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The Steiner problem on the regular tetrahedron

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The Steiner problem involves finding a shortest path network connecting a specified set of points. In this paper, we examine the Steiner problem for three points on the surface of a regular tetrahedron. We prove several important properties about Steiner minimal trees on a regular tetrahedron. There are infinitely many ways to connect three points on a tetrahedron, so we present a way to eliminate all but a finite number of possible solutions. We provide an algorithm for finding a shortest network connecting three given points on a regular tetrahedron. The solution can be found by direct measurement of the remaining possible Steiner trees.

1. Introduction

The *Steiner problem* asks to find a shortest path network to connect a given set of points on a surface. In this paper we will study the three point Steiner problem on a regular tetrahedron. We will provide an algorithm in Section 10, Algorithm 10.1, that determines a solution to the three point Steiner problem on the regular tetrahedron.

On the Euclidean plane, the Steiner problem has been studied extensively; see [Gilbert and Pollak 1968; Hwang et al. 1992; Ivanov and Tuzhilin 1994, Chapter 9; Melzak 1961; Zacharias 1914–1921]. The Steiner problem for three points on the Euclidean plane was formally introduced in the seventeenth century by Fermat; see [Hwang et al. 1992; Kuhn 1974; Zacharias 1914–1921]. A general algorithm to find the solution to the Steiner problem for n points on the Euclidean plane was first developed by Melzak [1961] (see also [Hwang et al. 1992]).

The Steiner problem on the surface of the tetrahedron is not as straightforward as on the plane. In particular, a geodesic segment connecting any two points on the surface of the tetrahedron is not unique (see top part of Figure 1 on next page). Consequently, there are infinitely many locally stable shortest-length trees connecting any three points on the surface of the tetrahedron (see Figure 1, bottom). In this

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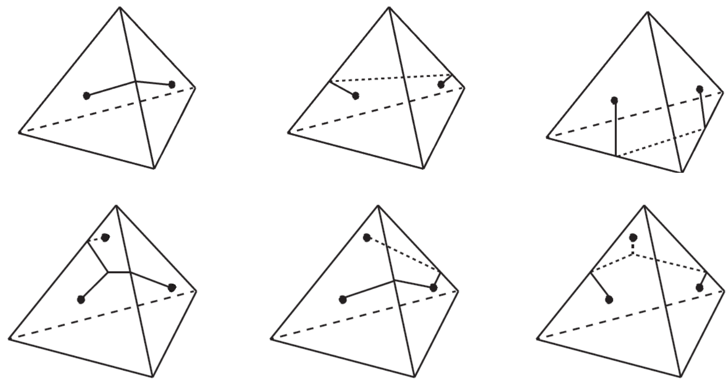


Figure 1. Candidates for a shortest path (top) and for a shortest tree (bottom).

paper, we provide an algorithm that eliminates all but a small number of path networks that need be considered as possible minimizers. Amongst these remaining candidates, a shortest path network can be found using direct measurement.

This research contributes to the growing set of strategies for solving Steiner problems on surfaces in general. Algorithms exist to find the solution for the Steiner problem on certain surfaces of constant curvature. The problem was studied in [Weng 2001; Litwhiler and Aly 1980; Brazil et al. 1998] for on curved surfaces, including spheres. March and Halverson [2005] studied Steiner trees in hyperbolic space. Lee et al. [2011] studied the Steiner problem on wide and narrow cones. Penrod [2007] and May and Mitchell [2007] developed algorithms to solve Steiner problems on the flat torus. Caffarelli et al. [2010] studied the Steiner problem on surfaces of revolution. Brune and Sipe [2009] developed an algorithm to find a shortest path between two points on the surface of the regular tetrahedron. This research about the Steiner problem on the regular tetrahedron may provide further insight into the Steiner problem on more general piecewise linear surfaces.

2. Preliminaries

We begin by setting up the basic framework for the Steiner problem on a regular tetrahedron $\mathcal{T}$. Let $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$ be a set of given points on $\mathcal{T}$ called *terminal points*, and let L be a path network (also on $\mathcal{T}$) connecting the points in $\mathcal{A}$. A *path network* connects a collection of arcs, only possibly meeting at the endpoints such that the network contains a path connecting any two points of $\mathcal{A}$. If L is a shortest path network, the edges must be geodesics. L must also be a tree since if L contained a cycle, one of the edges could be removed. The goal of the Steiner problem is to find a shortest path network L connecting the points of $\mathcal{A}$. A shortest path network may have additional vertices called *Steiner points*. The solution to the Steiner problem is called the *Steiner minimal tree*, which will be denoted by SMT($\mathscr{A}$).

As defined in [Hwang and Weng 1986], a tree with n fixed points is called a *Steiner tree* on n fixed points if it satisfies the following conditions:

(1) There are at most $n-2$ Steiner points.

(2) Each Steiner point has exactly three incident edges.

(3) Any pair of edges meeting at any vertex of the tree form an angle with measure at least 120°.

Note that for a tree with no degree-two Steiner points, the number of edges minus the number of vertices is 1, which in fact implies condition 1. A tree that has exactly $n-2$ Steiner points is called a *full Steiner tree*. A tree that has fewer than $n-2$ Steiner points is called a *degenerate Steiner tree*.

The Steiner problem for n fixed points on the plane can be solved in finite time using Melzak's algorithm [1961]. We will utilize these results for the regular tetrahedron since the plane can be viewed as a branched cover of the regular tetrahedron. The Steiner problem on $\mathscr{T}$ is more complex than on the plane because there are infinitely many geodesics that could connect two points. Thus, the process of solving the Steiner problem on $\mathscr{T}$ is initially a problem of narrowing down potential path networks.

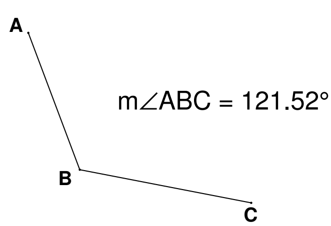
The algorithm used to solve the 3-point Steiner problem in Euclidean space was developed by Torricelli, Cavalieri, Simpson, Heinen, and Bertrand (see [Hwang et al. 1992]). For convenience, we repeat it here.

Algorithm 2.1. This algorithm provides a shortest network connecting three given points in Euclidean space.

(1) Let A, B, and C be given. Label A, B, and C so that $m\angle ABC \geq m\angle ACB$ and $m\angle ABC \geq m\angle BAC$.

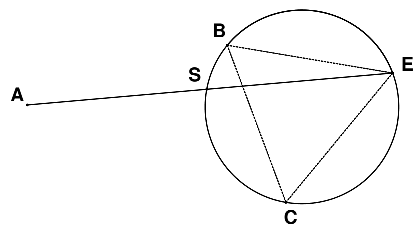
(2) Determine whether Case 1 or 2 applies.

Case 1. If $m\angle ABC > 120°$, the Steiner minimal tree is degenerate and it is $\overline{AB} \cup \overline{BC}$. The algorithm is complete (see figure for example).



Case 2. If $m\angle ABC \leq 120°$, proceed to Steps (3)–(6).

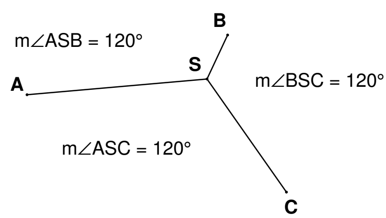
(3) Create an equilateral triangle $\triangle BCE$ where E is on the opposite side of $\overleftrightarrow{BC}$ from A.



(4) Construct $\overline{EA}$. This line segment is called the Simpson line. (The length of the Simpson Line is known to have the same length as the SMT(A, B, C) [Hwang et al. 1992].)

(5) Next, circumscribe a circle about $\triangle BCE$. The point of intersection of that circle and $\overline{EA}$ is the Steiner point S.

(6) Connect each of A, B, and C to S to form SMT(A, B, C). By construction, every two edges of the tree which meet at the Steiner point have angle 120° [Gilbert and Pollak 1968]. The algorithm is complete.

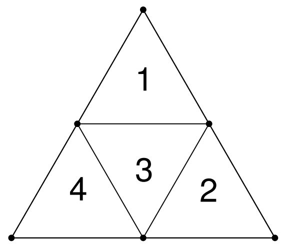


Another observation relevant to our discussion of the Steiner problem on the regular tetrahedron is that no geodesic passes through the vertices of a narrow cone [Lee et al. 2011]. Since a small neighborhood of a vertex is a narrow cone, no shortest path network will pass through any vertices of $\mathcal{T}$. Hence, a shortest path network can only meet a vertex of $\mathcal{T}$ if a fixed point is placed on that vertex [Ivanov and Tuzhilin 1994, Chapter 9].

3. Tiling the plane

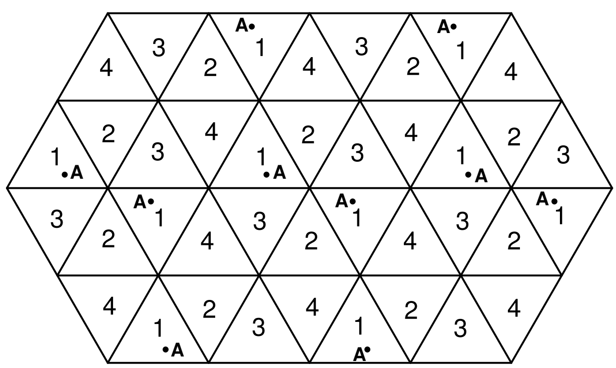
In this section we will show how to construct a branched covering of the plane onto the regular tetrahedron. For further reference, see [Ivanov and Tuzhilin 1994, Chapter 9].

Consider a regular tetrahedron with faces labeled 1, 2, 3, and 4. Cut along the edges common to faces 1 and 2, 1 and 4, and 2 and 4 and lay it on the plane, as shown in the figure. We will use this configuration to tile the plane.

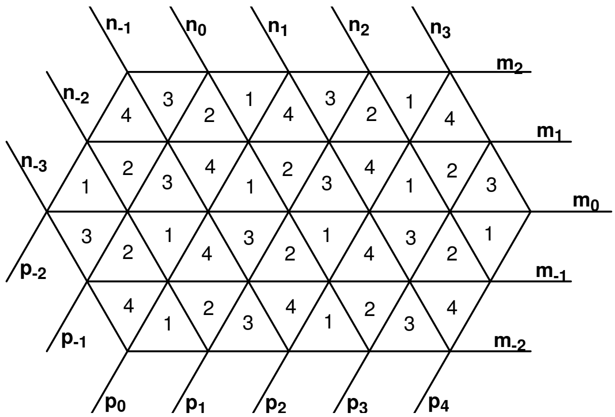


Notice that face 1 is adjacent to face 2 on the tetrahedron. Thus, in order to represent that on the plane, we must place a tile corresponding to face 2 so it becomes adjacent to a tile corresponding to face 1. This is accomplished by placing a tile corresponding to face 2 that is an 180° rotation about a common vertex. Similarly, we must place a tile corresponding to face 4 so that the tile corresponding to face 1 and a tile corresponding to face 4 have a common edge in the plane as they do on the tetrahedron. Since each face on the tetrahedron is adjacent to the other three faces, then each face should be adjacent to all of the other faces on the plane. If copies of each face are placed at 180° rotations about each of their respective vertices, this results in a comprehensive tiling of the Euclidean plane.

Points on $\mathcal{T}$ will be represented by lower case letters. The corresponding points in the tiling will be represented by corresponding capital letters. Assume a is on face 1 on $\mathcal{T}$. Then for each tile corresponding to face 1, there is a copy of A on the tile. Two adjacent tiles contain copies of A which are 180° rotations about the common vertex of the tile containing A. A small section of the tiling can be seen here:



We introduce a coordinate system to notate the different faces of the tiling. In the tiling, the horizontal lines that separate the triangles will be known as m_i, for $i = \dots, -2, -1, 0, 1, 2, \dots$. Similarly define n_i as the lines with the slope equal to $-\sqrt{3}$. Finally define p_i as the lines with slope $\sqrt{3}$. We thus obtain the following arrangement:



Using this coordinate system, we can identify individual tiles. For any tile that is bounded by m_x, n_y, and p_z, we will denote it as $T_{(x,y,z)}$. Without loss of generality, we will assume that $T_{(0,0,0)}$ corresponds to face 1, $T_{(1,-1,0)}$ corresponds to face 2, $T_{(0,-1,1)}$ corresponds to face 3, and $T_{(1,0,1)}$ corresponds to face 4. Though each face of the tetrahedron is replicated infinitely many times, each tile in the tiling has a unique labeling according to the lines that bound it.

We now show that this tiling is a branched covering of the plane onto the regular tetrahedron. Let $\Pi : \mathbb{R}^2 \to \mathcal{T}$ be the natural continuous map that takes each tile of the plane to its corresponding face in $\mathcal{T}$ homeomorphically. Let $\mathcal{V}$ be the vertex set of $\mathcal{T}$. Note that Π is a branched covering map with branch set $\mathcal{V}$. Then the map

$$\pi : \mathbb{R}^2 - \Pi^{-1}(V) \to \mathcal{T} - \mathcal{V}$$

(which is a restriction of Π) is a covering map of $\mathcal{T} - \mathcal{V}$. Since π is a covering map, it has the following lifting property: Suppose $a \in \mathcal{T} - \mathcal{V}$ and $A \in \Pi^{-1}(a)$. Then any path $\alpha : [0, 1] \to \mathcal{T} - \mathcal{V}$ so that $\alpha(0) = a$ has a unique lift to a path $\tilde{\alpha} : [0, 1] \to \mathbb{R}^2 - \Pi^{-1}(\mathcal{V})$ with $\tilde{\alpha}(a) = A$. The map $\tilde{\alpha}$ is a lift in the sense that $\pi \circ \tilde{\alpha} = \alpha$. It follows that any embedded path network in $\mathcal{T} - \mathcal{V}$ containing a can be uniquely lifted to a path network containing A.

Note that in the case that $a \in \mathcal{V}$ and $\Pi(A) = a$, for any embedded path network containing a in $\mathcal{T}$, there are two lifts of the path network containing A. These lifts are 180° rotations of each other about A.

