibly be degenerate). Using Lemma 7.3, we may assume without loss of generality that $\Gamma_{CV}$ separates $C$ from both $A$ and $B$. Reflect the part of the path on the $C$ side of $\Gamma_{CV}$ across $\Gamma_{CV}$. Let $C'$ be the reflection of $C$ across $\Gamma_{CV}$. Note that the partially reflected path connects $A$, $B$, and $C'$ and is equal in length to $SMT(A, B, C)$. Thus, there is an alternate choice of points in $\Pi^{-1}(a)$, $\Pi^{-1}(b)$, and $\Pi^{-1}(c)$ which is at least as short as $SMT(A, B, C)$. If $S$ is on the opposite side of $\Gamma_{CV}$ as $C$, we can shorten the tree by replacing $SC$ with $SC'$ (see figure on the left). If $S$ is on the same side of $\Gamma_{CV}$ as $C$, we can shorten the tree by replacing $SA$ with $SA'$ and $SB$ with $SB'$, where $A'$ and $B'$ are the reflections of $A$ and $B$ across $\Gamma_{CV}$, respectively. If $S$ is on $\Gamma_{CV}$, then $SC = SC'$, so either tree is the same length. However, the tree containing $A$, $B$,
and $C'$ will no longer meet the $120^\circ$ condition for Steiner trees, and will not be $\text{SMT}(A, B, C')$. Thus $\mathcal{L}(\text{SMT}(A, B, C')) < \mathcal{L}(\text{SMT}(A, B, C))$, which implies that $\Pi(\text{SMT}(A, B, C)) \neq \text{SMT}(a, b, c)$. □

**Theorem 7.5** (perpendicular bisector rule II). *Let $A, A' \in \Pi^{-1}(a)$ on the tiled plane be distinct. If $P_{AA'}$ separates $\{B, C\}$ from $A$, then

$$\mathcal{L}(\text{SMT}(A', B, C)) < \mathcal{L}(\text{SMT}(A, B, C)).$$

Hence, $\Pi(\text{SMT}(A, B, C)) \neq \text{SMT}(a, b, c)$.

**Proof.** Let $\lambda$ be the reflection of the part of $\text{SMT}(A, B, C)$ on the $A$ side of $P_{AA'}$ across $P_{AA'}$:

Note that $\lambda$ uses the point $A'$ as a terminal, thus it is a path network connecting $A', B,$ and $C$. By a similar argument as in Theorem 7.4, we obtain

$$\mathcal{L}(\text{SMT}(A, B, C)) = \mathcal{L}(\lambda) > \mathcal{L}(\text{SMT}(A, B, C')).$$

□

**Sectors and half-planes.**

**Definition 7.6.** Fix a vertex $V$ of the tiled plane, and let $T_1$ and $T_2$ be tiles (not necessarily adjacent to $V$) that are mapped to one another with respect to $180^\circ$ rotation about $V$. Define the sector $S_{T_2 T_1}$ as the intersection of all half-planes $\tilde{H}_{X_2 X_1}$, where $X_1$ runs over all points in $T_1$ and $X_2$ is its image under a $180^\circ$ rotation about $V$. Clearly $\tilde{H}_{X_2 X_1}$ is fully determined by the direction of the vector $VX_1$; thus by considering two extreme cases for this direction, as here:
we conclude that the $S_{T_2T_1}$ is the intersection of the half-planes $\tilde{H}_{X_2X_1}$ obtained in these two cases:

Next, if $Y$ and $Z$ are arbitrary points belonging to tiles $T_1$ and $T_2$, respectively, we set $S_{YZ} = S_{T_2T_1}$.

**Definition 7.7.** Let $T_1, T_2$ be tiles that are translates of each other on the tiled plane, satisfying $\Pi(T_1) = \Pi(T_2)$. Then the intersection of all half-planes $\tilde{H}_{X_2X_1}$ where $X_i \in T_i$ and $\Pi(X_1) = \Pi(X_2)$, is denoted by $H_{T_2T_1}$.

If $Y$ and $Z$ are arbitrary points belonging to tiles $T_1$ and $T_2$, respectively, we set $S_{YZ} = S_{T_2T_1}$.

**Theorem 7.8** (Steiner point rule). Let $A, B, and C$ be points in the tiled plane such that $\Pi(SMT(A, B, C))$ is a Steiner minimal tree on the tetrahedron. Suppose that $S$ is the Steiner point of $SMT(A, B, C)$. If $S'$ is any other point of $\Pi^{-1}(\Pi(s))$, then $XS \leq XS'$ for $X = A, B, and C$.

**Proof.** Without loss of generality, assume that $X = C$. By way of contradiction, suppose $CS' < CS$. Then there exists a point $C' \in \Pi^{-1}(c)$ such that $CS' = C'S$. This implies that

$$\mathcal{L}(SMT(A, B, C)) = AS + BS + CS > AS + BS + C'S \geq \mathcal{L}(SMT(A, B, C')),$$

as needed. (See Figure 3 on next page.)
8. Case 2: Three points on three distinct faces

When the three points can be viewed to lie on three distinct faces, we use the following procedure to determine the possible configurations of the points on the tiled plane which may realize the Steiner minimal tree. Our arguments apply also when the three points can be viewed to lie on two or one face, as may be the case if one or more of the points lie on vertices or edges. For example, if one point is in the interior of a face, another point is in the interior of another face, and the third point is on a vertex shared by both faces, then we can assign the third point to the third face which shares that vertex, and the configuration is in the realm of Case 2.

**Triple ribbon region.** Recall the labeling system introduced in Section 3, in which $m_i$, $n_i$, and $p_i$ represent the horizontal, negative slope, and positive slope lines, respectively. Also recall that the triangle that is bounded by $m_x$, $n_y$, and $p_z$ will be denoted as $T(x, y, z)$.

Let $a$, $b$, and $c$ be points on the tetrahedron such that $s$ is the Steiner point for SMT$(a, b, c)$. Let $\tau_0$ be the shaded region in Figure 4. Since $\tau_0$ contains copies of the tiles corresponding to all four faces, a copy of $S \in \Pi^{-1}(s)$ must lie within $\tau_0$.

![Figure 3. Toward the proof of Theorem 7.8.](image)

![Figure 4. The region $\tau_0$.](image)
Let $S^* = \Pi^{-1}(\Pi(s)) - \{S\}$. We will determine a region $R$ such that given a point $P \in R$, $PS \leq PS'$ for any $S' \in S^*$. It follows from Theorem 7.8 that any points not in $R$ cannot be the fixed points of the Steiner minimal tree that contains $S$ and realizes SMT$(a, b, c)$.

In order to simplify the process, we will first determine the region $R_i$ that contains all points closer to $T_i$ than to any other tile corresponding to face 1. Then $R = \bigcup R_i$. We will call $R = \bigcup R_i$ the triple ribbon region.

**Reductions.** Let $i = 1$. Let $S'$ be the $180^\circ$ rotation of $S$ about the vertex $V = T_1 \cap T_{(2,0,0)}$. Then any point $X \in S_{T_{(2,0,0)}}$ is closer to $S'$ than $S$. Thus no fixed point is in $S_{T_{(2,0,0)}}$:

Likewise, no fixed points will be found in $S_{T_{(0,-2,0)}}$ or $S_{T_{(0,0,2)}}$:

There are also no fixed points to be found in $S_{T_{(2,-2,-2)}}$, $S_{T_{(-2,-2,2)}}$, and $S_{T_{(2,2,2)}}$:
$R_1$ is the closure of the region remaining when the shaded regions in the six figures of the previous page are cut away. It is shown in white here:

Regions $R_2$, $R_3$, and $R_4$ are found similarly. The union of all these regions, $\mathcal{R} = \bigcup_{i=1}^{4} \mathcal{R}_i$, is the triple ribbon region (Figure 5).

**Figure 5.** The triple ribbon region (in white).

Regardless of the location of $s$ on the tetrahedron, a copy of $\Pi^{-1}(\text{SMT}(a, b, c))$ is contained within the triple ribbon region. Thus, it is sufficient to check only the combinations of fixed points in the triple ribbon region.

Although the number of potential path networks needed to be checked to find $\text{SMT}(a, b, c)$ is a finite number, it is still a significant number. Note that there are six tiles meeting the triple ribbon region corresponding to face $i$ for $i = 2, 3, 4$. Thus there are $6 \times 6 \times 6 = 216$ combinations to consider given the specification of points in certain faces of $\tau_0$. Hence, we continue to make further reductions.

**Horn removal.** We subdivide the triple ribbon region as follows. The closure of the bounded white region in Figure 6 (on the next page) is called the **badge region**. The small black triangles, which make up the difference between the triple ribbon region and the badge region, are called the **horns**.
Figure 6. The badge region (closure of the polygon in white) and the horns (in black).

Proposition 8.1. Suppose $a$, $b$, and $c$ are three points on distinct faces of $\mathcal{T}$, none of which are chosen to be face 1. Then there is a copy of

$$\text{SMT}(A, B, C) \in \Pi^{-1}(\text{SMT}(a, b, c))$$

on the tiled plane which is contained in the badge region centered about a tile corresponding to face 1 with Steiner point $S$ contained in the triangular region $\tau_0$ (see Figure 4).

