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Potentially eventually positive (PEP) sign patterns were introduced by Berman et al. (*Electron. J. Linear Algebra* **19** (2010), 108–120), where it was noted that a matrix is PEP if its positive part is primitive, and an example was given of a  $3 \times 3$  PEP sign pattern with reducible positive part. We extend these results by constructing  $n \times n$  PEP sign patterns with reducible positive part, for every  $n \geq 3$ .

## 1. Introduction

A *sign pattern matrix* (or *sign pattern*) is a matrix having entries in  $\{+, -, 0\}$ . For a real matrix  $A$ ,  $\text{sgn}(A)$  is the sign pattern having entries that correspond to the signs of the entries in  $A$ . If  $\mathcal{A}$  is an  $n \times n$  sign pattern, the *qualitative class* of  $\mathcal{A}$ , denoted  $Q(\mathcal{A})$ , is the set of all  $A \in \mathbb{R}^{n \times n}$  such that  $\text{sgn}(A) = \mathcal{A}$ , where  $\text{sgn}(A) = [\text{sgn}(a_{ij})]$ ; such a matrix  $A$  is called a *realization* of  $\mathcal{A}$ . Qualitative matrix problems were introduced by Samuelson [1947] in the mathematical modeling of problems from economics. Sign pattern matrices have useful applications in economics, population biology, chemistry and sociology. If  $P$  is a property of a real matrix, then a sign pattern  $\mathcal{A}$  is *potentially P* (or *allows P*) if there is some  $A \in Q(\mathcal{A})$  that has property  $P$ .

The *spectrum* of a square matrix  $A$ , denoted  $\sigma(A)$ , is the multiset of the eigenvalues of  $A$ , and the *spectral radius* of  $A$  is defined as  $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ . Matrix  $A$  has the *strong Perron–Frobenius property* if  $\rho(A) > 0$  is a simple strictly dominant eigenvalue of  $A$  that has a positive eigenvector. A matrix  $A \in \mathbb{R}^{n \times n}$  is *eventually positive* if there exists a  $k_0 \in \mathbb{Z}^+$  such that for all  $k \geq k_0$ ,  $A^k > 0$ , where the inequality is entrywise. Handelman developed the following test for eventual positivity in [Handelman 1981]: a matrix  $A$  is eventually positive if and only if both  $A$  and  $A^T$  satisfy the strong Perron–Frobenius property. If there exists a  $k$  such

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that  $A^k > 0$  and  $A^{k+1} > 0$ , then  $A$  is eventually positive [Johnson and Tarazaga 2004]. A sign pattern  $\mathcal{A}$  is *potentially eventually positive* (PEP) if there exists an eventually positive realization  $A \in \mathcal{Q}(\mathcal{A})$ .

For a sign pattern  $\mathcal{A} = [\alpha_{ij}]$ , define the *positive part* of  $\mathcal{A}$  to be  $\mathcal{A}^+ = [\alpha_{ij}^+]$  and the *negative part* of  $\mathcal{A}$  to be  $\mathcal{A}^- = [\alpha_{ij}^-]$ , where

$$\alpha_{ij}^+ = \begin{cases} + & \text{if } \alpha_{ij} = +, \\ 0 & \text{if } \alpha_{ij} = 0 \text{ or } \alpha_{ij} = -, \end{cases} \quad \alpha_{ij}^- = \begin{cases} - & \text{if } \alpha_{ij} = -, \\ 0 & \text{if } \alpha_{ij} = 0 \text{ or } \alpha_{ij} = +. \end{cases}$$

Clearly  $\mathcal{A} = \mathcal{A}^+ + \mathcal{A}^-$ . For a matrix  $A \in \mathbb{R}^{n \times n}$ , the positive part  $A^+$  of  $A$  and negative part  $A^-$  of  $A$  are defined analogously, and  $A = A^+ + A^-$ .

A *digraph*  $\Gamma = (V, E)$  consists of a finite, nonempty set  $V$  of vertices, together with a set  $E \subseteq V \times V$  of arcs. Note that a digraph allows loops (arcs of the form  $(v, v)$ ) and may have both arcs  $(v, w)$  and  $(w, v)$  but not multiple copies of the same arc. Let  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ . The *digraph of  $A$* , denoted  $\Gamma(A)$ , has vertex set  $\{1, \dots, n\}$  and arc set  $\{(i, j) : a_{ij} \neq 0\}$ . If  $\mathcal{A}$  is a sign pattern, then  $\Gamma(\mathcal{A}) = \Gamma(A)$  where  $A \in \mathcal{Q}(\mathcal{A})$ . A digraph  $\Gamma$  is *strongly connected* if for any two distinct vertices  $v$  and  $w$  of  $\Gamma$ , there is a path in  $\Gamma$  from  $v$  to  $w$ .