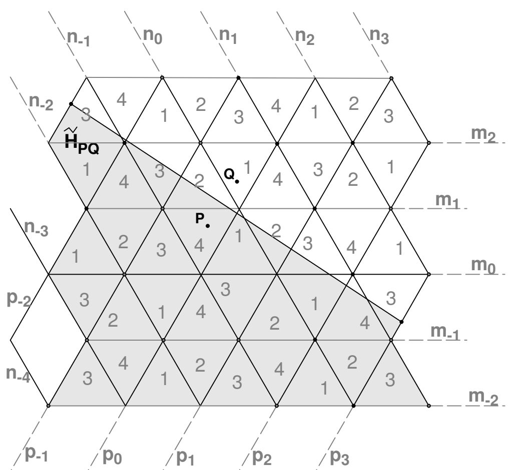
4. The two point problem

This section will briefly describe an algorithm used to construct a shortest path between any two points on a regular tetrahedron. For further details on this process, refer to [Brune and Sipe 2009]. The algorithm detailed here will depend heavily on the following basic geometric property:

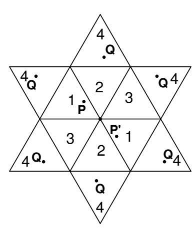
Property 4.1. *Given any two points A and B on the plane, construct the perpendicular bisector of $\overline{AB}$ and call it P_{AB}. If X is on the A side of P_{AB}, then X is closer to A. If X is on the B side of P_{AB}, then X is closer to B.*

Definition 4.2. Given two points P and Q on the plane, define $\tilde{H}_{PQ}$ to be the half-plane cut by the perpendicular bisector of P and Q on the P side; that is,

$$\tilde{H}_{PQ} = \{X \mid PX \leq QX\}.$$



The algorithm: a brief synopsis. Suppose there are two points p and q on distinct faces of the tetrahedron. Suppose $\mathbb{R}^2$ is tiled as in Section 3. Recall that $\Pi : \mathbb{R}^2 \to \mathcal{T}$ is the covering map and $\mathbb{R}^2$ is tiled as in Section 3. Then $\Pi^{-1}(p)$ and $\Pi^{-1}(q)$ contain infinitely many points. Let $P \in \Pi^{-1}(p)$. We want to find a point $Q \in \Pi^{-1}(q)$ that realizes a shortest path from p to q. The points of $\Pi^{-1}(q)$ that could realize a shortest path to P can be restricted to a star-shaped region. The region consists of an interior hexagon which contains the point P, outlined by six tiles which contains points of $\Pi^{-1}(q)$. This region is called an *i-star* for $i = 1, 2, 3$, or 4, where i is the face of the tetrahedron containing q. We illustrate a 4-star when p is on face 1 and q is on face 4:



It was proved in [Brune and Sipe 2009] that this i-star always contains a shortest path between two points.

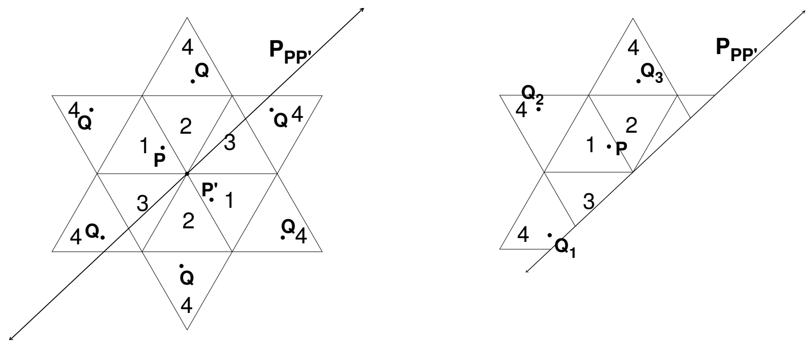
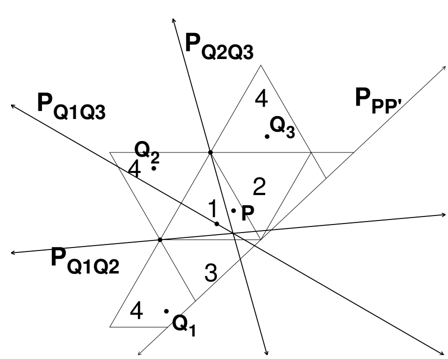


Figure 2. Reducing the number of possible points of $\Pi^{-1}(q)$ that can realize a shortest path.

There is a cutting technique that has been shown to reduce the number of possible points of $\Pi^{-1}(q)$ that could realize a shortest path. Begin by constructing the line segment from point P to the point $P' \in \Pi^{-1}(p)$, also located within the 4-star. Then, construct the perpendicular bisector of $\overline{PP'}$, denoted $P_{PP'}$ (see Figure 2, left). Every point of $\Pi^{-1}(q)$ within the star that falls on the same side of l as P will now be the only copies of $\Pi^{-1}(q)$ considered for the shortest path. The portion of the star-shaped region which is on the P side of $P_{PP'}$ is called τ (see Figure 2, right).

There are three points of $\Pi^{-1}(q)$ in τ which we will label as Q_1, Q_2, and Q_3, as shown in Figure 2, right. (If $P_{PP'}$ contains a point of $\Pi^{-1}(q)$ in τ, then it contains another point of $\Pi^{-1}(q)$ and either point in $\Pi^{-1}(q)$ in τ can be discarded.) To find $\min\{PQ_i\}$ where $i = 1, 2, 3$, we construct $\tilde{H}_{Q_iQ_j}$ for $i = 1, 2, 3$ and $j \neq i$.

Note that the boundary of $\tilde{H}_{Q_iQ_j}$ is $P_{Q_iQ_j}$. If Q_i is closest to P, then P must lie in $\tilde{H}_{Q_iQ_j} \cap \tilde{H}_{Q_iQ_k}$. Note that if P is equally close to Q_i and Q_j, then P lies in both $\tilde{H}_{Q_iQ_j} \cap \tilde{H}_{Q_iQ_k}$ and $\tilde{H}_{Q_jQ_i} \cap \tilde{H}_{Q_jQ_k}$. In the figure below, a shortest path is realized by $\overline{PQ_3}$. Hence, P lies in $\tilde{H}_{Q_3Q_1} \cap \tilde{H}_{Q_3Q_2}$. In particular, $\Pi(\overline{PQ_3})$ is the minimal geodesic connecting p and q and will traverse faces 1, 2, and 4.



5. Overview

Suppose $\{x, y, z\} \in \mathcal{T}$. Recall that Π is the branched covering map described in Section 3. Thus $\Pi^{-1}(x)$, $\Pi^{-1}(y)$, and $\Pi^{-1}(z)$ contain infinitely many points. Hence, there are also infinitely many distinct Steiner trees connecting points x, y and z. Our goal in this paper is to narrow down the number of combinations in the tiled plane which may realize the solution.

As stated earlier, we will divide our discussion of this problem into three cases:

Case 1: Three points that can be considered to be on one face of $\mathcal{T}$.

Case 2: Three points that can be considered to be on three distinct faces of $\mathcal{T}$.

Case 3: Any configuration of three points that does not fit into the first two cases (i.e., three points that can only be considered to be on two distinct faces).

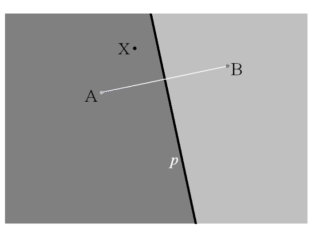
Section 6 will address the simplest case where all three points are on a common face of the tetrahedron. Section 7 will introduce the strategies needed for Sections 8 and 9. In Section 8, we will discuss case 2, and in Section 9 we will discuss case 3. We will discuss how to solve the problem for any specific positioning of the points in Section 10.

6. Case 1: Three points on one face

We know by a theorem proved in [Brune and Sipe 2009] that a shortest path network connecting n points contained on the same face of a regular tetrahedron is contained within that face. Thus, the Steiner minimal tree for three points on the same face of a tetrahedron can be constructed in that face using the algorithm described in Algorithm 2.1.

7. Geometric properties of Steiner minimal trees

Given $a, b, c \in \mathcal{T}$ and the corresponding point sets on the tiled plane, there are many ways that points can be selected, each corresponding to a Steiner tree on $\mathcal{T}$. However, only certain of the combinations realize the Steiner minimal tree on the tetrahedron. The next several results represent strategies that help eliminate fruitless combinations. At this point the reader is encouraged to reread Property 4.1, describing the situation illustrated here:



Lemma 7.1 (perpendicular bisector rule I). *Suppose $A, A' \in \Pi^{-1}(a)$ such that A is on tile T and A' is on tile T'. Then for any point B on T, $AB \le A'B$. If neither A nor B are a common vertex of T and T', then $AB < A'B$.*

Proof. Let $b = \Pi(B)$. Note that a and b are on the same face. We know from a theorem proved in [Brune and Sipe 2009] that a shortest path network connecting n points on the same face is in that same face and here is $\overline{ab}$, which is realized by $\overline{AB}$ in T. Since $\overline{AB}$ is a minimum of all paths $A'B$ where $A' \in \Pi^{-1}(a)$, then for all $A' \neq A$, $AB \le A'B$. If A is not a common vertex of T and T', then $A \neq A'$, so $P_{AA'}$ is defined. If B is not a common vertex of T, then $B \in P_{AA'}$. Thus $AB < A'B$. □

Next, let A, B, and C be points in the tiled plane such that

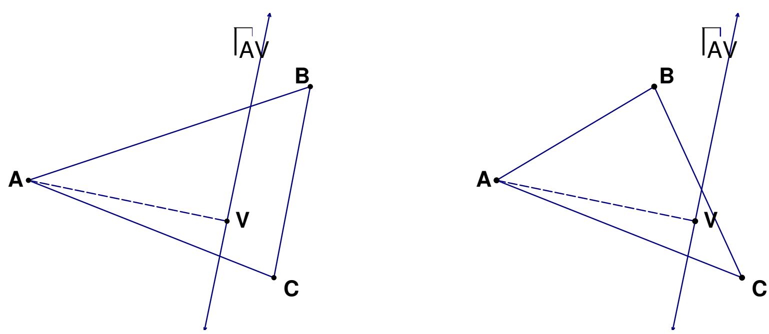
$$\Pi(\mathrm{SMT}(A, B, C)) = \mathrm{SMT}(a, b, c).$$

We will show that the convex hull of the triangular region formed from A, B, and C cannot contain a vertex of the tiled plane unless that vertex is one of A, B, or C. However, before we prove this, we introduce a definition and a property of triangular regions in general.

Definition 7.2. Given two points X and V, let Γ_{XV} be the line perpendicular to $\overline{XV}$ through V.

Lemma 7.3. *Suppose there is a triangular region with vertices A, B, and C that contains the point V in the interior. Then there is an $X \in \{A, B, C\}$ such that Γ_{XV} separates X from $\{A, B, C\} - \{X\}$.*

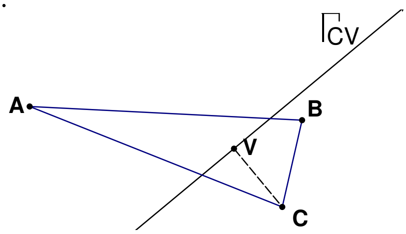
Proof. If Γ_{AV} separates A from BC, the proof is done (left figure):



Otherwise, one of B or C is on the same side of Γ_{AV} as A.

Without loss of generality, suppose B is on the same side of Γ_{AV} as A (right figure). Then $m\angle AVB \le 90°$. Then if Γ_{CV} separates C from A and B, the proof is done.

If not, one of A or B is on the same side of Γ_{CV} as C. In the former case we have $m\angle CVA \le 90°$, while in the latter we have $m\angle CVB \le 90°$. Here is an illustration of the second possibility:



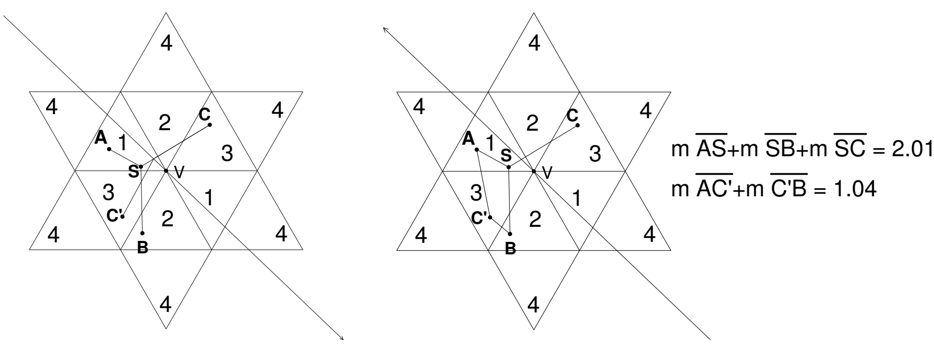
Thus, either $m\angle AVC + m\angle AVB \leq 180°$ or $m\angle CVB + m\angle AVB \leq 180°$. In either case, we are in contradiction with the hypothesis that V is in the interior of $\triangle ABC$. Thus, there exists an $X \in \{A, B, C\}$ such that Γ_{XV} separates X from $\{A, B, C\} - \{X\}$. $\square$

Theorem 7.4 (vertex rule). *Suppose a, b, and $c \in \mathcal{T}$ and*

$$\Pi(\mathrm{SMT}(A, B, C)) = \mathrm{SMT}(a, b, c).$$

Then the image of the convex hull of $\triangle ABC$ under Π cannot contain a vertex v, unless v is one of a, b, or c.

Proof. By way of contradiction, suppose a vertex V of the tiling is contained in the interior of the convex hull of $\triangle ABC$. Construct $\mathrm{SMT}(A, B, C)$, and label the Steiner point S (the Steiner tree may possibly be degenerate). Using Lemma 7.3, we may assume without loss of generality that Γ_{CV} separates C from both A and B. Reflect the part of the path on the C side of Γ_{CV} across Γ_{CV}. Let C' be the reflection of C across Γ_{CV}. Note that the partially reflected path connects A, B, and C' and is equal in length to $\mathrm{SMT}(A, B, C)$. Thus, there is an alternate choice of points in $\Pi^{-1}(a)$, $\Pi^{-1}(b)$, and $\Pi^{-1}(c)$ which is at least as short as $\mathrm{SMT}(A, B, C)$. If S is on the opposite side of Γ_{CV} as C, we can shorten the tree by replacing SC with SC' (see figure on the left). If S is on the same side of Γ_{CV} as C, we can



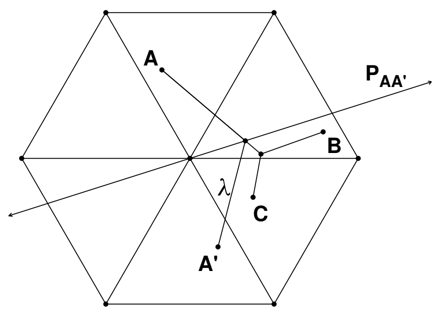
shorten the tree by replacing SA with SA' and SB with SB', where A' and B' are the reflections of A and B across Γ_{CV}, respectively. If S is on Γ_{CV}, then $SC = SC'$, so either tree is the same length. However, the tree containing A, B, and C' will no longer meet the 120° condition for Steiner trees, and will not be $\mathrm{SMT}(A, B, C')$. Thus $\mathscr{L}(\mathrm{SMT}(A, B, C')) < \mathscr{L}(\mathrm{SMT}(A, B, C))$, which implies that $\Pi(\mathrm{SMT}(A, B, C)) \neq \mathrm{SMT}(a, b, c)$. □

Theorem 7.5 (perpendicular bisector rule II). *Let $A, A' \in \Pi^{-1}(a)$ on the tiled plane be distinct. If $P_{AA'}$ separates $\{B, C\}$ from A, then*

$$\mathscr{L}(\mathrm{SMT}(A', B, C)) < \mathscr{L}(\mathrm{SMT}(A, B, C)).$$

Hence, $\Pi(\mathrm{SMT}(A, B, C)) \neq \mathrm{SMT}(a, b, c)$.