Proof. Without loss of generality, assume that $a$ is contained on face 3, $b$ is contained on face 4, and $c$ is contained on face 2. Let $A \in \Pi^{-1}(a)$, $B \in \Pi^{-1}(b)$, and $C \in \Pi^{-1}(c)$ lie in the triple ribbon region such that $\Pi(\text{SMT}(A, B, C)) = \text{SMT}(a, b, c)$. Note that no portion of the horns contains any points of $\Pi^{-1}(a)$, $\Pi^{-1}(b)$, or $\Pi^{-1}(c)$ and therefore cannot contain $A$, $B$, or $C$. Let $H_1$ be the horn bounded by $m_2$, $n_0$, and $p_{-1}$ that is outside the badge region.

Suppose an edge of $\text{SMT}(A, B, C)$ meets $H_1$ outside the badge region. If the interior of an edge passes through either side of the horn not on $m_2$, the edge must meet the shaded region. But by hypothesis, $\text{SMT}(A, B, C)$ must lie entirely within the triple ribbon region. Thus the edge may only pass through the boundary of the horn on $m_2$. If so, the only possibility is that one of the endpoints of the edges is contained in $H_1$. Thus a fixed point is contained in the interior of the horn, and hence contained in the interior of face 1. But by hypothesis, face 1 was not selected as one of the faces containing fixed points. Therefore, an edge of $\text{SMT}(A, B, C)$ does not meet $H_1$. By a similar argument, $\text{SMT}(A, B, C)$ cannot meet any horn.

Reduction to the piping region. Using Theorem 7.4 and Theorem 7.5, we will now demonstrate that a lift of the Steiner minimal tree can be contained in a subset of the badge region called the piping region (Figure 7). What is left over of the
Figure 7. The piping region (closure of the polygon in white) and the flaps (in black).

The badge region is called the (top) flaps. We will show that if SMT(A, B, C) realizes SMT(a, b, c) and is contained in the badge region, then SMT(A, B, C) does not meet the flaps outside the piping region.

**Theorem 8.2.** Suppose $a$, $b$, and $c$ are three points on distinct faces of $\mathcal{T}$, none of which chosen to be face 1. Suppose SMT(A, B, C) $\in \Pi^{-1}(a, b, c)$ is contained in the badge region. Then SMT(A, B, C) $\in \Pi^{-1}(a, b, c)$ is also contained in the piping region centered about a tile corresponding to face 1.

**Proof.** Assume the setup given in the proof of Proposition 8.1. We will show that the Steiner minimal tree need not meet any of the flaps. By way of contradiction, suppose that SMT(A, B, C) meets the top flap, the flap contained in $T_{(2,1,-1)}$, outside the piping region. If SMT(A, B, C) meets the top flap, then at least one fixed point or vertex of SMT(A, B, C) must lie above $m_2$. Note that by construction, $S$ is contained in $T_0$ and cannot be this point. Since the only tile in the badge region which lies above $m_2$ is a tile corresponding to face 3, the fixed point must lie in the interior of face 3. Thus, $A$ must lie in the top flap outside the piping region. For the remainder of the argument, we will denote $A$ by $A_1$ and label the other copies of $\Pi^{-1}(a)$, $\Pi^{-1}(b)$, and $\Pi^{-1}(c)$ contained in tiles meeting the badge region as shown in the figure on the top of the next page. We will show either that any Steiner tree SMT($A_1, B_i, C_j$) with $S$ in $\tau_0$ contained within the badge region cannot realize SMT($a, b, c$) or that there exists another copy of the tree within the piping region.

We will first determine which combinations cannot realize SMT($a, b, c$). Once those combinations are determined, we will show that the remaining combinations have an equivalent copy contained in the piping region.

Construct the sector $S_{A_2A_1}$. If any points $B_i$ and $C_j$ are both contained in $S_{A_2A_1}$, they must both be separated from $A_1$ by $P_{A_2A_1}$. Thus, by Theorem 7.5, we know that $\Pi(\text{SMT}(A_1, B_i, C_j)) \neq \text{SMT}(a, b, c)$ for $B_i$ and $C_j$ contained in these sectors.
By this argument, the combinations \((B_i, C_j)\), for \(i = 4, 5\) and \(j = 4, 5, 6\), cannot be used with \(A_1\) to realize \(\text{SMT}(a, b, c)\).

Construct the half-plane \(H_{A_3A_1}\). If any points \(B_i\) and \(C_j\) are both contained in \(H_{A_3A_1}\), they must be separated from \(A_1\) by \(P_{A_3A_1}\). Thus, by Theorem 7.5, we know that \(\Pi(\text{SMT}(A_1, B_i, C_j)) \neq \text{SMT}(a, b, c)\) for \(B_i\) and \(C_j\) contained in these sectors.

By this argument, the combinations \((B_i, C_j)\), for \(i = 4, 6\) and \(j = 3, 4, 5, 6\), cannot be used with \(A_1\) to realize \(\text{SMT}(a, b, c)\).

Construct the half-plane \(H_{A_4A_1}\). If any points \(B_i\) and \(C_j\) are both contained in \(H_{A_4A_1}\), they must be separated from \(A_1\) by \(P_{A_4A_1}\). Thus, by Theorem 7.5, we know that \(\Pi(\text{SMT}(A_1, B_i, C_j)) \neq \text{SMT}(a, b, c)\) for \(B_i\) and \(C_j\) contained in these sectors.

By this argument, the combinations \((B_i, C_j)\), for \(i = 3, 4, 5, 6\) and \(j = 4, 6\), cannot be used with \(A_1\) to realize \(\text{SMT}(a, b, c)\).

Consider \(\text{SMT}(A_1, B_1, C_3)\). Note that both \(A_1\) and \(B_1\) must be contained in \(S_{C_1C_3}\). Thus, \(A_1\) and \(B_1\) must be separated from \(C_3\) by \(P_{C_1C_3}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_1, C_3)) \neq \text{SMT}(a, b, c)\).

Consider \(\text{SMT}(A_1, B_1, C_4)\). Note that both \(A_1\) and \(B_1\) must be contained in \(S_{C_2C_4}\). Thus, \(A_1\) and \(B_1\) must be separated from \(C_4\) by \(P_{C_2C_4}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_1, C_4)) \neq \text{SMT}(a, b, c)\).

Consider \(\text{SMT}(A_1, B_1, C_5)\). Note that both \(A_1\) and \(B_1\) must be contained in \(H_{C_1C_5}\). Thus, \(A_1\) and \(B_1\) must be separated from \(C_5\) by \(P_{C_1C_5}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_1, C_5)) \neq \text{SMT}(a, b, c)\).

Consider \(\text{SMT}(A_1, B_1, C_6)\). Note that both \(A_1\) and \(B_1\) must be contained in \(S_{C_1C_6}\). Thus, \(A_1\) and \(B_1\) must be separated from \(C_6\) by \(P_{C_1C_6}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_1, C_6)) \neq \text{SMT}(a, b, c)\).
Consider $\text{SMT}(A_1, B_2, C_2)$. Let $V$ be the intersection of $m_1$ and $n_0$. Note that $V$ and $A_1$ are on the same side of $\overrightarrow{B_2C_2}$, $V$ and $B_2$ are on the same side of $\overrightarrow{A_1C_2}$, and $V$ and $C_2$ are on the same side of $\overrightarrow{A_1B_2}$. Thus $V$ is contained in $\triangle A_1B_2C_2$.

By Theorem 7.4, $\Pi(\text{SMT}(a, b, c)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_1, B_2, C_3)$. Note that $C_3$ lies in $H_{A_2A_1}$ and that $A_1$ lies in $S_{C_1C_3}$. $B_2$ must lie in at least one of $S_{A_2A_1}$ and $H_{C_1C_3}$. Suppose $B_2$ lies in $S_{A_2A_1}$. Then both $B_2$ and $C_3$ must be separated from $A_1$ by $P_{A_2A_1}$. If $B_2$ does not lie in $S_{A_2A_1}$, then $B_2$ must lie in $H_{C_1C_3}$. But then both $B_2$ and $A_1$ must be separated from $C_3$ by $P_{C_1C_3}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_2, C_3)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_1, B_2, C_4)$. Note that $C_4$ lies in $H_{A_3A_1}$ and that $A_1$ lies in $S_{C_2C_4}$. $B_2$ must lie in at least one of $H_{A_3A_1}$ and $S_{C_2C_4}$. Suppose $B_2$ lies in $H_{A_3A_1}$. Then both $B_2$ and $C_4$ must be separated from $A_1$ by $P_{A_3A_1}$. If $B_2$ does not lie in $H_{A_3A_1}$, then $B_2$ must lie in $S_{C_2C_4}$. But then both $B_2$ and $A_1$ must be separated from $C_4$ by $P_{C_1C_4}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_2, C_4)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_1, B_2, C_5)$. Note that $C_5$ lies in $H_{A_3A_1}$ and that $A_1$ lies in $H_{C_1C_5}$. $B_2$ must lie in at least one of $H_{A_3A_1}$ and $H_{C_1C_5}$. Suppose $B_2$ lies in $H_{A_3A_1}$. Then both $B_2$ and $C_5$ must be separated from $A_1$ by $P_{A_3A_1}$. If $B_2$ does not lie in $H_{A_3A_1}$, then $B_2$ must lie in $H_{C_1C_5}$. But then both $B_2$ and $A_1$ must be separated from $C_5$ by $P_{C_1C_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_2, C_5)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_1, B_2, C_6)$. Note that both $A_1$ and $B_2$ must be contained in $S_{C_1C_6}$. Thus, $A_1$ and $B_2$ must be separated from $C_6$ by $P_{C_1C_6}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_2, C_6)) \neq \text{SMT}(a, b, c)$.