A square matrix  $A$  is *reducible* if there exists a permutation matrix  $P$  such that

$$PAP^T = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

where  $A_{11}$  and  $A_{22}$  are nonempty square matrices and  $0$  is a (possibly rectangular) block consisting entirely of zero entries, or  $A$  is the  $1 \times 1$  zero matrix. If  $A$  is not reducible, then  $A$  is called *irreducible*. It is well known that for  $n \geq 2$ ,  $A$  is irreducible if and only if  $\Gamma(A)$  is strongly connected. For a strongly connected digraph  $\Gamma$ , the *index of imprimitivity* is the greatest common divisor of the lengths of the cycles in  $\Gamma$ . A strongly connected digraph is *primitive* if its index of imprimitivity is one; otherwise it is *imprimitive*. The *index of imprimitivity* of a nonnegative sign pattern  $\mathcal{A}$  is the index of imprimitivity of  $\Gamma(\mathcal{A})$  and  $\mathcal{A} \geq 0$  is *primitive* if  $\Gamma(\mathcal{A})$  is primitive, or equivalently, if the index of imprimitivity of  $\mathcal{A}$  is one.

The study of PEP sign patterns was introduced in [Berman et al. 2010], where it was shown that if  $\mathcal{A}^+$  is primitive, then  $\mathcal{A}$  is PEP, and where the first example of a PEP sign pattern with reducible positive part was given: the  $3 \times 3$  pattern

$$\mathcal{B} = \begin{bmatrix} + & - & 0 \\ + & 0 & - \\ - & + & + \end{bmatrix}.$$

In Section 2 we extend the results of [Berman et al. 2010] by generalizing the  $3 \times 3$  pattern  $\mathcal{B}$  given there to a family of PEP sign patterns having reducible positive part for every order  $n \geq 3$ .

In [Section 3](#) we examine the effect of the Kronecker product on PEP sign patterns and obtain another method of constructing PEP sign patterns with reducible positive part.

### 2. A family of sign patterns generalizing $\mathcal{B}$

The sign pattern  $\mathcal{B}$  from [[Berman et al. 2010](#)] was the first PEP sign pattern with a reducible positive part. This sign pattern may be generalized by defining the  $n \times n$  sign pattern

$$\mathcal{B}_n = \begin{bmatrix} + & - & \cdots & - & 0 \\ + & 0 & \cdots & 0 & - \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ + & 0 & \cdots & 0 & - \\ - & + & \cdots & + & + \end{bmatrix}.$$

The following result, which is a special case of the *Schur–Cohn criterion* (see, e.g., [[Marden 1949](#)]), will be used in the proof that  $\mathcal{B}_n$  is PEP.

**Lemma 2.1.** *If the polynomial  $f(x) = x^2 - \beta x + \alpha$  satisfies  $|\beta| < 1 + \alpha < 2$ , then all zeros of  $f(x)$  lie strictly inside the unit circle.*

It is well known that if the characteristic polynomial of  $A$  is  $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$  then  $a_{n-k} = (-1)^k E_k(A)$ , where  $E_k(A)$  is the sum of the  $k \times k$  principal minors of  $A$  (see, e.g., [[Horn and Johnson 1985](#)]).

**Theorem 2.2.** *For  $n \geq 3$  the  $n \times n$  sign pattern  $\mathcal{B}_n$  is PEP.*

*Proof.* For  $t > 0$ , let  $B_n(t)$  be the  $n \times n$  matrix

$$B_n(t) = \begin{bmatrix} 1 + (n-2)t & -t & \cdots & -t & 0 \\ 1 + t & 0 & \cdots & 0 & -t \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 + t & 0 & \cdots & 0 & -t \\ -(n-2)t - \frac{1}{2}t^2 & t & \cdots & t & 1 + \frac{1}{2}t^2 \end{bmatrix}.$$

Then  $B_n(t) \in Q(\mathcal{B}_n)$ , and 1 is an eigenvalue of  $B_n(t)$  with positive right eigenvector  $\mathbb{1}$  (the all ones vector) and positive left eigenvector

$$\mathbf{w} = \left[ \frac{2n-5}{t} \quad 1 \quad \cdots \quad 1 \quad \frac{2n-4}{t} \right]^T.$$