Proof. Let λ be the reflection of the part of $\mathrm{SMT}(A, B, C)$ on the A side of $P_{AA'}$ across $P_{AA'}$:

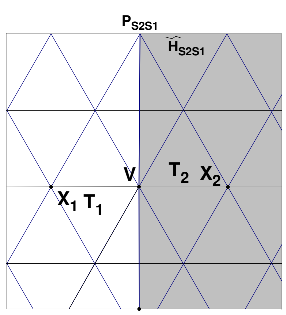


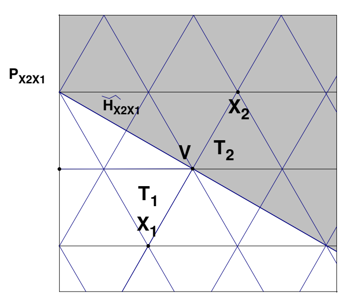
Note that λ uses the point A' as a terminal, thus it is a path network connecting A', B, and C. By a similar argument as in Theorem 7.4, we obtain

$$\mathscr{L}(\mathrm{SMT}(A, B, C)) = \mathscr{L}(\lambda) > \mathscr{L}(\mathrm{SMT}(A, B, C')). \qquad \square$$

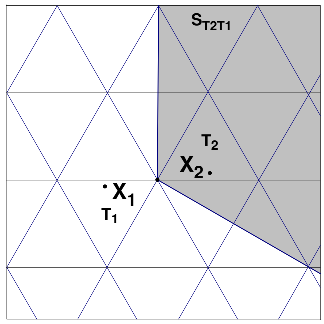
Sectors and half-planes.

Definition 7.6. Fix a vertex V of the tiled plane, and let T_1 and T_2 be tiles (not necessarily adjacent to V) that are mapped to one another with respect to 180° rotation about V. Define the sector $S_{T_2T_1}$ as the intersection of all half-planes $\tilde{H}_{X_2X_1}$, where X_1 runs over all points in T_1 and X_2 is it image under a 180° rotation about V. Clearly $\tilde{H}_{X_2X_1}$ is fully determined by the direction of the vector VX_1; thus by considering two extreme cases for this direction, as here:



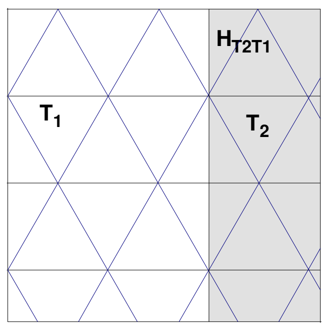


we conclude that the $S_{T_2T_1}$ is the intersection of the half-planes $\tilde{H}_{X_2X_1}$ obtained in these two cases:



Next, if Y and Z are arbitrary points belonging to tiles T_1 and T_2, respectively, we set $S_{YZ} = S_{T_2T_1}$.

Definition 7.7. Let T_1, T_2 be tiles that are translates of each other on the tiled plane, satisfying $\Pi(T_1) = \Pi(T_2)$. Then the intersection of all half-planes $\tilde{H}_{X_2X_1}$ where $X_i \in T_i$ and $\Pi(X_1) = \Pi(X_2)$, is denoted by $H_{T_2T_1}$.



If Y and Z are arbitrary points belonging to tiles T_1 and T_2, respectively, we set $S_{YZ} = S_{T_2T_1}$.

Theorem 7.8 (Steiner point rule). *Let A, B, and C be points in the tiled plane such that* $\Pi(\text{SMT}(A, B, C))$ *is a Steiner minimal tree on the tetrahedron. Suppose that S is the Steiner point of* $\text{SMT}(A, B, C)$. *If S' is any other point of* $\Pi^{-1}(\Pi(s))$, *then $XS \leq XS'$ for $X = A, B,$ and C.*

Proof. Without loss of generality, assume that $X = C$. By way of contradiction, suppose $CS' < CS$. Then there exists a point $C' \in \Pi^{-1}(c)$ such that $CS' = C'S$. This implies that

$$\begin{aligned}\mathcal{L}(\text{SMT}(A, B, C)) &= AS + BS + CS > AS + BS + C'S \\ &\geq \mathcal{L}(\text{SMT}(A, B, C')),\end{aligned}$$

as needed. (See Figure 3 on next page.) □

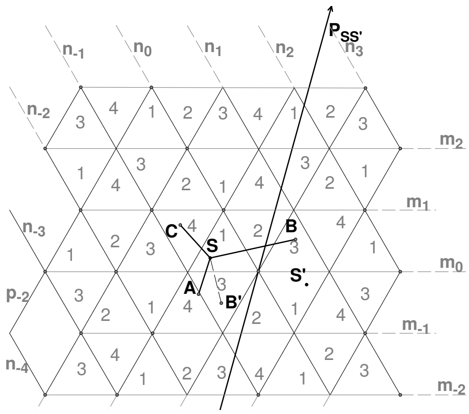


Figure 3. Toward the proof of Theorem 7.8.

8. Case 2: Three points on three distinct faces

When the three points can be viewed to lie on three distinct faces, we use the following procedure to determine the possible configurations of the points on the tiled plane which may realize the Steiner minimal tree. Our arguments apply also when the three points can be viewed to lie on two or one face, as may be the case if one or more of the points lie on vertices or edges. For example, if one point is in the interior of a face, another point is in the interior of another face, and the third point is on a vertex shared by both faces, then we can assign the third point to the third face which shares that vertex, and the configuration is in the realm of Case 2.

Triple ribbon region. Recall the labeling system introduced in Section 3, in which m_i, n_i, and p_i represent the horizontal, negative slope, and positive slope lines, respectively. Also recall that the triangle that is bounded by m_x, n_y, and p_z will be denoted as $T_{(x,y,z)}$.

Let a, b, and c be points on the tetrahedron such that s is the Steiner point for SMT(a, b, c). Let τ_0 be the shaded region in Figure 4. Since τ_0 contains copies of the tiles corresponding to all four faces, a copy of $S \in \Pi^{-1}(s)$ must lie within τ_0.

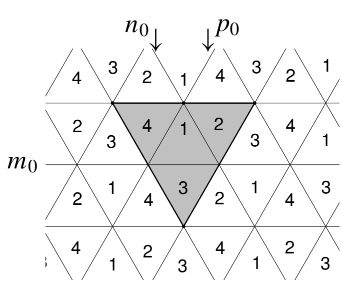
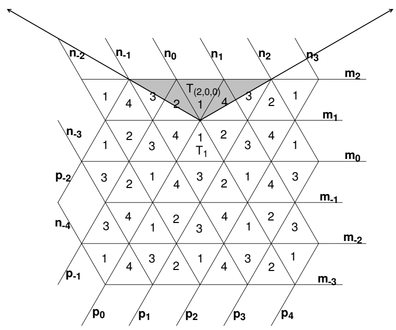


Figure 4. The region τ_0.

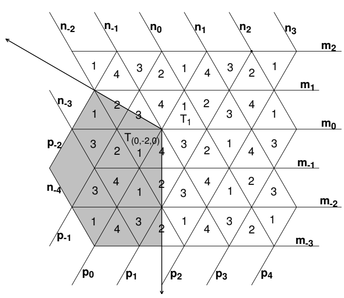
Let $\mathcal{S}^* = \Pi^{-1}(\Pi(s)) - \{S\}$. We will determine a region $\mathcal{R}$ such that given a point $P \in \mathcal{R}$, $PS \leq PS'$ for any $S' \in \mathcal{S}^*$. It follows from Theorem 7.8 that any points not in $\mathcal{R}$ cannot be the fixed points of the Steiner minimal tree that contains S and realizes $\mathrm{SMT}(a, b, c)$.

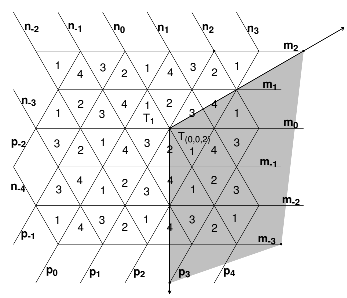
In order to simplify the process, we will first determine the region $\mathcal{R}_i$ that contains all points closer to T_i than to any other tile corresponding to face 1. Then $\mathcal{R} = \bigcup \mathcal{R}_i$. We will call $\mathcal{R} = \bigcup \mathcal{R}_i$ the *triple ribbon region*.

Reductions. Let $i = 1$. Let S' be the 180° rotation of S about the vertex $V = T_1 \cap T_{(2,0,0)}$. Then any point $X \in S_{T_{(2,0,0)}T_1}$ is closer to S' than S. Thus no fixed point is in $S_{T_{(2,0,0)}T_1}$:

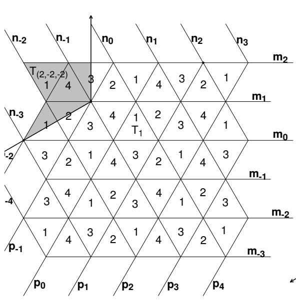


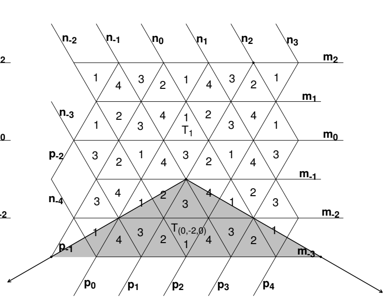
Likewise, no fixed points will be found in $S_{T_{(0,-2,0)}T_1}$ or $S_{T_{(0,0,2)}T_1}$:

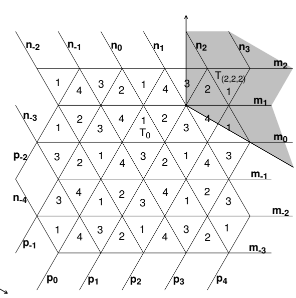




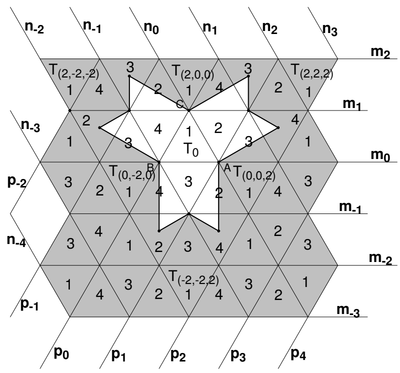
There are also no fixed points to be found in $S_{T_{(2,-2,-2)}T_1}$, $S_{T_{(-2,-2,2)}T_1}$, and $S_{T_{(2,2,2)}T_1}$:







R_1 is the closure of the region remaining when the shaded regions in the six figures of the previous page are cut away. It is shown in white here:



Regions R_2, R_3, and R_4 are found similarly. The union of all these regions, $\mathcal{R} = \bigcup_{i=1}^{4} \mathcal{R}_i$, is the triple ribbon region (Figure 5).

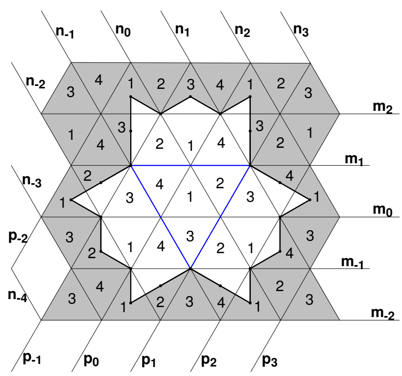


Figure 5. The triple ribbon region (in white).

Regardless of the location of s on the tetrahedron, a copy of $\Pi^{-1}(\mathrm{SMT}(a, b, c))$ is contained within the triple ribbon region. Thus, it is sufficient to check only the combinations of fixed points in the triple ribbon region.

Although the number of potential path networks needed to be checked to find $\mathrm{SMT}(a, b, c)$ is a finite number, it is still a significant number. Note that there are six tiles meeting the triple ribbon region corresponding to face i for $i = 2, 3, 4$. Thus there are $6 \times 6 \times 6 = 216$ combinations to consider given the specification of points in certain faces of τ_0. Hence, we continue to make further reductions.

Horn removal. We subdivide the triple ribbon region as follows. The closure of the bounded white region in Figure 6 (on the next page) is called the *badge region*. The small black triangles, which make up the difference between the triple ribbon region and the badge region, are called the *horns*.

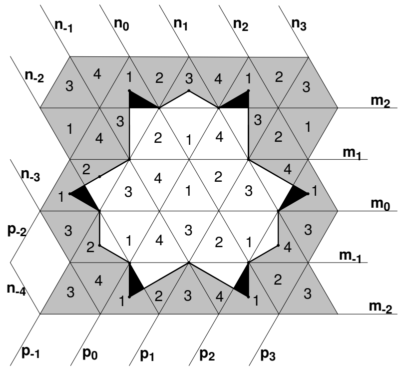


Figure 6. The badge region (closure of the polygon in white) and the horns (in black).

Proposition 8.1. *Suppose a, b, and c are three points on distinct faces of $\mathcal{T}$, none of which are chosen to be face* 1. *Then there is a copy of*

$$\mathrm{SMT}(A, B, C) \in \Pi^{-1}(\mathrm{SMT}(a, b, c))$$

on the tiled plane which is contained in the badge region centered about a tile corresponding to face 1 *with Steiner point S contained in the triangular region τ_0* (see Figure 4).