We now consider the combinations $(A_1, B_i, C_j)$ for $i = 4, 5, 6$ and $j = 1, 2$. By arguments of symmetry, $\Pi(\text{SMT}(A_1, B_i, C_j)) \neq \text{SMT}(a, b, c)$ for $i = 4, 5, 6$ and $j = 1, 2$.

Consider $\text{SMT}(A_1, B_3, C_3)$. Let $V$ be the intersection of $m_1$ and $n_0$. Note that $V$ and $A_1$ are on the same side of $\overrightarrow{B_3C_3}$, $V$ and $B_3$ are on the same side of $\overrightarrow{A_1C_3}$, and $V$ and $C_3$ are on the same side of $\overrightarrow{A_1B_3}$. Thus $V$ is contained in $\triangle A_1B_3C_3$.

By Theorem 7.4, $\Pi(\text{SMT}(A_1, B_1, C_1)) \neq \text{SMT}(a, b, c)$.

The only remaining cases are $(A_1, B_1, C_1)$, $(A_1, B_1, C_2)$, and $(A_1, B_2, C_1)$. We will show that copies of these three trees exist within the piping region. However, we will not claim that the Steiner point $S$ must remain in $\tau_0$.

For $(A_1, B_1, C_1)$, note that $\Pi(\text{SMT}(A_1, B_1, C_1)) = \Pi(\text{SMT}(A_2, B_2, C_2))$ since $\text{SMT}(A_2, B_2, C_2)$ is a rotation of $\text{SMT}(A_1, B_1, C_1)$ about $V$. $\text{SMT}(A_2, B_2, C_2)$ is contained within the piping region.

For $(A_1, B_1, C_2)$, note that $\Pi(\text{SMT}(A_1, B_1, C_2)) = \Pi(\text{SMT}(A_2, B_2, C_1))$ since $\text{SMT}(A_2, B_2, C_1)$ is a rotation of $\text{SMT}(A_1, B_1, C_2)$ about $V$. $\text{SMT}(A_2, B_2, C_1)$ is contained within the piping region.

For $(A_1, B_2, C_1)$, note that $\Pi(\text{SMT}(A_1, B_2, C_1)) = \Pi(\text{SMT}(A_2, B_1, C_2))$ since $\text{SMT}(A_2, B_1, C_2)$ is a rotation of $\text{SMT}(A_1, B_2, C_1)$ about $V$. Also, $\text{SMT}(A_2, B_1, C_2)$
is contained within the piping region.

Thus, each possible combination \((A_1, B_i, C_j)\) does not realize \(\text{SMT}(a, b, c)\) or has a copy within the piping region. Likewise, each possible combination involving \(B_5\) or \(C_5\) does not realize \(\text{SMT}(a, b, c)\) or has a copy within the piping region. Therefore, there is a solution contained in the piping region.

\[\square\]

The region resulting from Theorem 8.2 is the piping region, which we illustrated in Figure 7.

**Reduction to the truncated triangle region.** We further subdivide the piping region into the **truncated triangle region** and the **side flaps** (Figure 8).

\[\text{Figure 8. The truncated triangle region (closure of white polygon) and the side flaps (in black).}\]

**Theorem 8.3.** Suppose \(a, b,\) and \(c\) are three points on distinct faces of \(\mathcal{T}\), none of which are in the interior of face 1. Suppose \(\text{SMT}(A, B, C) \in \Pi^{-1}(a, b, c)\) is contained in the piping region. Then either \(\text{SMT}(A, B, C) \in \Pi^{-1}(a, b, c)\) is also contained in the truncated triangle region centered about a tile corresponding to face 1 or there is a copy of \(\text{SMT}(A, B, C)\) contained within the truncated triangle region that is a rotation of \(\text{SMT}(A, B, C)\).

**Proof.** Assume the setup in the proof of Proposition 8.1. Without loss of generality, suppose that \(\text{SMT}(A, B, C)\) is in the piping region. We will show that the Steiner minimal tree need not meet any of the side flaps. Although the final cases of the proof of Theorem 8.2 did not guarantee that \(S\) was contained in \(\tau_0\), \(S\) must be contained in the truncated triangle region. This is because all the trees which could be rotated to lie within the piping region contained fixed points contained within the truncated triangle region. Because \(S\) must be contained in the convex hull of
the triangular region formed from the fixed points, $S$ must be contained within the truncated triangle region.

By way of contradiction, suppose the Steiner minimal tree meets the flap contained in $T_{(2,-1,-1)}$. If $SMT(A, B, C)$ meets this side flap, then at least one fixed point or vertex of $SMT(A, B, C)$ must lie above to the left of $p_{-1}$ and above $m_1$. Since $S$ is contained in the truncated triangle region (Figure 8), $S$ cannot be this point. Since the only tile in the piping region which lies to the left of $p_{-1}$ and above $m_1$ is a tile corresponding to face 3, the fixed point must lie in the interior of face 3. Thus, $A$ must lie in the specified side flap outside the truncated triangle region. For the remainder of the proof we will denote $A$ by $A_1$ and number the other points within the piping region as follows:

![Diagram](image-url)

Construct the sector $S_{A_2A_1}$. If any points $B_i$ and $C_j$ are both contained in $S_{A_2A_1}$, they must both be separated from $A_1$ by $P_{A_2A_1}$. Thus, by Theorem 7.5, we know that $\Pi(SMT(A_1, B_i, C_j)) \neq SMT(a, b, c)$ for $B_i$ and $C_j$ contained in these sectors. By this argument, the combinations $(B_i, C_j)$, for $i = 4, 5$ and $j = 2, 4, 5$, cannot be used with $A_1$ to realize $SMT(a, b, c)$.

Construct the half-plane $H_{A_3A_1}$. If any points $B_i$ and $C_j$ are both contained in $H_{A_3A_1}$, they must both be separated from $A_1$ by $P_{A_3A_1}$. Thus, by Theorem 7.5, we know that $\Pi(SMT(A_1, B_i, C_j)) \neq SMT(a, b, c)$ for $B_i$ and $C_j$ contained in these sectors. By this argument, the combinations $(B_i, C_j)$, for $i = 2, 4, 5$ and $j = 4, 5$, cannot be used with $A_1$ to realize $SMT(a, b, c)$.

Construct the sector $S_{A_4A_1}$. If any points $B_i$ and $C_j$ are both contained in $S_{A_4A_1}$, they must both be separated from $A_1$ by $P_{A_4A_1}$. Thus, by Theorem 7.5, we know that $\Pi(SMT(A_1, B_i, C_j)) \neq SMT(a, b, c)$ for $B_i$ and $C_j$ contained in these sectors.
By this argument, the combinations \((B_i, C_j)\), for \(i = 2, 5\) and \(j = 3, 4\), cannot be used with \(A_1\) to realize \(\text{SMT}(a, b, c)\).

For \(\text{SMT}(A_1, B_1, C_2)\), note that \(A_1\) and \(B_1\) are contained in \(S_{C_1C_2}\), so they are both separated from \(C_2\) by \(P_{C_1C_2}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_1, C_2)) \neq \text{SMT}(a, b, c)\).

For \(\text{SMT}(A_1, B_1, C_4)\), note that \(A_1\) and \(B_1\) are contained in \(S_{C_1C_4}\), so they are both separated from \(C_4\) by \(P_{C_1C_4}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_1, C_4)) \neq \text{SMT}(a, b, c)\).

For \(\text{SMT}(A_1, B_1, C_5)\), note that \(A_1\) and \(B_1\) are contained in \(S_{C_1C_5}\), so they are both separated from \(C_5\) by \(P_{C_1C_5}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_1, C_5)) \neq \text{SMT}(a, b, c)\).

Consider \(\text{SMT}(A_1, B_3, C_2)\). Let \(V_1\) be the intersection of \(m_1\) and \(n_{-1}\). Note that \(A_1\) and \(V_1\) are on the same side of \(\overrightarrow{B_3C_2}, B_3\) and \(V\) are on the same side of \(\overrightarrow{A_1C_2}\), and \(C_2\) and \(V\) are on the same side of \(\overrightarrow{A_1B_3}\). Thus, \(V_1\) is contained in \(\triangle ABC\). By Theorem 7.4, \(\Pi(\text{SMT}(A_1, B_3, C_2)) \neq \text{SMT}(a, b, c)\).

Consider \(\text{SMT}(A_1, B_3, C_4)\). Note that \(A_1\) lies in \(S_{C_2C_4}\) and \(C_4\) lies in \(S_{A_2A_1}\). Note that \(B_3\) must lie in at least one of \(S_{C_2C_4}\) and \(S_{A_2A_1}\). If \(B_3\) lies in \(S_{C_2C_4}\), both \(B_3\) and \(A_1\) must be separated from \(C_4\) by \(P_{C_2C_4}\). If \(B_3\) lies in \(S_{A_2A_1}\), both \(B_3\) and \(C_4\) must be separated from \(A_1\) by \(P_{A_2A_1}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_3, C_4)) \neq \text{SMT}(a, b, c)\).

Consider \(\text{SMT}(A_1, B_3, C_5)\). Note that both \(A_1\) and \(B_3\) lie in \(S_{C_1C_5}\), so they are both separated from \(C_5\) by \(P_{C_1C_5}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_3, C_5)) \neq \text{SMT}(a, b, c)\).