Proof. Without loss of generality, assume that a is contained on face 3, b is contained on face 4, and c is contained on face 2. Let $A \in \Pi^{-1}(a)$, $B \in \Pi^{-1}(b)$, and $C \in \Pi^{-1}(c)$ lie in the triple ribbon region such that $\Pi(\mathrm{SMT}(A, B, C)) = \mathrm{SMT}(a, b, c)$. Note that no portion of the horns contains any points of $\Pi^{-1}(a)$, $\Pi^{-1}(b)$, or $\Pi^{-1}(c)$ and therefore cannot contain A, B, or C. Let H_1 be the horn bounded by m_2, n_0, and p_{-1} that is outside the badge region.

Suppose an edge of $\mathrm{SMT}(A, B, C)$ meets H_1 outside the badge region. If the interior of an edge passes through either side of the horn not on m_2, the edge must meet the shaded region. But by hypothesis, $\mathrm{SMT}(A, B, C)$ must lie entirely within the triple ribbon region. Thus the edge may only pass through the boundary of the horn on m_2. If so, the only possibility is that one of the endpoints of the edges is contained in H_1. Thus a fixed point is contained in the interior of the horn, and hence contained in the interior of face 1. But by hypothesis, face 1 was not selected as one of the faces containing fixed points. Therefore, an edge of $\mathrm{SMT}(A, B, C)$ does not meet H_1. By a similar argument, $\mathrm{SMT}(A, B, C)$ cannot meet any horn. □

Reduction to the piping region. Using Theorem 7.4 and Theorem 7.5, we will now demonstrate that a lift of the Steiner minimal tree can be contained in a subset of the badge region called the *piping region* (Figure 7). What is left over of the

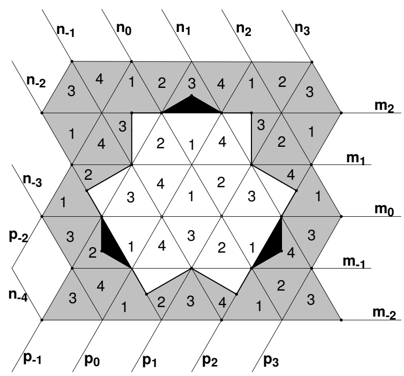


Figure 7. The piping region (closure of the polygon in white) and the flaps (in black).

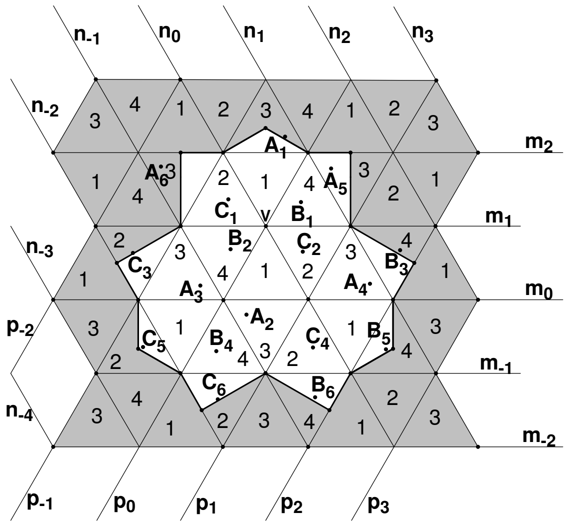
badge region is called the *(top) flaps*. We will show that if SMT(A, B, C) realizes SMT(a, b, c) and is contained in the badge region, then SMT(A, B, C) does not meet the flaps outside the piping region.

Theorem 8.2. *Suppose a, b, and c are three points on distinct faces of $\mathcal{T}$, none of which chosen to be face* 1. *Suppose SMT$(A, B, C) \in \Pi^{-1}(a, b, c)$ is contained in the badge region. Then SMT$(A, B, C) \in \Pi^{-1}(a, b, c)$ is also contained in the piping region centered about a tile corresponding to face* 1.

Proof. Assume the setup given in the proof of Proposition 8.1. We will show that the Steiner minimal tree need not meet any of the flaps. By way of contradiction, suppose that SMT(A, B, C) meets the top flap, the flap contained in $T_{(2,1,-1)}$, outside the piping region. If SMT(A, B, C) meets the top flap, then at least one fixed point or vertex of SMT(A, B, C) must lie above m_2. Note that by construction, S is contained in T_0 and cannot be this point. Since the only tile in the badge region which lies above m_2 is a tile corresponding to face 3, the fixed point must lie in the interior of face 3. Thus, A must lie in the top flap outside the piping region. For the remainder of the argument, we will denote A by A_1 and label the other copies of $\Pi^{-1}(a)$, $\Pi^{-1}(b)$, and $\Pi^{-1}(c)$ contained in tiles meeting the badge region as shown in the figure on the top of the next page. We will show either that any Steiner tree SMT(A_1, B_i, C_j) with S in τ_0 contained within the badge region cannot realize SMT(a, b, c) or that there exists another copy of the tree within the piping region.

We will first determine which combinations cannot realize SMT(a, b, c).Once those combinations are determined, we will show that the remaining combinations have an equivalent copy contained in the piping region.

Construct the sector $S_{A_2A_1}$. If any points B_i and C_j are both contained in $S_{A_2A_1}$, they must both be separated from A_1 by $P_{A_2A_1}$. Thus, by Theorem 7.5, we know that $\Pi(\text{SMT}(A_1, B_i, C_j)) \neq \text{SMT}(a, b, c)$ for B_i and C_j contained in these sectors.



By this argument, the combinations (B_i, C_j), for $i = 4, 5$ and $j = 4, 5, 6$, cannot be used with A_1 to realize SMT(a, b, c).

Construct the half-plane $H_{A_3A_1}$. If any points B_i and C_j are both contained in $H_{A_3A_1}$, they must be separated from A_1 by $P_{A_3A_1}$. Thus, by Theorem 7.5, we know that $\Pi(\text{SMT}(A_1, B_i, C_j)) \neq \text{SMT}(a, b, c)$ for B_i and C_j contained in these sectors. By this argument, the combinations (B_i, C_j), for $i = 4, 6$ and $j = 3, 4, 5, 6$, cannot be used with A_1 to realize SMT(a, b, c).

Construct the half-plane $H_{A_4A_1}$. If any points B_i and C_j are both contained in $H_{A_4A_1}$, they must be separated from A_1 by $P_{A_4A_1}$. Thus, by Theorem 7.5, we know that $\Pi(\text{SMT}(A_1, B_i, C_j)) \neq \text{SMT}(a, b, c)$ for B_i and C_j contained in these sectors. By this argument, the combinations (B_i, C_j), for $i = 3, 4, 5, 6$ and $j = 4, 6$, cannot be used with A_1 to realize SMT(a, b, c).

Consider SMT(A_1, B_1, C_3). Note that both A_1 and B_1 must be contained in $S_{C_1C_3}$. Thus, A_1 and B_1 must be separated from C_3 by $P_{C_1C_3}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_1, C_3)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_1, C_4). Note that both A_1 and B_1 must be contained in $S_{C_2C_4}$. Thus, A_1 and B_1 must be separated from C_4 by $P_{C_2C_4}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_1, C_4)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_1, C_5). Note that both A_1 and B_1 must be contained in $H_{C_1C_5}$. Thus, A_1 and B_1 must be separated from C_5 by $P_{C_1C_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_1, C_5)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_1, C_6). Note that both A_1 and B_1 must be contained in $S_{C_1C_6}$. Thus, A_1 and B_1 must be separated from C_3 by $P_{C_1C_6}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_1, C_6)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_2, C_2). Let V be the intersection of m_1 and n_0. Note that V and A_1 are on the same side of $\overleftrightarrow{B_2C_2}$, V and B_2 are on the same side of $\overleftrightarrow{A_1C_2}$, and V and C_2 are on the same side of $\overleftrightarrow{A_1B_2}$. Thus V is contained in $\triangle A_1B_2C_2$. By Theorem 7.4, $\Pi(\text{SMT}(a, b, c)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_2, C_3). Note that C_3 lies in $H_{A_2A_1}$ and that A_1 lies in $S_{C_1C_3}$. B_2 must lie in at least one of $S_{A_2A_1}$ and $H_{C_1C_3}$. Suppose B_2 lies in $S_{A_2A_1}$. Then both B_2 and C_3 must be separated from A_1 by $P_{A_2A_1}$. If B_2 does not lie in $S_{A_2A_1}$, then B_2 must lie in $H_{C_1C_3}$. But then both B_2 and A_1 must be separated from C_3 by $P_{C_1C_3}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_2, C_3)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_2, C_4). Note that C_4 lies in $H_{A_3A_1}$ and that A_1 lies in $S_{C_2C_4}$. B_2 must lie in at least one of $H_{A_3A_1}$ and $S_{C_2C_4}$. Suppose B_2 lies in $H_{A_3A_1}$. Then both B_2 and C_4 must be separated from A_1 by $P_{A_3A_1}$. If B_2 does not lie in $H_{A_3A_1}$, then B_2 must lie in $S_{C_2C_4}$. But then both B_2 and A_1 must be separated from C_4 by $P_{C_2C_4}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_2, C_4)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_2, C_5). Note that C_5 lies in $H_{A_3A_1}$ and that A_1 lies in $H_{C_1C_5}$. B_2 must lie in at least one of $H_{A_3A_1}$ and $H_{C_1C_5}$. Suppose B_2 lies in $H_{A_3A_1}$. Then both B_2 and C_5 must be separated from A_1 by $P_{A_3A_1}$. If B_2 does not lie in $H_{A_3A_1}$, then B_2 must lie in $H_{C_1C_5}$. But then both B_2 and A_1 must be separated from C_5 by $P_{C_1C_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_2, C_5)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_2, C_6). Note that both A_1 and B_2 must be contained in $S_{C_1C_6}$. Thus, A_1 and B_2 must be separated from C_6 by $P_{C_1C_6}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_2, C_6)) \neq \text{SMT}(a, b, c)$.

We now consider the combinations (A_1, B_i, C_j) for $i = 4, 5, 6$ and $j = 1, 2$. By arguments of symmetry, $\Pi(\text{SMT}(A_1, B_i, C_j)) \neq \text{SMT}(a, b, c)$ for $i = 4, 5, 6$ and $j = 1, 2$.

Consider SMT(A_1, B_3, C_3). Let V be the intersection of m_1 and n_0. Note that V and A_1 are on the same side of $\overleftrightarrow{B_3C_3}$, V and B_3 are on the same side of $\overleftrightarrow{A_1C_3}$, and V and C_3 are on the same side of $\overleftrightarrow{A_1B_3}$. Thus V is contained in $\triangle A_1B_3C_3$. By Theorem 7.4, $\Pi(\text{SMT}(A_1, B_1, C_1)) \neq \text{SMT}(a, b, c)$.

The only remaining cases are (A_1, B_1, C_1), (A_1, B_1, C_2), and (A_1, B_2, C_1). We will show that copies of these trees exist within the piping region. However, we will not claim that the Steiner point S must remain in τ_0.

For (A_1, B_1, C_1), note that $\Pi(\text{SMT}(A_1, B_1, C_1)) = \Pi(\text{SMT}(A_2, B_2, C_2))$ since SMT(A_2, B_2, C_2) is a rotation of SMT(A_1, B_1, C_1) about V. SMT(A_2, B_2, C_2) is contained within the piping region.

For (A_1, B_1, C_2), note that $\Pi(\text{SMT}(A_1, B_1, C_2)) = \Pi(\text{SMT}(A_2, B_2, C_1))$ since SMT(A_2, B_2, C_1) is a rotation of SMT(A_1, B_1, C_2) about V. SMT(A_2, B_2, C_1) is contained within the piping region.

For (A_1, B_2, C_1), note that $\Pi(\text{SMT}(A_1, B_2, C_1)) = \Pi(\text{SMT}(A_2, B_1, C_2))$ since SMT(A_2,B_1,C_2) is a rotation of SMT(A_1,B_2,C_1) about V. Also, SMT(A_2,B_1,C_2)

is contained within the piping region.

Thus, each possible combination (A_1, B_i, C_j) does not realize SMT(a, b, c) or has a copy within the piping region. Likewise, each possible combination involving B_5 or C_5 does not realize SMT(a, b, c) or has a copy within the piping region. Therefore, there is a solution contained in the piping region. □

The region resulting from Theorem 8.2 is the piping region, which we illustrated in Figure 7.

Reduction to the truncated triangle region. We further subdivide the piping region into the *truncated triangle region* and the *side flaps* (Figure 8).

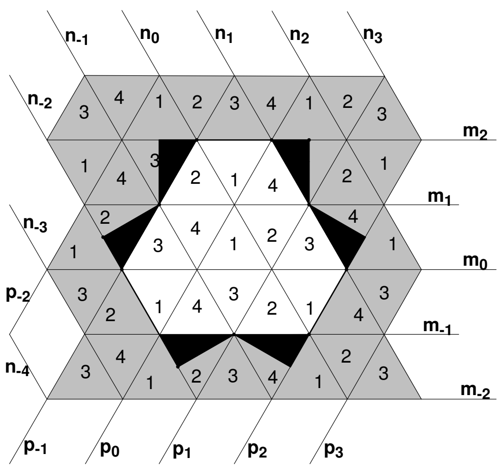
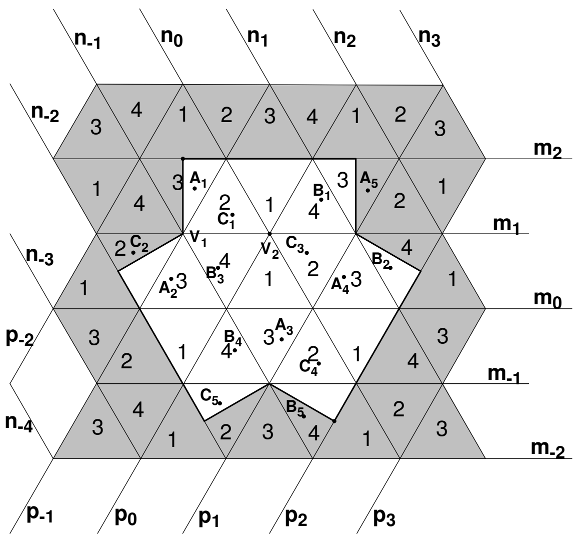


Figure 8. The truncated triangle region (closure of white polygon) and the side flaps (in black).

Theorem 8.3. *Suppose a, b, and c are three points on distinct faces of $\mathcal{T}$, none of which are in the interior of face* 1. *Suppose SMT$(A, B, C) \in \Pi^{-1}(a, b, c)$ is contained in the piping region. Then either SMT$(A, B, C) \in \Pi^{-1}(a, b, c)$ is also contained in the truncated triangle region centered about a tile corresponding to face* 1 *or there is a copy of* SMT(A, B, C) *contained within the truncated triangle region that is a rotation of* SMT(A, B, C).