Consider \(\text{SMT}(A_1, B_2, C_1)\). Note that both \(A_1\) and \(C_1\) lie in \(S_{B_1B_2}\), so they are both separated from \(B_2\) by \(P_{B_1B_2}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_2, C_1)) \neq \text{SMT}(a, b, c)\).

Consider \(\text{SMT}(A_1, B_2, C_2)\). Note that both \(A_1\) and \(C_2\) lie in \(S_{B_1B_2}\), so they are both separated from \(B_2\) by \(P_{B_1B_2}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_2, C_2)) \neq \text{SMT}(a, b, c)\).

Consider \(\text{SMT}(A_1, B_4, C_1)\). Note that both \(A_1\) and \(C_1\) lie in \(S_{B_3B_4}\), so they are both separated from \(B_4\) by \(P_{B_3B_4}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_4, C_1)) \neq \text{SMT}(a, b, c)\).

Consider \(\text{SMT}(A_1, B_4, C_3)\). Note that \(A_1\) lies in \(S_{B_3B_4}\) and \(B_4\) lies in \(H_{A_1A_3}\). Note that \(C_3\) must lie in at least one of \(S_{B_3B_4}\) and \(H_{A_1A_3}\). If \(C_3\) lies in \(S_{B_3B_4}\), both \(A_1\) and \(C_3\) are separated from \(B_4\) by \(P_{B_3B_4}\). If \(C_3\) lies in \(H_{A_1A_3}\), both \(B_4\) and \(C_3\) are separated from \(A_1\) by \(P_{A_3A_1}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_4, C_3)) \neq \text{SMT}(a, b, c)\).

Consider \(\text{SMT}(A_1, B_5, C_1)\). Note that both \(A_1\) and \(C_1\) lie in \(S_{B_5B_4}\), so they are both separated from \(B_5\) by \(P_{B_5B_4}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_5, C_1)) \neq \text{SMT}(a, b, c)\).
The only remaining cases are \((A_1, B_1, C_1), (A_1, B_1, C_3),\) and \((A_1, B_3, C_3).\) We will show that copies of these trees exist within the truncated triangle region.

For SMT\((A_1, B_1, C_1),\) we have \(5(\text{SMT}(A_1, B_1, C_1)) = 5(\text{SMT}(A_4, B_3, C_3))\) and SMT\((A_4, B_3, C_3)\) is contained within the truncated triangle region.

For SMT\((A_1, B_1, C_3),\) we have \(5(\text{SMT}(A_1, B_1, C_3)) = 5(\text{SMT}(A_4, B_1, C_1))\) and SMT\((A_4, B_1, C_1)\) is contained within the truncated triangle region.

For \((A_1, B_3, C_3),\) we have \(5(\text{SMT}(A_1, B_3, C_3)) = 5(\text{SMT}(A_4, B_1, C_1))\) and SMT\((A_4, B_1, C_1)\) is contained within the truncated triangle region.

Thus, each possible combination \((A_1, B_i, C_j)\) does not realize SMT\((a, b, c)\) or has a copy within the truncated triangle region. Likewise, each possible combination involving \(A_5, B_2, B_5, C_5,\) or \(C_2\) cannot realize SMT\((a, b, c)\) or has a copy within the truncated triangle region. Therefore, there is a solution contained in the truncated triangle region. \(\square\)

**Final reductions.** Within the truncated triangle region, there are three copies of every face that contains a terminal point (the center of each region does not contain any points; in this scenario, face 1). That means that there are three copies of each point:

If all combinations of three points were considered possible configurations for the Steiner minimal tree, there would be 27 different Steiner trees that could be considered. However, some of these possibilities may still be eliminated.

There are three remaining combinations that can be eliminated within the truncated triangle region. Let \(V_1\) be the intersection of \(m_0\) and \(n_{-2},\) \(V_2\) be the intersection of \(m_0\) and \(n_0,\) and \(V_3\) be the intersection of \(m_1\) and \(n_0.\)
Consider SMT\((A_1, B_3, C_2)\). Since \(A_1\) and \(V_1\) are on the same side of \(\overrightarrow{B_3C_2}\), \(B_3\) and \(V_1\) are on the same side of \(\overrightarrow{A_1C_2}\), and \(C_2\) and \(V_1\) are on the same \(\overrightarrow{A_1B_3}\), then \(V_1\) is contained in the interior of \(\triangle A_1B_3C_2\). By Theorem 7.4, \(\Pi(\text{SMT}(A_1, B_3, C_2)) \neq \text{SMT}(a, b, c)\).

Consider SMT\((A_2, B_1, C_3)\). Since \(A_2\) and \(V_2\) are on the same side of \(\overrightarrow{B_1C_3}\), \(B_1\) and \(V_2\) are on the same side of \(\overrightarrow{A_2C_3}\), and \(C_3\) and \(V_2\) are on the same \(\overrightarrow{A_2B_1}\), then \(V_2\) is contained in the interior of \(\triangle A_2B_1C_3\). By Theorem 7.4, \(\Pi(\text{SMT}(A_2, B_1, C_3)) \neq \text{SMT}(a, b, c)\).

Consider SMT\((A_3, B_2, C_1)\). Since \(A_3\) and \(V_3\) are on the same side of \(\overrightarrow{B_2C_1}\), \(B_2\) and \(V_3\) are on the same side of \(\overrightarrow{A_3C_1}\), and \(C_1\) and \(V_3\) are on the same \(\overrightarrow{A_3B_2}\), then \(V_3\) is contained in the interior of \(\triangle A_3B_2C_1\). By Theorem 7.4, \(\Pi(\text{SMT}(A_3, B_2, C_1)) \neq \text{SMT}(a, b, c)\).

**List of potential combinations in case 2.** The remaining possibilities are

\[
(A_1, B_1, C_1), \ (A_2, B_1, C_1), \ (A_3, B_1, C_1), \\
(A_1, B_1, C_2), \ (A_2, B_1, C_2), \ (A_3, B_1, C_2), \\
(A_1, B_1, C_3), \ (A_2, B_1, C_3), \ (A_3, B_1, C_3), \\
(A_1, B_2, C_1), \ (A_2, B_2, C_1), \ (A_3, B_2, C_1), \\
(A_1, B_2, C_2), \ (A_2, B_2, C_2), \ (A_3, B_2, C_2), \\
(A_1, B_2, C_3), \ (A_2, B_2, C_3), \ (A_3, B_2, C_3), \\
(A_1, B_3, C_1), \ (A_2, B_3, C_1), \ (A_3, B_3, C_1), \\
(A_1, B_3, C_2), \ (A_2, B_3, C_2), \ (A_3, B_3, C_2), \\
(A_1, B_3, C_3), \ (A_2, B_3, C_3), \ (A_3, B_3, C_3).
\]

Thus, the Steiner tree which realizes SMT\((a, b, c)\) will be formed from one of the 24 combinations in this list.

**9. Case 3: Three points on two faces**

In this section, we consider the cases that haven’t been addressed in the other sections, namely where three points lie on two faces and cannot be considered to lie on three faces or a single face. The two remaining possibilities are:

1. Two of the points are contained in the interior of one face with the third point anywhere not meeting that face.
2. One point is contained in the interior of a face \(f\), a second point is contained in the interior of an edge adjacent to \(f\), and the final point is in the complement of \(f\).

The arguments for both are the same.

We will assume \(a\) and \(b\) are on the same face and that at least \(a\) is in the interior of the face. Thus either \(b\) is in the interior of the face or in the interior of an edge of the face.
On the tiled plane, there are infinitely many copies of $A \in \Pi^{-1}(a)$ and $B \in \Pi^{-1}(b)$. Suppose $\text{SMT}(A, B, C)$ realizes $\text{SMT}(a, b, c)$. Then either $A$ and $B$ reside on the same tile, or they don’t. We will discuss each case separately. We will discuss the former case here and the latter starting on page 392.

**A and B on the same tile.** In this case, the following theorem provides a region containing the fixed points that can realize $\text{SMT}(a, b, c)$:

**Theorem 9.1.** Let $a, b, c \in \mathcal{F}$ and assume

$$A \in \Pi^{-1}(a), \quad B \in \Pi^{-1}(b), \quad C \in \Pi^{-1}(c)$$

are the points that determine $\text{SMT}(a, b, c)$. If $A$ and $B$ are on the same tile, the Steiner minimal tree must be contained in the ten-triangle region shown here in white and light gray:

![Diagram](image.png)

**Proof.** We can assume without loss of generality that suppose $c$ is on face 1, while $a$ and $b$ are on face 3. We suppose that $C$ is contained in the light gray tile in the figure above.

**Case 1:** Suppose $C$ is not on a vertex of a tile. The other copies of $C_i \in \Pi^{-1}(c)$ are located on the other tiles corresponding to face 1. We number them as in the figure above. We will now determine the tiles on which $A$ and $B$ could possibly reside.

Construct $S_{C_i, C}$. The points $A_i$ and $B_j$ which lie in $S_{C_i, C}$ must be separated from $C$ by $P_{C_i, C}$. By Theorem 7.5, $\text{SMT}(A_i, B_j, C)$ cannot realize $\text{SMT}(a, b, c)$. Thus, we can eliminate from consideration as a candidate for containing $A$ and $B$ any
tiles whose interior overlaps the region $S_{C_1 C}$, which we show in dark gray (left):

Construct $S_{C_2 C}$. Again, using Theorem 7.5, we can eliminate any tile contained in $S_{C_2 C}$, which is the reason shown in dark gray in the figure above and to the right.

Continue the process by constructing the sectors $S_{C_i C}$, where $i = 3, \ldots, 9$. Three of these are shown below, while the other four are obtainable by reflection in a vertical line (through the central triangle) from others already illustrated: $S_{C_3 C}$ from $S_{C_1 C}$, $S_{C_4 C}$ from $S_{C_2 C}$, $S_{C_5 C}$ from $S_{C_3 C}$, and $S_{C_6 C}$ from $S_{C_4 C}$.

The only copies of tile 3 not completely covered by the union of the shaded regions are those contained in the white region in the statement of Theorem 9.1.
By hypothesis, $A$ and $B$ are contained on the same tile. The convex hull of $\triangle ABC$ contains the tree realizing $\text{SMT}(a, b, c)$. The white region is the minimal collection of tiles containing all such possible convex hulls. Since there are five tiles containing copies of $A$ and $B$ in this region, there are five potential Steiner trees which must be tested within this region.

Case 2: Suppose $C$ is a vertex of a tile. It can only be the vertex at the intersection of $m_1$ and $n_0$, because the other vertices are adjacent to tiles containing $A$ and $B$, and this case has already been addressed in Section 6.

Construct $\tilde{H}_{C_i}$ for $i = 1, 3, 4, 5, 6, 7$. The union of the added “union of the” shaded regions $\tilde{H}_{C_i}$ is shown here:

If both $A_j$ and $B_k$ lie in any $\tilde{H}_{C_i}$, they must both be separated from $C$ by $P_{C_i}$. By Theorem 7.5, $A_j$ and $B_j$ cannot be used with $C$ to realize $\text{SMT}(a, b, c)$. Note that at least one of $A$ and $B$ must lie in the unshaded region, and $A$ and $B$ are on the same tile by hypothesis. Thus, there are six possible path networks that connect $C$ with a pair of points $A_j$ and $B_k$ which are contained on the same tile where at least one is not in the shaded region. Since each path has one identical path by reflection across $C$, there are only three distinct paths, and there exists a copy of each in the region stated in Theorem 9.1.

A and $B$ not on the same tile. We now study the case that $A$ and $B$ are not on the same tile. This will occupy us through page 399. We will determine the faces where the Steiner point can reside in Theorem 9.2. We will then find the region that must contains the fixed points. We will eliminate possibilities for fixed points in Theorems 9.3–9.6. We will then make final reductions and list the combinations that could realize $\text{SMT}(a, b, c)$.

Theorem 9.2. Assume the setup in Theorem 9.1. Suppose that $s$ is the Steiner point for $\text{SMT}(a, b, c)$. If $A$ and $B$ are not found on the same tile, then $s$ can not be on the face containing $a$ and $b$, including the interior of its edges.
Proof. By way of contradiction, suppose \( s \) is on the same face as \( a \) and \( b \). Suppose \( S \in \Pi^{-1}(s) \) is contained in the region bounded by \( n_{-1}, m_1, \) and \( p_1 \). Without loss of generality, \( c \) is on face 4, and \( a \) and \( b \) are on face 3.

**Case 1:** Suppose \( S \) is not on the same tile as \( B \). Then there exists a distinct point \( S' \in \Pi^{-1}(s) \) on the same tile as \( B \). By Lemma 7.1, \( S' B < S B \). Then \( P_{SS'} \) separates \( B \) from \( S \). By Theorem 7.8, \( S \) cannot be the Steiner point.

**Case 2:** Suppose \( S \) is on the same tile as \( B \), but \( S \) is not a vertex. Then there exists an \( S' \in \Pi^{-1}(s) \) on the same tile as \( A \). Since \( S \) is not a vertex, \( S' \neq S \). Then \( P_{SS'} \) separates \( A \) from \( S \). By Theorem 7.8, \( S \) cannot be the Steiner point.

It follows from Theorem 9.2 that \( s \) must be contained on at least one of faces 1, 2, or 4. Since \( S \) cannot be on any tile corresponding to face 3, we can fix \( S \) in the shaded region bounded by \( m_1, m_0, n_{-1}, \) and \( p_1 \), which we call the key trapezoid:

![Key Trapezoid Diagram](image)

By a similar procedure to that discussed on pages 378 and following, we can eliminate all points lying in the sectors \( S_{SS} \) or half-planes \( H_{SS} \) for all \( S' \neq S \), where \( S' \in \Pi^{-1}(s) \). The resulting region is this:

![Resulting Region Diagram](image)
Because no terminals are located on faces 1 or 2, the Steiner tree will never cross the copies of face 1 or 2 whose interior meets the edge of this region. We can eliminate these to obtain the region shaded in the figure below. Within this region, there are a maximum of four copies of A, four copies of B, and five copies of C, resulting in a maximum of 80 possible Steiner trees. However, we can reduce the region even further.

**Theorem 9.3.** Suppose S is contained in the key trapezoid (page 393). Let $C_1$ and $A_i, B_j$ (with $i, j = 1, \ldots, 5$) lie in the triangles specified in this diagram:

Then $\Pi(\text{SMT}(A_i, B_j, C_1)) \neq \text{SMT}(a, b, c)$ for all $i, j$ with $i \neq j$. Hence, the tile containing $C_1$ can be removed from the region of interest.

**Proof.** The last assertion follows immediately once we show that no combination $(A_i, B_j, C_1)$ which can be used to realize $\text{SMT}(a, b, c)$. We analyze each case:

Consider $\text{SMT}(A_1, B_2, C_1)$. Both $B_2$ and $C_1$ are contained in $S_{A_2A_1}$. Thus $B_2$ and $C_1$ are separated from $A_1$ by $P_{A_2A_1}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_2, C_1)) \neq \text{SMT}(a, b, c)$. Similarly, $\Pi(\text{SMT}(A_2, B_1, C_1)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_1, B_3, C_1)$. Both $B_3$ and $C_1$ are contained in $H_{A_3A_1}$. Thus $B_3$ and $C_1$ are separated from $A_1$ by $P_{A_3A_1}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_3, C_1)) \neq \text{SMT}(a, b, c)$. Similarly, $\Pi(\text{SMT}(A_3, B_1, C_1)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_1, B_4, C_1)$. Both $B_4$ and $C_1$ are contained in $H_{A_4A_1}$. Thus $B_4$ and $C_1$ are separated from $A_1$ by $P_{A_4A_1}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_4, C_1)) \neq \text{SMT}(a, b, c)$. Similarly, $\Pi(\text{SMT}(A_4, B_1, C_1)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_1, B_5, C_1)$. Both $A_1$ and $B_5$ are contained in $S_{C_5C_1}$. Thus $A_1$ and $B_5$ are separated from $C_1$ by $P_{C_5C_1}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_5, C_1)) \neq \text{SMT}(a, b, c)$. Similarly, $\Pi(\text{SMT}(A_5, B_1, C_1)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_2, B_3, C_1)$. Both $B_3$ and $C_1$ are contained in $S_{A_3A_2}$. Thus $B_3$ and $C_1$ are separated from $A_1$ by $P_{A_3A_2}$. By Theorem 7.5, $\Pi(\text{SMT}(A_2, B_3, C_1)) \neq \text{SMT}(a, b, c)$. Similarly, $\Pi(\text{SMT}(A_3, B_2, C_1)) \neq \text{SMT}(a, b, c)$.
Consider SMT($A_2$, $B_4$, $C_1$). We claim that $V_3$ is contained in the interior of $\triangle A_2B_4C_1$. Note that $V_3$ and $C_1$ are on the same side of $\overline{A_2B_4}$, $V_3$ and $B_4$ are on the same side of $\overline{A_2C_1}$, and $V_3$ and $A_2$ are on the same side of $\overline{A_2B_4}$. Thus $\triangle A_2B_4C_1$ contains $V_3$. By Theorem 7.4, $\Pi(SMT(A_2, B_4, C_1)) \neq SMT(a, b, c)$. Similarly, $\Pi(SMT(A_4, B_2, C_1)) \neq SMT(a, b, c)$.

Consider SMT($A_2$, $B_3$, $C_1$). Both $A_2$ and $C_1$ are contained in $H_{B_3B_5}$. Thus $A_2$ and $C_1$ are separated from $B_5$ by $P_{B_3B_5}$. By Theorem 7.5, $\Pi(SMT(A_2, B_5, C_1)) \neq SMT(a, b, c)$. Similarly, $\Pi(SMT(A_5, B_2, C_1)) \neq SMT(a, b, c)$.

Consider SMT($A_3$, $B_4$, $C_1$). Both $A_3$ and $C_1$ are contained in $S_{B_3B_4}$. Thus $A_3$ and $C_1$ are separated from $B_4$ by $P_{B_3B_4}$. By Theorem 7.5, $\Pi(SMT(A_3, B_4, C_1)) \neq SMT(a, b, c)$. Similarly, $\Pi(SMT(A_5, B_2, C_1)) \neq SMT(a, b, c)$.

Theorem 9.4. Suppose $S$ is contained in the key trapezoid (page 393). Let $A_i$, $B_j$ (with $i, j = 1, \ldots, 5$) lie in the triangles specified in the diagram of Theorem 9.3. Then $\Pi(SMT(A_i, B_j, C_2)) \neq SMT(a, b, c)$ for all $i, j$ with $i \neq j$. That is, the tile containing $C_2$ can be removed from the region of interest.

Proof. Again we apply a case-by-case analysis.