Proof. Assume the setup in the proof of Proposition 8.1. Without loss of generality, suppose that SMT(A, B, C) is in the piping region. We will show that the Steiner minimal tree need not meet any of the side flaps. Although the final cases of the proof of Theorem 8.2 did not guarantee that S was contained in τ_0, S must be contained in the truncated triangle region. This is because all the trees which could be rotated to lie within the piping region contained fixed points contained within the truncated triangle region. Because S must be contained in the convex hull of the triangular region formed from the fixed points, S must be contained within the truncated triangle region.

By way of contradiction, suppose the Steiner minimal tree meets the flap contained in $T_{(2,-1,-1)}$. If $\mathrm{SMT}(A, B, C)$ meets this side flap, then at least one fixed point or vertex of $\mathrm{SMT}(A, B, C)$ must lie above to the left of p_{-1} and above m_1. Since S is contained in the truncated triangle region (Figure 8), S cannot be this point. Since the only tile in the piping region which lies to the left of p_{-1} and above m_1 is a tile corresponding to face 3, the fixed point must lie in the interior of face 3. Thus, A must lie in the specified side flap outside the truncated triangle region. For the remainder of the proof we will denote A by A_1 and number the other points within the piping region as follows:



Construct the sector $S_{A_2 A_1}$. If any points B_i and C_j are both contained in $S_{A_2 A_1}$, they must both be separated from A_1 by $P_{A_2 A_1}$. Thus, by Theorem 7.5, we know that $\Pi(\mathrm{SMT}(A_1, B_i, C_j)) \neq \mathrm{SMT}(a, b, c)$ for B_i and C_j contained in these sectors. By this argument, the combinations (B_i, C_j), for $i = 4, 5$ and $j = 2, 4, 5$, cannot be used with A_1 to realize $\mathrm{SMT}(a, b, c)$.

Construct the half-plane $H_{A_3 A_1}$. If any points B_i and C_j are both contained in $H_{A_3 A_1}$, they must both be separated from A_1 by $P_{A_3 A_1}$. Thus, by Theorem 7.5, we know that $\Pi(\mathrm{SMT}(A_1, B_i, C_j)) \neq \mathrm{SMT}(a, b, c)$ for B_i and C_j contained in these sectors. By this argument, the combinations (B_i, C_j), for $i = 2, 4, 5$ and $j = 4, 5$, cannot be used with A_1 to realize $\mathrm{SMT}(a, b, c)$.

Construct the sector $S_{A_4 A_1}$. If any points B_i and C_j are both contained in $S_{A_4 A_1}$, they must both be separated from A_1 by $P_{A_4 A_1}$. Thus, by Theorem 7.5, we know that $\Pi(\mathrm{SMT}(A_1, B_i, C_j)) \neq \mathrm{SMT}(a, b, c)$ for B_i and C_j contained in these sectors.

By this argument, the combinations (B_i, C_j), for $i = 2, 5$ and $j = 3, 4$, cannot be used with A_1 to realize $\mathrm{SMT}(a, b, c)$.

For $\mathrm{SMT}(A_1, B_1, C_2)$, note that A_1 and B_1 are contained in $S_{C_1C_2}$, so they are both separated from C_2 by $P_{C_1C_2}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_1, C_2)) \neq \mathrm{SMT}(a, b, c)$.

For $\mathrm{SMT}(A_1, B_1, C_4)$, note that A_1 and B_1 are contained in $S_{C_3C_4}$, so they are both separated from C_4 by $P_{C_3C_4}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_1, C_4)) \neq \mathrm{SMT}(a, b, c)$.

For $\mathrm{SMT}(A_1, B_1, C_5)$, note that A_1 and B_1 are contained in $S_{C_1C_5}$, so they are both separated from C_5 by $P_{C_1C_5}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_1, C_5)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_1, B_3, C_2)$. Let V_1 be the intersection of m_1 and n_{-1}. Note that A_1 and V_1 are on the same side of $\overleftrightarrow{B_3C_2}$, B_3 and V are on the same side of $\overleftrightarrow{A_1C_2}$, and C_2 and V are on the same side of $\overleftrightarrow{A_1B_3}$. Thus, V_1 is contained in $\triangle ABC$. By Theorem 7.4, $\Pi(\mathrm{SMT}(A_1, B_3, C_2)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_1, B_3, C_4)$. Note that A_1 lies in $S_{C_2C_4}$ and C_4 lies in $S_{A_2A_1}$. Note that B_3 must lie in at least one of $S_{C_2C_4}$ and $S_{A_2A_1}$. If B_3 lies in $S_{C_2C_4}$, both B_3 and A_1 must be separated from C_4 by $P_{C_2C_4}$. If B_3 lies in $S_{A_2A_1}$, both B_3 and C_4 must be separated from A_1 by $P_{A_2A_1}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_3, C_4)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_1, B_3, C_5)$. Note that both A_1 and B_3 lie in $S_{C_1C_5}$, so they are both separated from C_5 by $P_{C_1C_5}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_3, C_5)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_1, B_2, C_1)$. Note that both A_1 and C_1 lie in $S_{B_4B_2}$, so they are both separated from B_2 by $P_{B_4B_2}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_2, C_1)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_1, B_2, C_2)$. Note that both A_1 and C_2 lie in $S_{B_4B_2}$, so they are both separated from B_2 by $P_{B_4B_2}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_2, C_2)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_1, B_4, C_1)$. Note that both A_1 and C_1 lie in $S_{B_3B_4}$, so they are both separated from B_4 by $P_{B_3B_4}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_4, C_1)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_1, B_4, C_3)$. Note that A_1 lies in $S_{B_3B_4}$ and B_4 lies in $H_{A_3A_1}$. Note that C_3 must lie in at least one of $S_{B_3B_4}$ and $H_{A_3A_1}$. If C_3 lies in $S_{B_3B_4}$, both A_1 and C_3 are separated from B_4 by $P_{B_3B_4}$. If C_3 lies in $H_{A_1A_3}$, both B_4 and C_3 are separated from A_1 by $P_{A_3A_1}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_4, C_3)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_1, B_5, C_1)$. Note that both A_1 and C_1 lie in $S_{B_5B_4}$, so they are both separated from B_5 by $P_{B_4B_5}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_5, C_1)) \neq \mathrm{SMT}(a, b, c)$.

The only remaining cases are (A_1, B_1, C_1), (A_1, B_1, C_3), and (A_1, B_3, C_3). We will show that copies of these trees exist within the truncated triangle region.

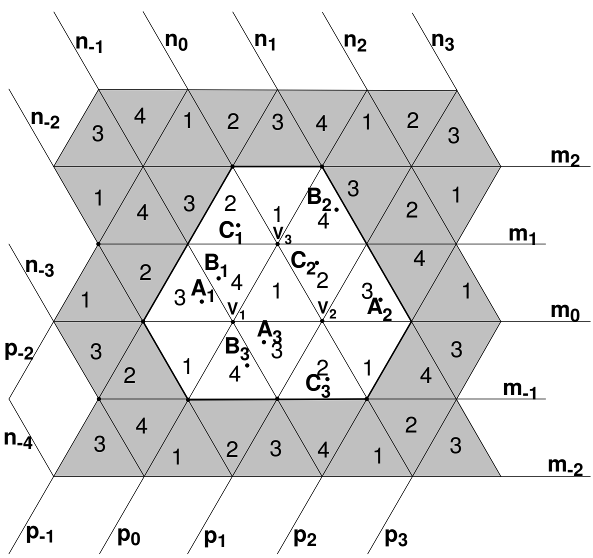
For SMT(A_1, B_1, C_1), we have $\Pi(\text{SMT}(A_1, B_1, C_1)) = \Pi(\text{SMT}(A_4, B_3, C_3))$ and SMT(A_4, B_3, C_3) is contained within the truncated triangle region.

For SMT(A_1, B_1, C_3), we have $\Pi(\text{SMT}(A_1, B_1, C_3)) = \Pi(\text{SMT}(A_4, B_3, C_1))$ and SMT(A_4, B_3, C_1) is contained within the truncated triangle region.

For (A_1, B_3, C_3), we have $\Pi(\text{SMT}(A_1, B_3, C_3)) = \Pi(\text{SMT}(A_4, B_1, C_1))$ and SMT(A_4, B_1, C_1) is contained within the truncated triangle region.

Thus, each possible combination (A_1, B_i, C_j) does not realize SMT(a, b, c) or has a copy within the truncated triangle region. Likewise, each possible combination involving A_5, B_2, B_5, C_5, or C_2 cannot realize SMT(a, b, c) or has a copy within the truncated triangle region. Therefore, there is a solution contained in the truncated triangle region. □

Final reductions. Within the truncated triangle region, there are three copies of every face that contains a terminal point (the center of each region does not contain any points; in this scenario, face 1). That means that there are three copies of each point:



If all combinations of three points were considered possible configurations for the Steiner minimal tree, there would be 27 different Steiner trees that could be considered. However, some of these possibilities may still be eliminated.

There are three remaining combinations that can be eliminated within the truncated triangle region. Let V_1 be the intersection of m_0 and n_{-2}, V_2 be the intersection of m_0 and n_0, and V_3 be the intersection of m_1 and n_0.

Consider SMT(A_1, B_3, C_2). Since A_1 and V_1 are on the same side of $\overleftrightarrow{B_3C_2}$, B_3 and V_1 are on the same side of $\overleftrightarrow{A_1C_2}$, and C_2 and V_1 are on the same $\overleftrightarrow{A_1B_3}$, then V_1 is contained in the interior of $\triangle A_1B_3C_2$. By Theorem 7.4, $\Pi(\mathrm{SMT}(A_1, B_3, C_2)) \neq \mathrm{SMT}(a, b, c)$.

Consider SMT(A_2, B_1, C_3). Since A_2 and V_2 are on the same side of $\overleftrightarrow{B_1C_3}$, B_1 and V_2 are on the same side of $\overleftrightarrow{A_2C_3}$, and C_3 and V_2 are on the same $\overleftrightarrow{A_2B_1}$, then V_2 is contained in the interior of $\triangle A_2B_1C_3$. By Theorem 7.4, $\Pi(\mathrm{SMT}(A_2, B_1, C_3)) \neq \mathrm{SMT}(a, b, c)$.

Consider SMT(A_3, B_2, C_1). Since A_3 and V_3 are on the same side of $\overleftrightarrow{B_2C_1}$, B_2 and V_3 are on the same side of $\overleftrightarrow{A_3C_1}$, and C_1 and V_3 are on the same $\overleftrightarrow{A_3B_2}$, then V_3 is contained in the interior of $\triangle A_3B_2C_1$. By Theorem 7.4, $\Pi(\mathrm{SMT}(A_3, B_2, C_1)) \neq \mathrm{SMT}(a, b, c)$.

List of potential combinations in case 2. The remaining possibilities are

$$\begin{array}{lll}(A_1, B_1, C_1), & (A_2, B_1, C_1), & (A_3, B_1, C_1),\\ (A_1, B_1, C_2), & (A_2, B_1, C_2), & (A_3, B_1, C_2),\\ (A_1, B_1, C_3), & (A_2, B_2, C_1), & (A_3, B_1, C_3),\\ (A_1, B_2, C_1), & (A_2, B_2, C_2), & (A_3, B_2, C_2),\\ (A_1, B_2, C_2), & (A_2, B_2, C_3), & (A_3, B_2, C_3),\\ (A_1, B_2, C_3), & (A_2, B_3, C_1), & (A_3, B_3, C_1),\\ (A_1, B_3, C_1), & (A_2, B_3, C_2), & (A_3, B_3, C_2),\\ (A_1, B_3, C_3), & (A_2, B_3, C_3), & (A_3, B_3, C_3).\end{array}$$

Thus, the Steiner tree which realizes SMT(a, b, c) will be formed from one of the 24 combinations in this list.

9. Case 3: Three points on two faces

In this section, we consider the cases that haven't been addressed in the other sections, namely where three points lie on two faces and cannot be considered to lie on three faces or a single face. The two remaining possibilities are:

(1) Two of the points are contained in the interior of one face with the third point anywhere not meeting that face.

(2) One point is contained in the interior of a face f, a second point is contained in the interior of an edge adjacent to f, and the final point is in the complement of f.

The arguments for both are the same.

We will assume a and b are on the same face and that at least a is in the interior of the face. Thus either b is in the interior of the face or in the interior of an edge of the face.

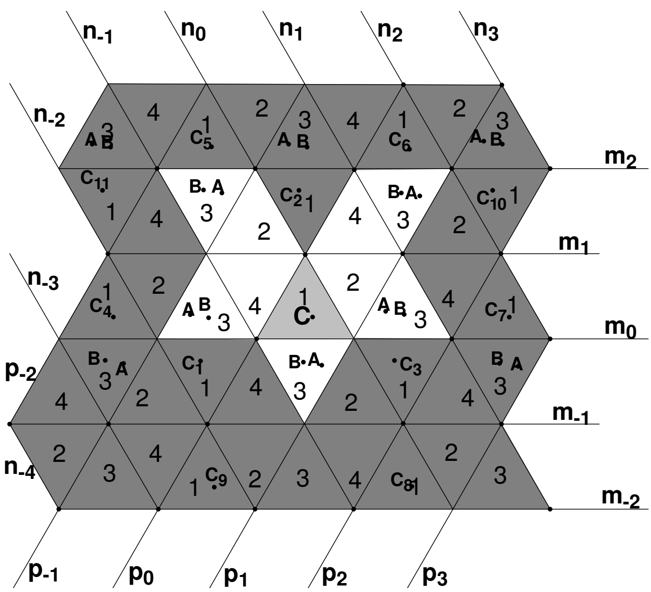
On the tiled plane, there are infinitely many copies of $A \in \Pi^{-1}(a)$ and $B \in \Pi^{-1}(b)$. Suppose $\mathrm{SMT}(A, B, C)$ realizes $\mathrm{SMT}(a, b, c)$. Then either A and B reside on the same tile, or they don't. We will discuss each case separately. We will discuss the former case here and the latter starting on page 392.