Consider SMT($A_1$, $B_2$, $C_2$). Both $B_2$ and $C_2$ are contained in $S_{A_2A_1}$. Thus, $B_2$ and $C_2$ are separated from $A_1$ by $P_{A_1A_2}$. By Theorem 7.5, $\Pi(SMT(A_1, B_2, C_2)) \neq (SMT(a, b, c))$. By a similar argument, $\Pi(SMT(A_2, B_1, C_2)) \neq SMT(a, b, c)$.

Consider SMT($A_1$, $B_3$, $C_2$). Both $B_3$ and $C_2$ are contained in $H_{A_3A_1}$. Thus, $B_3$ and $C_2$ are separated from $A_1$ by $P_{A_1A_3}$. By Theorem 7.5, $\Pi(SMT(A_1, B_3, C_2)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_3, B_1, C_2)) \neq SMT(a, b, c)$.

Consider SMT($A_1$, $B_4$, $C_2$). Both $B_4$ and $C_2$ are contained in $H_{A_3A_1}$. Thus, $B_4$ and $C_2$ are separated from $A_1$ by $P_{A_1A_4}$. By Theorem 7.5, $\Pi(SMT(A_1, B_4, C_2)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_4, B_1, C_2)) \neq SMT(a, b, c)$.

Consider SMT($A_1$, $B_5$, $C_2$). We claim that $V_1$ is contained in the interior of $\triangle A_1B_5C_2$. Both $C_2$ and $V_1$ lie on the same side of $\overline{A_1B_5}$, $B_5$ and $V_1$ lie on the same side of $\overline{A_1C_2}$, and $A_1$ and $V_1$ lie on the same side of $\overline{B_5C_2}$. Thus $V_1$ must be contained in the interior of $\triangle A_1B_5C_2$. By Theorem 7.4, $\Pi(SMT(A_1, B_5, C_2)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_5, B_1, C_2)) \neq SMT(a, b, c)$.

Consider SMT($A_2$, $B_3$, $C_2$). Recall that $S$ are contained in the convex hull of $\triangle A_2B_3C_2$. By hypothesis, $S$ is contained in the key trapezoid (page 393). These two conditions are satisfied only if $A_2B_3$ lies above the vertex $V_2$. Thus, $C_2$ and
$V_2$ lie on the same side of $\overline{A_2B_3}$, $B_3$ and $V_2$ lie on the same side of $\overline{A_2C_2}$, and $A_2$ and $V_2$ lie on the same side of $\overline{B_3C_2}$. Thus $V_2$ are contained in the interior of $\triangle A_2B_3C_2$. By Theorem 7.4, $\Pi(SMT(A_2, B_3, C_2)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_3, B_2, C_2)) \neq SMT(a, b, c)$.

Consider SMT($A_2, B_4, C_2$). Both $A_2$ and $C_2$ are contained in $S_{B_3B_4}$. Thus, $A_2$ and $C_2$ are separated from $B_4$ by $P_{B_3B_4}$. By Theorem 7.5, $\Pi(SMT(A_2, B_4, C_2)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_4, B_2, C_2)) \neq SMT(a, b, c)$.

Consider SMT($A_2, B_5, C_2$). Both $A_2$ and $C_2$ are contained in $H_{B_3B_4}$. Thus, $A_2$ and $C_2$ are separated from $B_5$ by $P_{B_3B_5}$. By Theorem 7.5, $\Pi(SMT(A_2, B_5, C_2)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_5, B_2, C_2)) \neq SMT(a, b, c)$.

Consider SMT($A_3, B_4, C_2$). Both $A_3$ and $C_2$ are contained in $S_{B_3B_4}$. Thus, $A_3$ and $C_2$ are separated from $B_4$ by $P_{B_3B_4}$. By Theorem 7.5, $\Pi(SMT(A_3, B_4, C_2)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_4, B_3, C_2)) \neq SMT(a, b, c)$.

Consider SMT($A_3, B_5, C_2$). Both $A_3$ and $C_2$ are contained in $H_{B_3B_4}$. Thus, $A_3$ and $C_2$ are separated from $B_5$ by $P_{B_3B_5}$. By Theorem 7.5, $\Pi(SMT(A_3, B_5, C_2)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_5, B_3, C_2)) \neq SMT(a, b, c)$.

Consider SMT($A_4, B_5, C_2$). Both $A_4$ and $C_2$ are contained in $S_{B_3B_5}$. Thus, $A_4$ and $C_2$ are separated from $B_5$ by $P_{B_4B_5}$. By Theorem 7.5, $\Pi(SMT(A_4, B_5, C_2)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_5, B_4, C_2)) \neq SMT(a, b, c)$.

\[\square\]

**Theorem 9.5.** Suppose $S$ is contained in the key trapezoid (page 393). Let $C_i$ and $A_j = B_j$ (with $i, j = 1, \ldots, 5$) lie in the triangles specified in the diagram of Theorem 9.3. Then $\Pi(SMT(A_i, B_j, C_4)) \neq SMT(a, b, c)$ for all $i, j$ with $i \neq j$. That is, the tile containing $C_4$ can be removed from the region of interest.

**Proof.** Consider SMT($A_1, B_2, C_4$). Both $B_2$ and $C_4$ are contained in $S_{A_2A_1}$, so $B_2$ and $C_4$ are separated from $A_1$ by $P_{A_1A_2}$. By Theorem 7.5, $\Pi(SMT(A_1, B_2, C_4)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_2, B_1, C_4)) \neq SMT(a, b, c)$.

Consider SMT($A_1, B_3, C_4$). Let $V_4$ be the intersection of $m_1$ and $p_1$. Note that $C_4$ and $V_4$ lie on the same side of $\overline{A_1B_3}$, $B_3$ and $V_4$ lie on the same side of $\overline{A_1C_4}$, and $A_1$ and $V_4$ lie on the same side of $\overline{B_3C_4}$. Thus $V_4$ are contained in the interior of $\triangle A_1B_3C_4$. By Theorem 7.4, $\Pi(SMT(A_1, B_3, C_4)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_3, B_1, C_4)) \neq SMT(a, b, c)$.

Consider SMT($A_1, B_4, C_4$). Both $A_1$ and $C_4$ are contained in $H_{B_3B_4}$. Thus, $A_1$ and $C_4$ are separated from $B_4$ by $P_{B_3B_4}$. By Theorem 7.5, $\Pi(SMT(A_1, B_4, C_4)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_4, B_1, C_4)) \neq SMT(a, b, c)$.

Consider SMT($A_1, B_5, C_4$). Both $A_1$ and $C_4$ are contained in $H_{B_3B_5}$. Thus, $A_1$ and $C_4$ are separated from $B_5$ by $P_{B_3B_5}$. By Theorem 7.5, $\Pi(SMT(A_1, B_5, C_4)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_5, B_1, C_4)) \neq SMT(a, b, c)$.

Consider SMT($A_2, B_3, C_4$). Both $A_2$ and $C_4$ are contained in $S_{B_3B_5}$. Thus, $A_2$ and $C_4$ are separated from $B_5$ by $P_{B_3B_5}$. By Theorem 7.5, $\Pi(SMT(A_2, B_3, C_4)) \neq SMT(a, b, c)$. By a similar argument, $\Pi(SMT(A_5, B_2, C_4)) \neq SMT(a, b, c)$.
and $C_4$ are separated from $B_3$ by $P_{B_2B_3}$. By Theorem 7.5, $\Pi(\text{SMT}(A_2, B_3, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_3, B_2, C_4)) \neq \text{SMT}(a, b, c)$.

Consider SMT($A_2, B_4, C_4$). Both $A_2$ and $C_4$ are contained in $H_{B_2B_4}$. Thus, $A_2$ and $C_4$ are separated from $B_4$ by $P_{B_2B_4}$. By Theorem 7.5, $\Pi(\text{SMT}(A_2, B_4, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_4, B_2, C_4)) \neq \text{SMT}(a, b, c)$.

Consider SMT($A_2, B_3, C_4$). Both $A_2$ and $C_4$ are contained in $H_{B_2B_3}$. Thus, $A_2$ and $C_4$ are separated from $B_5$ by $P_{B_2B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_2, B_3, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_2, C_4)) \neq \text{SMT}(a, b, c)$.

Consider SMT($A_3, B_4, C_4$). Both $A_3$ and $C_4$ are contained in $S_{B_3B_4}$. Thus, $A_3$ and $C_4$ are separated from $B_4$ by $P_{B_3B_4}$. By Theorem 7.5, $\Pi(\text{SMT}(A_3, B_4, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_4, B_3, C_4)) \neq \text{SMT}(a, b, c)$.

Consider SMT($A_3, B_5, C_4$). Both $A_3$ and $C_4$ are contained in $H_{B_3B_5}$. Thus, $A_3$ and $C_4$ are separated from $B_5$ by $P_{B_3B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_3, B_5, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_3, C_4)) \neq \text{SMT}(a, b, c)$.

Consider SMT($A_4, B_5, C_4$). Both $A_4$ and $B_5$ are contained in $H_{C_5C_4}$. Thus, $A_4$ and $B_5$ are separated from $C_4$ by $P_{C_5C_4}$. By Theorem 7.5, $\Pi(\text{SMT}(A_4, B_5, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_4, C_4)) \neq \text{SMT}(a, b, c)$.

**Theorem 9.6.** Suppose $S$ is contained in the key trapezoid (page 393). Let $C_3$ and $A_1$, $B_j$ (with $i, j = 1, \ldots, 5$) lie in the triangles specified in the diagram of Theorem 9.3. Then $\Pi(\text{SMT}(A_i, B_j, C_3)) \neq \text{SMT}(a, b, c)$ for all $i, j$ with $i \neq j$. That is, the tile containing $C_4$ can be removed from the region of interest.

**Proof.** Consider SMT($A_1, B_2, C_3$). Both $B_2$ and $C_3$ are contained in $S_{A_2A_1}$, so $B_2$ and $C_3$ are separated from $A_1$ by $P_{A_1A_2}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_2, C_3)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_2, B_1, C_3)) \neq \text{SMT}(a, b, c)$.

Consider SMT($A_1, B_3, C_3$). Assume $B_3$ is not a vertex. We claim that $V_2$ is contained in $\triangle A_1B_3C_2$. Let $V_2$ be the intersection of $m_0$ and $p_1$. Note that $V_2$ and $C_3$ are on the same side of $\overline{A_1B_3}$, $V_2$ and $B_3$ are on the same side of $\overline{A_1C_3}$, and $V_2$ and $A_1$ are on the same side of $\overline{B_3C_3}$. Thus $\triangle A_1B_3C_3$ must contain $V_2$. By Theorem 7.4, $\Pi(\text{SMT}(A_1, B_3, C_3)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_3, B_1, C_3)) \neq \text{SMT}(a, b, c)$.

Consider SMT($A_1, B_4, C_3$). Both $A_1$ and $B_4$ are contained in $H_{C_6C_3}$. Thus, $A_1$ and $B_4$ are separated from $C_3$ by $P_{C_6C_3}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_4, C_3)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_4, B_1, C_3)) \neq \text{SMT}(a, b, c)$.

Consider SMT($A_1, B_5, C_3$). Both $A_1$ and $C_3$ are contained in $H_{B_5B_3}$. Thus, $A_1$ and $C_3$ are separated from $B_5$ by $P_{B_1B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_5, C_3)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_1, B_5, C_3)) \neq \text{SMT}(a, b, c)$.

Consider SMT($A_2, B_3, C_3$). Recall that $S$ are contained in the convex hull of $\triangle A_2B_3C_3$. By hypothesis, $S$ is contained in the key trapezoid (page 393). These two conditions are satisfied only if $\overrightarrow{A_2B_3}$ lies above the vertex $V_2$, the intersection
of \( m_0 \) and \( p_1 \). Thus, \( C_3 \) and \( V_2 \) lie on the same side of \( \overline{A_2B_3}, B_3 \) and \( V_2 \) lie on the same side of \( \overline{A_2C_3} \), and \( A_2 \) and \( V_2 \) lie on the same side of \( \overline{B_3C_3} \). Thus \( V_2 \) are contained in the interior of \( \triangle A_2B_3C_3 \). By Theorem 7.4, \( \Pi(\text{smt}(A_2, B_3, C_3)) \neq \text{smt}(a, b, c) \). By a similar argument, \( \Pi(\text{smt}(A_3, B_3, C_3)) \neq \text{smt}(a, b, c) \).

Consider \( \text{smt}(A_2, B_4, C_3) \). Both \( A_2 \) and \( C_3 \) are contained in \( H_{B_2B_4} \). Thus, \( A_2 \) and \( C_3 \) are separated from \( B_4 \) by \( P_{B_2B_4} \). By Theorem 7.5, \( \Pi(\text{smt}(A_2, B_4, C_3)) \neq \text{smt}(a, b, c) \). By a similar argument, \( \Pi(\text{smt}(A_4, B_2, C_3)) \neq \text{smt}(a, b, c) \).

Consider \( \text{smt}(A_2, B_5, C_3) \). Both \( A_2 \) and \( C_3 \) are contained in \( H_{B_2B_5} \). Thus \( A_2 \) and \( C_3 \) are separated from \( B_5 \) by \( P_{B_2B_5} \). By Theorem 7.5, \( \Pi(\text{smt}(A_2, B_5, C_3)) \neq \text{smt}(a, b, c) \). By a similar argument, \( \Pi(\text{smt}(A_5, B_2, C_3)) \neq \text{smt}(a, b, c) \).

Consider \( \text{smt}(A_3, B_4, C_3) \). Both \( A_3 \) and \( C_3 \) are contained in \( S_{B_3B_4} \). Thus, \( A_3 \) and \( C_3 \) are separated from \( B_4 \) by \( P_{B_3B_4} \). By Theorem 7.5, \( \Pi(\text{smt}(A_3, B_4, C_3)) \neq \text{smt}(a, b, c) \). By a similar argument, \( \Pi(\text{smt}(A_4, B_3, C_3)) \neq \text{smt}(a, b, c) \).

Consider \( \text{smt}(A_3, B_5, C_3) \). Both \( A_3 \) and \( C_3 \) are contained in \( H_{B_3B_5} \). Thus \( A_3 \) and \( C_3 \) are separated from \( B_5 \) by \( P_{B_3B_5} \). By Theorem 7.5, \( \Pi(\text{smt}(A_3, B_5, C_3)) \neq \text{smt}(a, b, c) \). By a similar argument, \( \Pi(\text{smt}(A_5, B_3, C_3)) \neq \text{smt}(a, b, c) \).

Consider \( \text{smt}(A_4, B_5, C_3) \). Both \( A_4 \) and \( B_5 \) are contained in \( H_{C_1C_3} \). Thus \( A_4 \) and \( B_5 \) are separated from \( C_3 \) by \( P_{C_1C_3} \). By Theorem 7.5, \( \Pi(\text{smt}(A_4, B_5, C_3)) \neq \text{smt}(a, b, c) \). By a similar argument, \( \Pi(\text{smt}(A_4, B_3, C_3)) \neq \text{smt}(a, b, c) \). \( \square \)

The final region of interest, after the removal of the tiles containing \( C_1, C_2, C_3 \) and \( C_4 \), is shown in Figure 9. This region also contains each of the five Steiner trees that can be considered when \( A \) and \( B \) are on the same tile (see Theorem 9.1).

**Final reductions.** The region shown in Figure 9 must contain at least one copy of the tree \( \text{smt}(A, B, C) \) that realizes \( \text{smt}(a, b, c) \) where \( A \) and \( B \) come from different tiles. Within this region, there are still combinations that can never realize

![Figure 9. The final region of interest.](image-url)
SMT\((a, b, c)\) and thus do not need to be considered. In this section we will eliminate these combinations and then provide a list of all the trees SMT\((A_i, B_j, C_k)\) that must be considered to determine the SMT\((A, B, C)\) realizing SMT\((a, b, c)\).

Consider SMT\((A_1, B_5, C_6)\). Both \(B_5\) and \(C_6\) lie in \(H_{A_1 A_5}\). Thus, \(B_5\) and \(C_6\) must be separated from \(A_5\) by \(P_{A_1 A_5}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_5, C_6)) \neq \text{SMT}(a, b, c)\). By a similar argument, \(\Pi(\text{SMT}(A_5, B_1, C_6)) \neq \text{SMT}(a, b, c)\).

Consider SMT\((A_2, B_5, C_6)\). Both \(A_2\) and \(C_6\) lie in \(S_{B_2 B_5}\). Thus, \(A_2\) and \(C_6\) must be separated from \(B_5\) by \(P_{B_2 B_5}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_2, B_5, C_6)) \neq \text{SMT}(a, b, c)\). By a similar argument, \(\Pi(\text{SMT}(A_5, B_2, C_6)) \neq \text{SMT}(a, b, c)\).

Consider SMT\((A_3, B_5, C_6)\), and let \(V_1 = m_1 \cap n_0\). Then \(A_3\) and \(V_1\) are on the same side of \(\overline{B_5 C_6}\), \(B_5\) and \(V_1\) are on the same side of \(\overline{A_3 C_6}\), and \(C_6\) and \(V_1\) are on the same side of \(\overline{A_3 B_5}\). Thus, \(V_1 \subset \triangle A_3 B_5 C_6\). By Theorem 7.4, \(\Pi(\text{SMT}(A_3, B_5, C_6)) \neq \text{SMT}(a, b, c)\). Similarly, \(\Pi(\text{SMT}(A_5, B_3, C_6)) \neq \text{SMT}(a, b, c)\).

Consider SMT\((A_4, B_5, C_6)\). Note that if \(B_5\) is within the shaded region (which is required for it to even be considered), then both \(B_5\) and \(A_4\) lie in \(S_{C_5 C_6}\). Thus, \(B_5\) and \(A_4\) are separated from \(C_6\) by \(P_{C_5 C_6}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_4, B_5, C_6)) \neq \text{SMT}(a, b, c)\). By a similar argument, \(\Pi(\text{SMT}(A_5, B_4, C_6)) \neq \text{SMT}(a, b, c)\).

Consider SMT\((A_1, B_4, C_5)\). Both \(B_4\) and \(C_5\) lie in \(S_{A_1 A_4}\). Thus, both \(B_4\) and \(C_5\) must be separated from \(A_1\) by \(P_{A_4 A_1}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_4, C_5)) \neq \text{SMT}(a, b, c)\). By a similar argument, \(\Pi(\text{SMT}(A_4, B_1, C_6)) \neq \text{SMT}(a, b, c)\).

Consider SMT\((A_2, B_5, C_5)\). Both \(B_5\) and \(C_5\) lie in \(S_{A_4 A_2}\). Thus, both \(B_5\) and \(C_5\) must be separated from \(A_2\) by \(P_{A_5 A_2}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_2, B_5, C_5)) \neq \text{SMT}(a, b, c)\). By a similar argument, \(\Pi(\text{SMT}(A_5, B_2, C_6)) \neq \text{SMT}(a, b, c)\).