A and B on the same tile. In this case, the following theorem provides a region containing the fixed points that can realize $\mathrm{SMT}(a, b, c)$:

Theorem 9.1. *Let $a, b, c \in \mathcal{T}$ and assume*

$$A \in \Pi^{-1}(a), \quad B \in \Pi^{-1}(b), \quad C \in \Pi^{-1}(c)$$

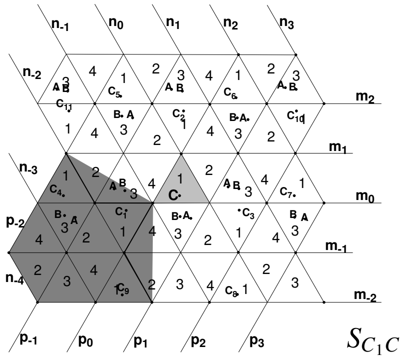
are the points that determine $\mathrm{SMT}(a, b, c)$. *If A and B are on the same tile*, *the Steiner minimal tree must be contained in the ten-triangle region shown here in white and light gray*:

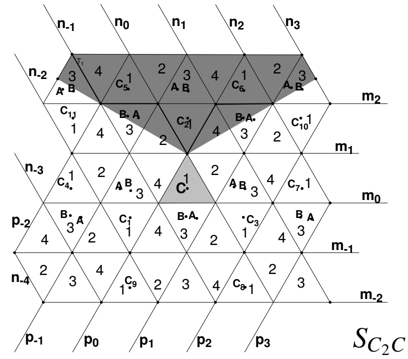


Proof. We can assume without loss of generality that suppose c is on face 1, while a and b are on face 3. We suppose that C is contained in the light gray tile in the figure above.

<u>*Case 1:*</u> Suppose C is not on a vertex of a tile. The other copies of $C_i \in \Pi^{-1}(c)$ are located on the other tiles corresponding to face 1. We number them as in the figure above. We will now determine the tiles on which A and B could possibly reside.

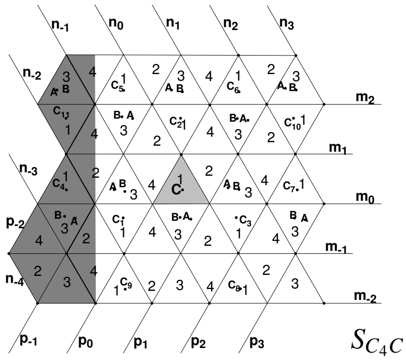
Construct S_{C_1C}. The points A_i and B_j which lie in S_{C_1C} must be separated from C by P_{C_1C}. By Theorem 7.5, $\mathrm{SMT}(A_i, B_j, C)$ cannot realize $\mathrm{SMT}(a, b, c)$. Thus, we can eliminate from consideration as a candidate for containing A and B any tiles whose interior overlaps the region S_{C_1C}, which we show in dark gray (left):

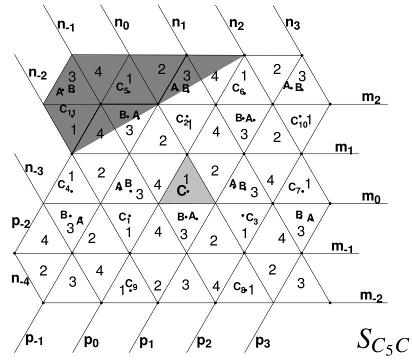


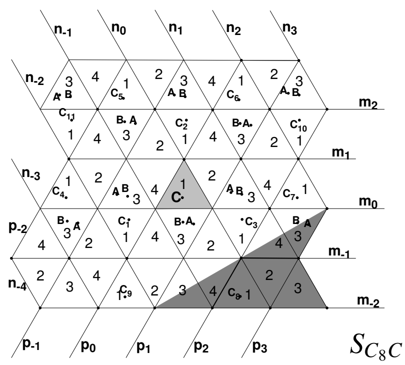


Construct S_{C_2C}. Again, using Theorem 7.5, we can eliminate any tile contained in S_{C_2C}, which is the reason shown in dark gray in the figure above and to the right.

Continue the process by constructing the sectors S_{C_iC}, where $i = 3, \ldots, 9$. Three of these are shown below, while the other four are obtainable by reflection in a vertical line (through the central triangle) from others already illustrated: S_{C_3C} from S_{C_1C}, S_{C_6C} from S_{C_5C}, S_{C_7C} from S_{C_4C}, and S_{C_9C} from S_{C_8C}.





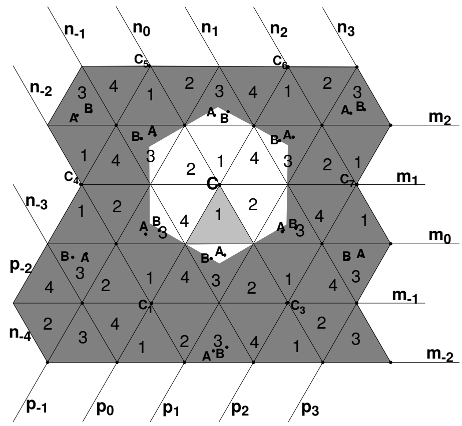


The only copies of tile 3 not completely covered by the union of the shaded regions are those contained in the white region in the statement of Theorem 9.1.

By hypothesis, A and B are contained on the same tile. The convex hull of $\triangle ABC$ contains the tree realizing $\mathrm{SMT}(a, b, c)$. The white region is the minimal collection of tiles containing all such possible convex hulls. Since there are five tiles containing copies of A and B in this region, there are five potential Steiner trees which must be tested within this region.

<u>Case 2:</u> Suppose C is a vertex of a tile. It can only be the vertex at the intersection of m_1 and n_0, because the other vertices are adjacent to tiles containing A and B, and this case has already been addressed in Section 6.

Construct $\tilde{H}_{C_i C}$ for $i = 1, 3, 4, 5, 6, 7$. The union of the added "union of the" shaded regions $\tilde{H}_{C_i C}$ is shown here:



If both A_j and B_k lie in any $\tilde{H}_{C_i C}$, they must both be separated from C by $P_{C_i C}$. By Theorem 7.5, A_j and B_j cannot be used with C to realize $\mathrm{SMT}(a, b, c)$. Note that at least one of A and B must lie in the unshaded region, and A and B are on the same tile by hypothesis. Thus, there are six possible path networks that connect C with a pair of points A_j and B_k which are contained on the same tile where at least one is not in the shaded region. Since each path has one identical path by reflection across C, there are only three distinct paths, and there exists a copy of each in the region stated in Theorem 9.1. □

A and B not on the same tile. We now study the case that A and B are not on the same tile. This will occupy us through page 399. We will determine the faces where the Steiner point can reside in Theorem 9.2. We will then find the region that must contains the fixed points. We will eliminate possibilities for fixed points in Theorems 9.3–9.6. We will then make final reductions and list the combinations that could realize $\mathrm{SMT}(a, b, c)$.

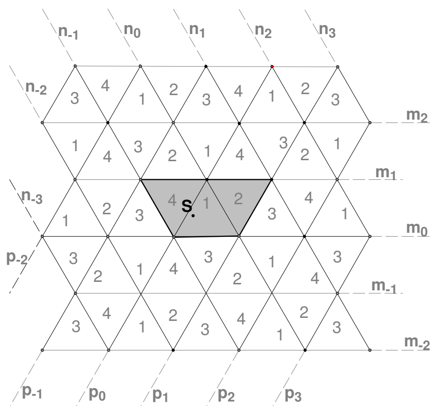
Theorem 9.2. *Assume the setup in Theorem 9.1. Suppose that s is the Steiner point for* $\mathrm{SMT}(a, b, c)$. *If A and B are not found on the same tile, then s can not be on the face containing a and b, including the interior of its edges.*

Proof. By way of contradiction, suppose s is on the same face as a and b. Suppose $S \in \Pi^{-1}(s)$ is contained in the region bounded by n_{-1}, m_1, and p_1. Without loss of generality, c is on face 4, and a and b are on face 3.

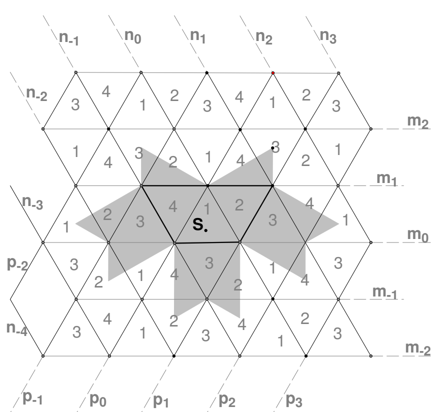
Case 1: Suppose S is not on the same tile as B. Then there exists a distinct point $S' \in \Pi^{-1}(s)$ on the same tile as B. By Lemma 7.1, $S'B < SB$. Then $P_{SS'}$ separates B from S. By Theorem 7.8, S cannot be the Steiner point.

Case 2: Suppose S is on the same tile as B, but S is not a vertex. Then there exists an $S' \in \Pi^{-1}(s)$ on the same tile as A. Since S is not a vertex, $S' \neq S$. Then $P_{SS'}$ separates A from S. By Theorem 7.8, S cannot be the Steiner point. □

It follows from Theorem 9.2 that s must be contained on at least one of faces 1, 2, or 4. Since S cannot be on any tile corresponding to face 3, we can fix S in the shaded region bounded by m_1, m_0, n_{-1}, and p_1, which we call the *key trapezoid*:

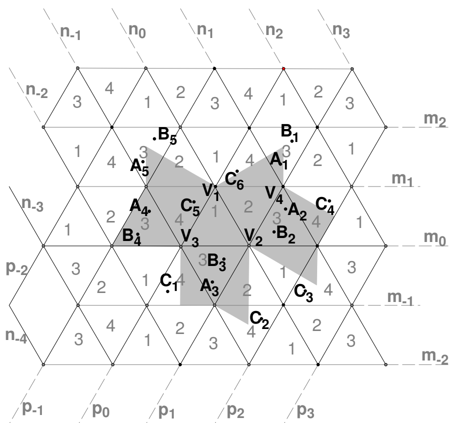


By a similar procedure to that discussed on pages 378 and following, we can eliminate all points lying in the sectors $S_{S'S}$ or half-planes $H_{S'S}$ for all $S' \neq S$, where $S' \in \Pi^{-1}(s)$. The resulting region is this:



Because no terminals are located on faces 1 or 2, the Steiner tree will never cross the copies of face 1 or 2 whose interior meets the edge of this region. We can eliminate these to obtain the region shaded in the figure below. Within this region, there are a maximum of four copies of A, four copies of B, and five copies of C, resulting in a maximum of 80 possible Steiner trees. However, we can reduce the region even further.

Theorem 9.3. *Suppose S is contained in the key trapezoid* (page 393)*. Let C_1 and A_i, B_j (with $i, j = 1, \ldots, 5$) lie in the triangles specified in this diagram*:



Then $\Pi(\mathrm{SMT}(A_i, B_j, C_1)) \neq \mathrm{SMT}(a, b, c)$ for all i, j with $i \neq j$. Hence, the tile containing C_1 can be removed from the region of interest.

Proof. The last assertion follows immediately once we show that no combination (A_i, B_j, C_1) which can be used to realize $\mathrm{SMT}(a, b, c)$. We analyze each case:

Consider $\mathrm{SMT}(A_1, B_2, C_1)$. Both B_2 and C_1 are contained in $S_{A_2A_1}$. Thus B_2 and C_1 are separated from A_1 by $P_{A_2A_1}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_2, C_1)) \neq \mathrm{SMT}(a, b, c)$. Similarly, $\Pi(\mathrm{SMT}(A_2, B_1, C_1)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_1, B_3, C_1)$. Both B_3 and C_1 are contained in $H_{A_3A_1}$. Thus B_3 and C_1 are separated from A_1 by $P_{A_3A_1}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_3, C_1)) \neq \mathrm{SMT}(a, b, c)$. Similarly, $\Pi(\mathrm{SMT}(A_3, B_1, C_1)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_1, B_4, C_1)$. Both B_4 and C_1 are contained in $H_{A_4A_1}$. Thus B_4 and C_1 are separated from A_1 by $P_{A_4A_1}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_4, C_1)) \neq \mathrm{SMT}(a, b, c)$. Similarly, $\Pi(\mathrm{SMT}(A_4, B_1, C_1)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_1, B_5, C_1)$. Both A_1 and B_5 are contained in $S_{C_5C_1}$. Thus A_1 and B_5 are separated from C_1 by $P_{C_5C_1}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_1, B_5, C_1)) \neq \mathrm{SMT}(a, b, c)$. Similarly, $\Pi(\mathrm{SMT}(A_5, B_1, C_1)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_2, B_3, C_1)$. Both B_3 and C_1 are contained in $S_{A_3A_2}$. Thus B_3 and C_1 are separated from A_1 by $P_{A_3A_2}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_2, B_3, C_1)) \neq \mathrm{SMT}(a, b, c)$. Similarly, $\Pi(\mathrm{SMT}(A_3, B_2, C_1)) \neq \mathrm{SMT}(a, b, c)$.

Consider SMT(A_2, B_4, C_1). We claim that V_3 is contained in the interior of $\triangle A_2B_4C_1$. Note that V_3 and C_1 are on the same side of $\overleftrightarrow{A_2B_4}$, V_3 and B_4 are on the same side of $\overleftrightarrow{A_2C_1}$, and V_3 and A_2 are on the same side of $\overleftrightarrow{A_2B_4}$. Thus $\triangle A_2B_4C_1$ contains V_3. By Theorem 7.4, $\Pi(\text{SMT}(A_2, B_4, C_1)) \neq \text{SMT}(a, b, c)$. Similarly, $\Pi(\text{SMT}(A_4, B_2, C_1)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_2, B_5, C_1). Both A_2 and C_1 are contained in $H_{B_3B_5}$. Thus A_2 and C_1 are separated from B_5 by $P_{B_3B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_2, B_5, C_1)) \neq \text{SMT}(a, b, c)$. Similarly, $\Pi(\text{SMT}(A_5, B_2, C_1)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_3, B_4, C_1). Both A_3 and C_1 are contained in $S_{B_3B_4}$. Thus A_3 and C_1 are separated from B_4 by $P_{B_3B_4}$. By Theorem 7.5, $\Pi(\text{SMT}(A_3, B_4, C_1)) \neq \text{SMT}(a, b, c)$. Similarly, $\Pi(\text{SMT}(A_3, B_4, C_1)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_3, B_5, C_1). Both A_3 and C_1 are contained in $H_{B_3B_5}$. Thus A_3 and C_1 are separated from B_5 by $P_{B_3B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_3, B_5, C_1)) \neq \text{SMT}(a, b, c)$. Similarly, $\Pi(\text{SMT}(A_5, B_2, C_1)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_4, B_5, C_1). Both A_4 and C_1 are contained in $S_{B_4B_5}$. Thus A_4 and C_1 are separated from B_5 by $P_{B_4B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_4, B_5, C_1)) \neq \text{SMT}(a, b, c)$. Similarly, $\Pi(\text{SMT}(A_5, B_4, C_1)) \neq \text{SMT}(a, b, c)$. □

Theorem 9.4. *Suppose S is contained in the key trapezoid* (page 393)*. Let C_2 and A_i, B_j (with $i, j = 1, \dots, 5$) lie in the triangles specified in the diagram of Theorem 9.3. Then $\Pi(\text{SMT}(A_i, B_j, C_2)) \neq \text{SMT}(a, b, c)$ for all i, j with $i \neq j$. That is, the tile containing C_2 can be removed from the region of interest.*

Proof. Again we apply a case-by-case analysis.