Consider SMT\((A_1, B_5, C_5)\). Note that if \(B_5\) is within the shaded region, then both \(B_5\) and \(C_5\) are contained in \(S_{A_1 A_4}\). Thus, \(B_5\) and \(C_5\) are separated from \(A_1\) by \(P_{A_4 A_1}\). By Theorem 7.5, \(\Pi(\text{SMT}(A_1, B_5, C_5)) \neq \text{SMT}(a, b, c)\). By a similar argument, \(\Pi(\text{SMT}(A_5, B_1, C_6)) \neq \text{SMT}(a, b, c)\).

**List of potential combinations in Case 3.** The remaining combinations \((A_i, B_j, C_k)\) for both \(A\) and \(B\) on the same tile and \(A\) and \(B\) not on the same tile are

\[
\begin{align*}
(A_1, B_2, C_6) & \cong (A_4, B_5, C_5), & (A_2, B_1, C_6) & \cong (A_5, B_4, C_5), & (A_1, B_3, C_6), \\
(A_3, B_1, C_5), & & (A_1, B_4, C_6), & & (A_4, B_1, C_6), \\
(A_2, B_3, C_6), & & (A_3, B_2, C_6), & & (A_2, B_4, C_6), \\
(A_4, B_2, C_6), & & (A_3, B_4, C_6), & & (A_4, B_3, C_6), \\
(A_2, B_1, C_5), & & (A_1, B_2, C_5), & & (A_1, B_3, C_5), \\
(A_3, B_1, C_5), & & (A_2, B_3, C_5), & & (A_3, B_2, C_5), \\
(A_2, B_4, C_5), & & (A_4, B_2, C_5), & & (A_3, B_4, C_5), \\
(A_4, B_3, C_5), & & (A_3, B_5, C_5), & & (A_5, B_3, C_5), \\
(A_2, B_2, C_5), & & (A_3, B_3, C_5), & & (A_4, B_4, C_5) \cong (A_1, B_1, C_6), \\
(A_5, B_5, C_5) \cong (A_2, B_2, C_6), & & (A_3, B_3, C_6). & & 
\end{align*}
\]
Thus, the Steiner tree which realizes $\text{SMT}(a, b, c)$ will be formed from one of the 29 combinations included in this list.

10. An algorithm for finding a shortest network on three points

At the end of Sections 8 and 9 we provided lists of combinations which could realize $\text{SMT}(a, b, c)$ for the different cases. In this section we discuss how these lists can be further reduced by considerations of the specific positioning of the points within the faces. We provide two principles upon which the reductions are based. We also provide an algorithm that uses these principles. When the algorithm is applied, we have found that most point combinations can be eliminated.

Two principles allow us to eliminate potential combinations of points from consideration:

- We demonstrated that for Case 2 a solution must reside in the truncated triangle region (Figure 8) and for Case 3 it must reside in the shaded region in Figure 9. In either case, if a point lies outside the corresponding region, no combinations involving that particular point need to be considered.

- If any two points of a combination are separated from the third point by the perpendicular bisector of the third point and a rotation and/or translation of the third point, that combination does not need to be considered (see Theorem 7.5). Recall from Definition 4.2 that for any points $P$ and $Q$, $\tilde{H}_{PQ} = \{X \mid PX \leq QX\}$. Thus, equivalently, if $A$ and $B$ are contained in $\tilde{H}_{C'C}$ for some $C, C' \in \Pi^{-1}(c)$, then $(A, B, C)$ does not need to be considered.

Using these principles, point combinations within the list can be eliminated from consideration. A systematic approach to the elimination is introduced in the following algorithm.

**Algorithm 10.1**. The following algorithm provides a shortest network connecting three given points on a regular tetrahedron $\mathcal{T}$.

1. Determine whether Case 1, 2, or 3 applies.

   - **Case 1**: If all three points can be considered to lie on a common face, the Steiner tree is just a shortest network on that face (Section 6), and the Steiner tree can be constructed using Algorithm 2.1. The algorithm is complete.

   - **Case 2**: If the three points can be considered to lie on distinct faces of $\mathcal{T}$, define the region of interest to be the truncated triangle region (Figure 8). Define the list of potential combinations to be the list on page 389. Label the faces so that the face not considered to contain any points is face 1. Proceed to Steps (2)–(4).
**Case 3:** Otherwise, define the shaded region to be that shown in Figure 9. Define the list of potential combinations to be the list on page 399. Label the faces so that the face considered to contain two points is face 3, and the face considered to contain one point is face 4. Proceed to Steps (2)–(4).

(2) Eliminate any combinations within the list of potential combinations that contain points which are not contained within the shaded region.

(3) For all $C_m$ contained in the shaded region:
   
   (a) For all $C_i \neq C_m$ in the shaded region, construct $\tilde{H}_{C_iC_m}$. Eliminate any combinations $(A_k, B_l, C_m)$ where $A_k$ and $B_l$ are both contained in $\tilde{H}_{C_iC_m}$.
   
   (b) For the remaining $B_l$ that appear in combinations which have not yet been eliminated:
      
      (i) For all $B_l \neq B_l$ in the shaded region, construct $\tilde{H}_{B_lB_l}$. If both $C_m$ and $A_k$ are contained in $\tilde{H}_{B_lB_l}$ for any $B_l$, eliminate the combination $(A_k, B_l, C_m)$.
      
      (ii) For the $A_k$ that appear in a remaining combination with $B_l$ and $C_m$:
            
            For all $A_i \neq A_k$ in the shaded region construct $\tilde{H}_{A_iA_k}$. If both $C_m$ and $B_l$ are contained in $\tilde{H}_{A_iA_k}$, eliminate the combination $(A_k, B_l, C_m)$.

(4) Measure the lengths of the Steiner minimal trees formed from the remaining combinations using Algorithm 2.1. The Steiner minimal tree with shortest length realizes $SMT(a, b, c)$. The algorithm is complete.

We will now demonstrate how to apply the algorithm for the configuration shown in Figure 9, which clearly corresponds to Case 3.

$B_5, A_1$ and $A_5$ are not contained within the shaded region, so none of $(A_1, B_2, C_5)$, $(A_1, B_3, C_6)$, $(A_1, B_4, C_6)$, $(A_5, B_4, C_5)$, $(A_5, B_3, C_5)$, $(A_2, B_5, C_5)$ and $(A_4, B_5, C_5)$ need to be considered.

Construct $\tilde{H}_{C_6C_5}$:
Since both $A_2$ and $B_2$ are contained in $\tilde{H}_{C_6}C_5$, the combination $(A_2, B_2, C_5)$ can be eliminated.

Construct $\tilde{H}_{B_iB_1}$ for all $i \neq 1$ (left diagram). Since $C_5$ and $A_3$ are contained in $\tilde{H}_{B_3B_1}$, the combination $(A_3, B_1, C_5)$ can be eliminated. There are no remaining combinations which use both $B_1$ and $C_5$.

Construct $\tilde{H}_{B_iB_2}$ for all $i \neq 2$ (right diagram above). Since both $C_5$ and $A_3$ are contained in $\tilde{H}_{B_3B_2}$, the combination $(A_3, B_2, C_5)$ can be eliminated. Since both $C_5$ and $A_4$ are contained in $\tilde{H}_{B_3B_2}$, the combination $(A_4, B_2, C_5)$ can be eliminated. There are no remaining combinations which use both $B_2$ and $C_5$.

Construct $\tilde{H}_{B_iB_3}$ for all $i \neq 3$ (left diagram below). Since both $C_5$ and $A_4$ are contained in $\tilde{H}_{B_3B_3}$, the combination $(A_4, B_3, C_5)$ can be eliminated. The only remaining combinations the list are $(A_2, B_3, C_5)$ and $(A_3, B_3, C_5)$. However, since both $C_5$ and $B_3$ are contained in $\tilde{H}_{A_1A_2}$, $(A_2, B_3, C_5)$ can be eliminated.

Construct $\tilde{H}_{B_iB_4}$ for all $i \neq 4$. Since $C_5$ is not contained in any $\tilde{H}_{B_iB_4}$ with $i \neq 4$, the remaining possibilities from the above list are $(A_2, B_4, C_5)$, $(A_3, B_4, C_5)$, and $(A_4, B_4, C_5)$. Since both $C_5$ and $B_4$ are contained in $\tilde{H}_{A_4A_2}$, $(A_2, B_4, C_5)$ can be eliminated (right diagram immediately above). Since both $C_5$ and $B_4$ are contained in $\tilde{H}_{A_3A_3}$, $(A_3, B_4, C_5)$ can be eliminated.
We have shown that the only remaining combinations in the list containing $C_5$ are $(A_3, B_3, C_5)$ and $(A_4, B_4, C_5)$. Using a similar procedure, we can show that the only remaining combination containing $C_6$ is $(A_2, B_2, C_6)$. Assuming $\mathcal{F}$ has edge-length 1, we construct the Steiner trees associated with each of these combinations, with the following results:

\[ \mathcal{L}(\text{SMT}(A_3, B_3, C_5)) = 1.04 \]
\[ \mathcal{L}(\text{SMT}(A_4, B_4, C_5)) = 0.87 \]
\[ \mathcal{L}(\text{SMT}(A_2, B_2, C_6)) = 1.43 \]

Hence, SMT$(A_4, B_4, C_5)$ realizes SMT$(a, b, c)$ with length 0.87, and the algorithm is complete with only three actual measurements.

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