Consider SMT(A_1, B_2, C_2). Both B_2 and C_2 are contained in $S_{A_2A_1}$. Thus, B_2 and C_2 are separated from A_1 by $P_{A_1A_2}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_2, C_2)) \neq (\text{SMT}(a, b, c))$. By a similar argument, $\Pi(\text{SMT}(A_2, B_1, C_2)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_3, C_2). Both B_3 and C_2 are contained in $H_{A_3A_1}$. Thus, B_3 and C_2 are separated from A_1 by $P_{A_1A_3}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_3, C_2)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_3, B_1, C_2)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_4, C_2). Both B_4 and C_2 are contained in $H_{A_3A_1}$. Thus, B_4 and C_2 are separated from A_1 by $P_{A_1A_3}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_4, C_2)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_4, B_1, C_2)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_5, C_2). We claim that V_1 is contained in the interior of $\triangle A_1B_5C_2$. Both C_2 and V_1 lie on the same side of $\overleftrightarrow{A_1B_5}$, B_5 and V_1 lie on the same side of $\overleftrightarrow{A_1C_2}$, and A_1 and V_1 lie on the same side of $\overleftrightarrow{B_5C_2}$. Thus V_1 must be contained in the interior of $\triangle A_1B_5C_2$. By Theorem 7.4, $\Pi(\text{SMT}(A_1, B_5, C_2)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_1, C_2)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_2, B_3, C_2). Recall that S are contained in the convex hull of $\triangle A_2B_3C_2$. By hypothesis, S is contained in the key trapezoid (page 393). These two conditions are satisfied only if $\overline{A_2B_3}$ lies above the vertex V_2. Thus, C_2 and

V_2 lie on the same side of $\overleftrightarrow{A_2B_3}$, B_3 and V_2 lie on the same side of $\overleftrightarrow{A_2C_2}$, and A_2 and V_2 lie on the same side of $\overleftrightarrow{B_3C_2}$. Thus V_2 are contained in the interior of $\triangle A_2B_3C_2$. By Theorem 7.4, $\Pi(\text{SMT}(A_2, B_3, C_2)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_3, B_2, C_2)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_2, B_4, C_2)$. Both A_2 and C_2 are contained in $S_{B_3B_4}$. Thus, A_2 and C_2 are separated from B_4 by $P_{B_3B_4}$. By Theorem 7.5, $\Pi(\text{SMT}(A_2, B_4, C_2)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_4, B_2, C_2)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_2, B_5, C_2)$. Both A_2 and C_2 are contained in $H_{B_3B_4}$. Thus, A_2 and C_2 are separated from B_5 by $P_{B_2B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_2, B_5, C_2)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_2, C_2)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_3, B_4, C_2)$. Both A_3 and C_2 are contained in $S_{B_3B_4}$. Thus, A_3 and C_2 are separated from B_4 by $P_{B_3B_4}$. By Theorem 7.5, $\Pi(\text{SMT}(A_3, B_4, C_2)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_4, B_3, C_2)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_3, B_5, C_2)$. Both A_3 and C_2 are contained in $H_{B_3B_5}$. Thus, A_3 and C_2 are separated from B_5 by $P_{B_3B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_3, B_5, C_2)) \neq \text{SMT}(a, b, c)$. By a similar argument,$\Pi(\text{SMT}(A_5, B_3, C_2)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_4, B_5, C_2)$. Both A_4 and C_2 are contained in $S_{B_4B_5}$. Thus, A_4 and C_2 are separated from B_5 by $P_{B_4B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_4, B_5, C_2)) \neq \text{SMT}(a, b, c)$. By a similar argument,$\Pi(\text{SMT}(A_5, B_4, C_2)) \neq \text{SMT}(a, b, c)$. □

Theorem 9.5. *Suppose S is contained in the key trapezoid* (page 393)*. Let C_4 and A_i, B_j (with $i, j = 1, \dots, 5$) lie in the triangles specified in the diagram of Theorem 9.3. Then $\Pi(\text{SMT}(A_i, B_j, C_4)) \neq \text{SMT}(a, b, c)$ for all i, j with $i \neq j$. That is, the tile containing C_4 can be removed from the region of interest.*

Proof. Consider $\text{SMT}(A_1, B_2, C_4)$. Both B_2 and C_4 are contained in $S_{A_2A_1}$, so B_2 and C_4 are separated from A_1 by $P_{A_1A_2}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_2, C_4)) \neq \text{SMT}(a, b, c))$. By a similar argument, $\Pi(\text{SMT}(A_2, B_1, C_4)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_1, B_3, C_4)$. Let V_4 be the intersection of m_1 and p_1. Note that C_4 and V_4 lie on the same side of $\overleftrightarrow{A_1B_3}$, B_3 and V_4 lie on the same side of $\overleftrightarrow{A_1C_4}$, and A_1 and V_4 lie on the same side of $\overleftrightarrow{B_3C_4}$. Thus V_4 are contained in the interior of $\triangle A_1B_3C_4$. By Theorem 7.4, $\Pi(\text{SMT}(A_1, B_3, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_3, B_1, C_4)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_1, B_4, C_4)$. Both A_1 and C_4 are contained in $H_{B_1B_4}$. Thus, A_1 and C_4 are separated from B_4 by $P_{B_1B_4}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_4, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_4, B_1, C_4)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_1, B_5, C_4)$. Both A_1 and C_4 are contained in $H_{B_1B_5}$. Thus, A_1 and C_4 are separated from B_5 by $P_{B_1B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_5, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_1, C_4)) \neq \text{SMT}(a, b, c)$.

Consider $\text{SMT}(A_2, B_3, C_4)$. Both A_2 and C_4 are contained in $S_{B_2B_3}$. Thus, A_2

and C_4 are separated from B_3 by $P_{B_2B_3}$. By Theorem 7.5, $\Pi(\text{SMT}(A_2, B_3, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_3, B_2, C_4)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_2, B_4, C_4). Both A_2 and C_4 are contained in $H_{B_2B_4}$. Thus, A_2 and C_4 are separated from B_4 by $P_{B_2B_4}$. By Theorem 7.5, $\Pi(\text{SMT}(A_2, B_4, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_4, B_2, C_4)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_2, B_5, C_4). Both A_2 and C_4 are contained in $H_{B_2B_5}$. Thus, A_2 and C_4 are separated from B_5 by $P_{B_2B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_2, B_5, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_2, C_4)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_3, B_4, C_4). Both A_3 and C_4 are contained in $S_{B_3B_4}$. Thus, A_3 and C_4 are separated from B_4 by $P_{B_3B_4}$. By Theorem 7.5, $\Pi(\text{SMT}(A_3, B_4, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_4, B_3, C_4)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_3, B_5, C_4). Both A_3 and C_4 are contained in $H_{B_3B_5}$. Thus, A_3 and C_4 are separated from B_5 by $P_{B_3B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_3, B_5, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_3, C_4)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_4, B_5, C_4). Both A_4 and B_5 are contained in $H_{C_5C_4}$. Thus, A_4 and B_5 are separated from C_4 by $P_{C_5C_4}$. By Theorem 7.5, $\Pi(\text{SMT}(A_4, B_5, C_4)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_4, C_4)) \neq \text{SMT}(a, b, c)$. □

Theorem 9.6. *Suppose S is contained in the key trapezoid* (page 393). *Let C_3 and A_i, B_j (with $i, j = 1, \dots, 5$) lie in the triangles specified in the diagram of Theorem 9.3. Then $\Pi(\text{SMT}(A_i, B_j, C_3)) \neq \text{SMT}(a, b, c)$ for all i, j with $i \neq j$. That is, the tile containing C_4 can be removed from the region of interest.*

Proof. Consider SMT(A_1, B_2, C_3). Both B_2 and C_3 are contained in $S_{A_2A_1}$, so B_2 and C_3 are separated from A_1 by $P_{A_1A_2}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_2, C_3)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_2, B_1, C_3)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_3, C_3). Assume B_3 is not a vertex. We claim that V_2 is contained in $\triangle A_1B_3C_2$. Let V_2 be the intersection of m_0 and p_1. Note that V_2 and C_3 are on the same side of $\overleftrightarrow{A_1B_3}$, V_2 and B_3 are on the same side of $\overleftrightarrow{A_1C_3}$, and V_2 and A_1 are on the same side of $\overleftrightarrow{B_3C_3}$. Thus $\triangle A_1B_3C_3$ must contain V_2. By Theorem 7.4, $\Pi(\text{SMT}(A_1, B_3, C_3)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_3, B_1, C_3)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_4, C_3). Both A_1 and B_4 are contained in $H_{C_6C_3}$. Thus, A_1 and B_4 are separated from C_3 by $P_{C_6C_3}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_4, C_3)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_4, B_1, C_3)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_5, C_3). Both A_1 and C_3 are contained in $H_{B_1B_5}$. Thus, A_1 and C_3 are separated from B_5 by $P_{B_1B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_5, C_3)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_1, B_5, C_3)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_2, B_3, C_3). Recall that S are contained in the convex hull of $\triangle A_2B_3C_3$. By hypothesis, S is contained in the key trapezoid (page 393). These two conditions are satisfied only if $\overline{A_2B_3}$ lies above the vertex V_2, the intersection of m_0 and p_1. Thus, C_3 and V_2 lie on the same side of $\overleftrightarrow{A_2B_3}$, B_3 and V_2 lie on the same side of $\overleftrightarrow{A_2C_3}$, and A_2 and V_2 lie on the same side of $\overleftrightarrow{B_3C_3}$. Thus V_2 are contained in the interior of $\triangle A_2B_3C_3$. By Theorem 7.4, $\Pi(\mathrm{SMT}(A_2, B_3, C_3)) \neq \mathrm{SMT}(a, b, c)$. By a similar argument, $\Pi(\mathrm{SMT}(A_3, B_2, C_3)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_2, B_4, C_3)$. Both A_2 and C_3 are contained in $H_{B_2B_4}$. Thus, A_2 and C_3 are separated from B_4 by $P_{B_2B_4}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_2, B_4, C_3)) \neq \mathrm{SMT}(a, b, c)$. By a similar argument, $\Pi(\mathrm{SMT}(A_4, B_2, C_3)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_2, B_5, C_3)$. Both A_2 and C_3 are contained in $H_{B_2B_5}$. Thus A_2 and C_3 are separated from B_5 by $P_{B_2B_5}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_2, B_5, C_3)) \neq \mathrm{SMT}(a, b, c)$. By a similar argument, $\Pi(\mathrm{SMT}(A_5, B_2, C_3)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_3, B_4, C_3)$. Both A_3 and C_3 are contained in $S_{B_3B_4}$. thus, A_3 and C_3 are separated from B_4 by $P_{B_3B_4}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_3, B_4, C_3)) \neq \mathrm{SMT}(a, b, c)$. By a similar argument, $\Pi(\mathrm{SMT}(A_4, B_3, C_3)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_3, B_5, C_1)$. Both A_3 and C_3 are contained in $H_{B_3B_5}$. Thus, A_3 and C_3 are separated from B_5 by $P_{B_3B_5}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_3, B_5, C_3)) \neq \mathrm{SMT}(a, b, c)$. By a similar argument, $\Pi(\mathrm{SMT}(A_5, B_3, C_3)) \neq \mathrm{SMT}(a, b, c)$.

Consider $\mathrm{SMT}(A_4, B_5, C_3)$. Both A_4 and B_5 are contained in $H_{C_1C_3}$. Thus A_4 and B_5 are separated from C_3 by $P_{C_1C_3}$. By Theorem 7.5, $\Pi(\mathrm{SMT}(A_4, B_5, C_3)) \neq \mathrm{SMT}(a, b, c)$. By a similar argument, $\Pi(\mathrm{SMT}(A_4, B_3, C_3)) \neq \mathrm{SMT}(a, b, c)$. □

The final region of interest, after the removal of the tiles containing C_1, C_2, C_3 and C_4, is shown in Figure 9. This region also contains each of the five Steiner trees that can be considered when A and B are on the same tile (see Theorem 9.1).

Final reductions. The region shown in Figure 9 must contain at least one copy of the tree $\mathrm{SMT}(A, B, C)$ that realizes $\mathrm{SMT}(a, b, c)$ where A and B come from different tiles. Within this region, there are still combinations that can never realize

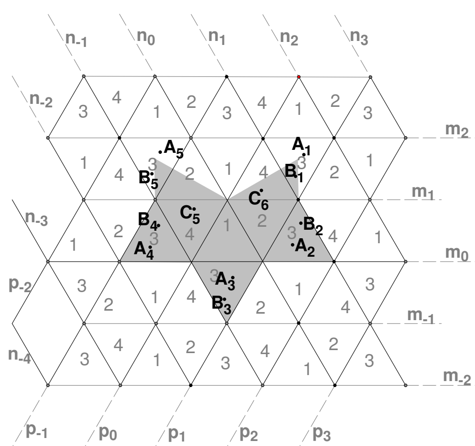


Figure 9. The final region of interest.

SMT(a, b, c) and thus do not need to be considered. In this section we will eliminate these combinations and then provide a list of all the trees SMT(A_i, B_j, C_k) that must be considered to determine the SMT(A, B, C) realizing SMT(a, b, c).

Consider SMT(A_1, B_5, C_6). Both B_5 and C_6 lie in $H_{A_1A_5}$. Thus, B_5 and C_6 must be separated from A_5 by $P_{A_1A_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_5, C_6)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_1, C_6)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_2, B_5, C_6). Both A_2 and C_6 lie in $S_{B_2B_5}$. Thus, A_2 and C_6 must be separated from B_5 by $P_{B_2B_5}$. By Theorem 7.5, $\Pi(\text{SMT}(A_2, B_5, C_6)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_2, C_6)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_3, B_5, C_6), and let $V_1 = m_1 \cap n_0$. Then A_3 and V_1 are on the same side of $\overleftrightarrow{B_5C_6}$, B_5 and V_1 on the same side of $\overleftrightarrow{A_3C_6}$, and C_6 and V_1 on the same side of $\overleftrightarrow{A_3B_5}$. Thus, $V_1 \subset \triangle A_3B_5C_6$. By Theorem 7.4, $\Pi(\text{SMT}(A_3, B_5, C_6)) \neq \text{SMT}(a, b, c)$. Similarly, $\Pi(\text{SMT}(A_5, B_3, C_6)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_4, B_5, C_6). Note that if B_5 is within the shaded region (which is required for it to even be considered), then both B_5 and A_4 lie in $S_{C_5C_6}$. Thus, B_5 and A_4 are separated from C_6 by $P_{C_5C_6}$. By Theorem 7.5, $\Pi(\text{SMT}(A_4, B_5, C_6)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_4, C_6)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_4, C_5). Both B_4 and C_5 lie in $S_{A_4A_1}$. Thus, both B_4 and C_5 must be separated from A_1 by $P_{A_4A_1}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_4, C_5)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_4, B_1, C_6)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_2, B_5, C_5). Both B_5 and C_5 lie in $S_{A_5A_2}$. Thus, both B_5 and C_5 must be separated from A_2 by $P_{A_5A_2}$. By Theorem 7.5, $\Pi(\text{SMT}(A_2, B_5, C_5)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_2, C_6)) \neq \text{SMT}(a, b, c)$.

Consider SMT(A_1, B_5, C_5). Note that if B_5 is within the shaded region, then both B_5 and C_5 are contained in $S_{A_4A_1}$. Thus, B_5 and C_5 are separated from A_1 by $P_{A_4A_1}$. By Theorem 7.5, $\Pi(\text{SMT}(A_1, B_5, C_5)) \neq \text{SMT}(a, b, c)$. By a similar argument, $\Pi(\text{SMT}(A_5, B_1, C_6)) \neq \text{SMT}(a, b, c)$.

List of potential combinations in Case 3. The remaining combinations (A_i, B_j, C_k) for both A and B on the same tile and A and B not on the same tile are

$(A_1, B_2, C_6) \cong (A_4, B_5, C_5)$,	$(A_2, B_1, C_6) \cong (A_5, B_4, C_5)$,	(A_1, B_3, C_6),
(A_3, B_1, C_6),	(A_1, B_4, C_6),	(A_4, B_1, C_6),
(A_2, B_3, C_6),	(A_3, B_2, C_6),	(A_2, B_4, C_6),
(A_4, B_2, C_6),	(A_3, B_4, C_6),	(A_4, B_3, C_6),
(A_2, B_1, C_5),	(A_1, B_2, C_5),	(A_1, B_3, C_5),
(A_3, B_1, C_5),	(A_2, B_3, C_5),	(A_3, B_2, C_5),
(A_2, B_4, C_5),	(A_4, B_2, C_5),	(A_3, B_4, C_5),
(A_4, B_3, C_5),	(A_3, B_5, C_5),	(A_5, B_3, C_5),
(A_2, B_2, C_5),	(A_3, B_3, C_5),	$(A_4, B_4, C_5) \cong (A_1, B_1, C_6)$,
$(A_5, B_5, C_5) \cong (A_2, B_2, C_6)$,	(A_3, B_3, C_6).	

Thus, the Steiner tree which realizes SMT(a, b, c) will be formed from one of the 29 combinations included in this list.

10. An algorithm for finding a shortest network on three points

At the end of Sections 8 and 9 we provided lists of combinations which could realize SMT(a, b, c) for the different cases. In this section we discuss how these lists can be further reduced by considerations of the specific positioning of the points within the faces. We provide two principles upon which the reductions are based. We also provide an algorithm that uses these principles. When the algorithm is applied, we have found that most point combinations can be eliminated.

Two principles allow us to eliminate potential combinations of points from consideration:

- We demonstrated that for Case 2 a solution must reside in the truncated triangle region (Figure 8) and for Case 3 it must resided in the shaded region in Figure 9. In either case, if a point lies outside the corresponding region, no combinations involving that particular point need to be considered.
- If any two points of a combination are separated from the third point by the perpendicular bisector of the third point and a rotation and/or translation of the third point, that combination does not need to be considered (see Theorem 7.5). Recall from Definition 4.2 that for any points P and Q, $\tilde{H}_{PQ} = \{X \mid PX \leq QX\}$. Thus, equivalently, if A and B are contained in $\tilde{H}_{C'C}$ for some $C, C' \in \Pi^{-1}(c)$, then (A, B, C) does not need to be considered.

Using these principles, point combinations within the list can be eliminated from consideration. A systematic approach to the elimination is introduced in the following algorithm.

Algorithm 10.1. The following algorithm provides a shortest network connecting three given points on a regular tetrahedron $\mathcal{T}$.

(1) Determine whether Case 1, 2, or 3 applies.

<u>*Case 1:*</u> If all three points can be considered to lie on a common face, the Steiner tree is just a shortest network on that face (Section 6), and the Steiner tree can be constructed using Algorithm 2.1. The algorithm is complete.

<u>*Case 2:*</u> If the three points can be considered to lie on distinct faces of $\mathcal{T}$, define the region of interest to be the truncated triangle region (Figure 8). Define the list of potential combinations to be the list on page 389. Label the faces so that the face not considered to contain any points is face 1. Proceed to Steps (2)–(4).

<u>*Case 3:*</u> Otherwise, define the shaded region to be that shown in Figure 9. Define the list of potential combinations to be the list on page 399. Label the faces so that the face considered to contain two points is face 3, and the face considered to contain one point is face 4. Proceed to Steps (2)–(4).

(2) Eliminate any combinations within the list of potential combinations that contain points which are not contained within the shaded region.

(3) For all C_m contained in the shaded region:

  (a) For all $C_i \neq C_m$ in the shaded region, construct $\tilde{H}_{C_i C_m}$. Eliminate any combinations (A_k, B_l, C_m) where A_k and B_l are both contained in $\tilde{H}_{C_i C_m}$.

  (b) For the remaining B_l that appear in combinations which have not yet been eliminated:

    (i) For all $B_i \neq B_l$ in the shaded region, construct $\tilde{H}_{B_i B_l}$. If both C_m and A_k are contained in $\tilde{H}_{B_i B_l}$ for any B_l, eliminate the combination (A_k, B_l, C_m).

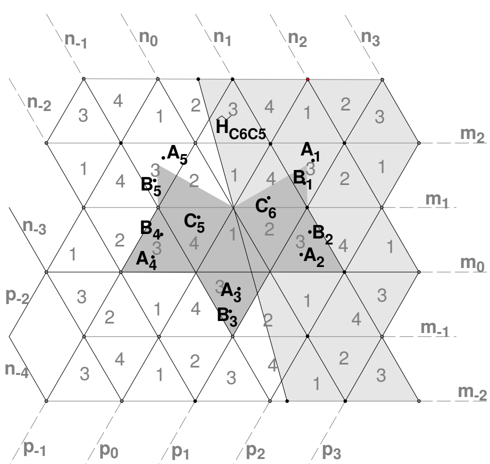
    (ii) For the A_k that appear in a remaining combination with B_l and C_m: For all $A_i \neq A_k$ in the shaded region construct $\tilde{H}_{A_i A_k}$. If both C_m and B_l are contained in $\tilde{H}_{A_i A_k}$, eliminate the combination (A_k, B_l, C_m).

(4) Measure the lengths of the Steiner minimal trees formed from the remaining combinations using Algorithm 2.1. The Steiner minimal tree with shortest length realizes SMT(a, b, c). The algorithm is complete.

We will now demonstrate how to apply the algorithm for the configuration shown in Figure 9, which clearly corresponds to Case 3.

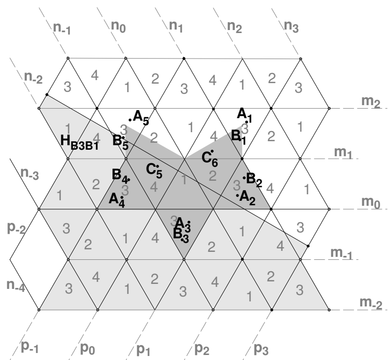
B_5, A_1 and A_5 are not contained within the shaded region, so none of (A_1, B_2, C_5), (A_1, B_3, C_6), (A_1, B_4, C_6), (A_5, B_4, C_5), (A_5, B_3, C_5), (A_3, B_5, C_5) and (A_4, B_5, C_5) need to be considered.

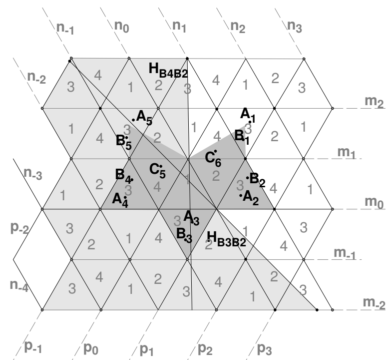
Construct $\tilde{H}_{C_6 C_5}$:



Since both A_2 and B_2 are contained in $\tilde{H}_{C_6C_5}$, the combination (A_2, B_2, C_5) can be eliminated.

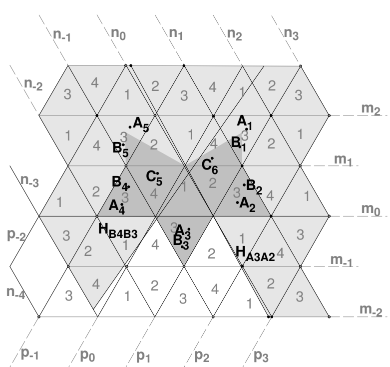
Construct $\tilde{H}_{B_iB_1}$ for all $i \neq 1$ (left diagram). Since C_5 and A_3 are contained in $\tilde{H}_{B_3B_1}$, the combination (A_3, B_1, C_5) can be eliminated. There are no remaining combinations which use both B_1 and C_5.

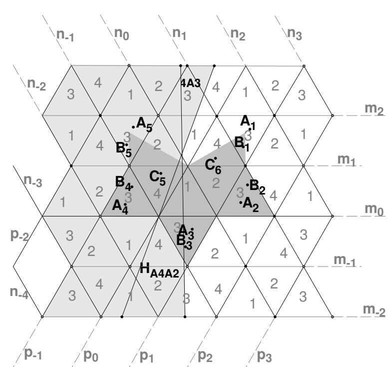




Construct $\tilde{H}_{B_iB_2}$ for all $i \neq 2$ (right diagram above). Since both C_5 and A_3 are contained in $\tilde{H}_{B_3B_2}$, the combination (A_3, B_2, C_5) can be eliminated. Since both C_5 and A_4 are contained in $\tilde{H}_{B_4B_2}$, the combination (A_4, B_2, C_5) can be eliminated. There are no remaining combinations which use both B_2 and C_5.

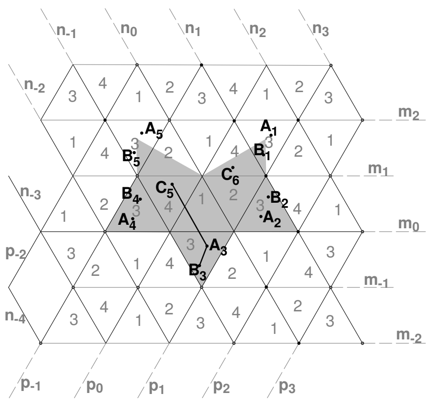
Construct $\tilde{H}_{B_iB_3}$ for all $i \neq 3$ (left diagram below). Since both C_5 and A_4 are contained in $\tilde{H}_{B_4B_3}$, the combination (A_4, B_3, C_5) can be eliminated. The only remaining combinations the list are (A_2, B_3, C_5) and (A_3, B_3, C_5). However, since both C_5 and B_3 are contained in $\tilde{H}_{A_3A_2}$, (A_2, B_3, C_5) can be eliminated.



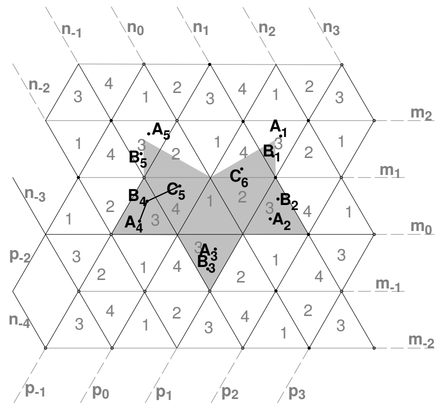


Construct $\tilde{H}_{B_iB_4}$ for all $i \neq 4$. Since C_5 is not contained in any $\tilde{H}_{B_iB_4}$ with $i \neq 4$, the remaining possibilities from the above list are (A_2, B_4, C_5), (A_3, B_4, C_5), and (A_4, B_4, C_5). Since both C_5 and B_4 are contained in $\tilde{H}_{A_4A_2}$, (A_2, B_4, C_5) can be eliminated (right diagram immediately above). Since both C_5 and B_4 are contained in $\tilde{H}_{A_4A_3}$, (A_3, B_4, C_5) can be eliminated.

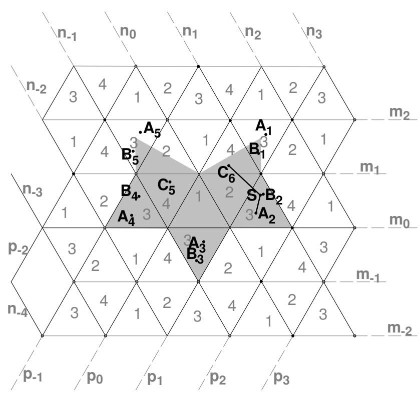
We have shown that the only remaining combinations in the list containing C_5 are (A_3, B_3, C_5) and (A_4, B_4, C_5). Using a similar procedure, we can show that the only remaining combination containing C_6 is (A_2, B_2, C_6). Assuming $\mathcal{T}$ has edge-length 1, we construct the Steiner trees associated with each of these combinations, with the following results:



$\mathcal{L}(\mathrm{SMT}(A_3, B_3, C_5)) = 1.04$



$\mathcal{L}(\mathrm{SMT}(A_4, B_4, C_5)) = 0.87$



$\mathcal{L}(\mathrm{SMT}(A_2, B_2, C_6)) = 1.43$

Hence, $\mathrm{SMT}(A_4, B_4, C_5)$ realizes $\mathrm{SMT}(a, b, c)$ with length 0.87, and the algorithm is complete with only three actual measurements.

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