We show that for some choice of  $t > 0$ , 1 is a simple strictly dominant eigenvalue of  $B_n(t)$  and hence  $B_n(t)$  is eventually positive. Since  $1 \in \sigma(B_n(t))$  and  $\text{rank } B_n(t) \leq 3$ , the characteristic polynomial  $p_{B_n(t)}(x)$  of  $B_n(t)$  is of the form

$$p_{B_n(t)}(x) = x^{n-3}(x-1)(x^2 - \beta x + \alpha) = x^n - (1+\beta)x^{n-1} + (\alpha+\beta)x^{n-2} - \alpha x^{n-3}.$$

Computing  $\alpha$  and  $\beta$  using the sums of principal minors to evaluate the characteristic polynomial gives  $\beta = \frac{1}{2}t^2 + (n - 2)t + 1$  and  $\alpha = (n - 2)t(1 + 2t + \frac{1}{2}t^2)$ . For  $n > 3$ , setting  $t = 1/(2(n - 2))$  gives  $|\beta| < 1 + \alpha < 2$ , which, using [Lemma 2.1](#), guarantees that the two nonzero eigenvalues of  $B_n$  other than 1 have modulus strictly less than 1 (recall that a  $3 \times 3$  eventually positive matrix  $B_3 \in \mathcal{Q}(\mathcal{B}_3)$  was given in [\[Berman et al. 2010\]](#) so we have not been concerned with this case in choosing  $t$ ).  $\square$

We illustrate this theorem with an example.

**Example 2.3.** Let  $n = 5$ . Following the proof of [Theorem 2.2](#), we choose  $t = \frac{1}{6}$  and define

$$B_5 = B_5\left(\frac{1}{6}\right) = \frac{1}{6} \begin{bmatrix} 9 & -1 & -1 & -1 & 0 \\ 7 & 0 & 0 & 0 & -1 \\ 7 & 0 & 0 & 0 & -1 \\ 7 & 0 & 0 & 0 & -1 \\ -\frac{37}{12} & 1 & 1 & 1 & \frac{73}{12} \end{bmatrix}.$$

Moreover, we have

$$\begin{aligned} \sigma(B_5) &= \left\{1, \frac{1}{144}(109 + i\sqrt{2087}), \frac{1}{144}(109 - i\sqrt{2087}), 0, 0\right\} \\ &\approx \{1, 0.7569 + 0.3172i, 0.7569 - 0.3172i, 0, 0\}, \end{aligned}$$

and  $[1 \ 1 \ 1 \ 1 \ 1]^T$  and  $[\frac{5}{6} \ \frac{1}{36} \ \frac{1}{36} \ \frac{1}{36} \ 1]^T$  are right and left eigenvectors, respectively, corresponding to  $\rho(B_5) = 1$ . Therefore  $B_5$  and  $B_5^T$  have the strong Perron–Frobenius property, so  $B_5$  is eventually positive by [Handelman’s criterion](#).

In [\[Berman et al. 2010\]](#) it was shown that if the sign pattern  $\mathcal{A}$  is PEP, then any sign pattern achieved by changing one or more zero entries of  $\mathcal{A}$  to be nonzero is also PEP. Applying this to  $\mathcal{B}_n$  yields a variety of additional PEP sign patterns having reducible positive part.

### 3. Kronecker products

The Kronecker product (sometimes called the tensor product) is a useful tool for generating larger eventually positive matrices and thus PEP sign patterns. The *Kronecker product* of  $A = [a_{ij}]$  and  $B = [b_{ij}]$  is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}.$$

It is clear that if  $A > 0$  and  $B > 0$ , then  $A \otimes B > 0$ . The following facts can be found in many linear algebra books; see [\[Reams 2006\]](#), for example. For  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{m \times m}$ ,  $(A \otimes B)^k = A^k \otimes B^k$ . For  $A, C, B, D$  of appropriate dimensions,

we have  $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ . There exists a permutation matrix  $P$  such that  $B \otimes A = P(A \otimes B)P^T$ .

**Proposition 3.1.** *If  $A$  and  $B$  are eventually positive matrices, then  $A \otimes B$  is eventually positive.*

*Proof.* Assume that  $A$  and  $B$  are eventually positive matrices. Since  $A$  and  $B$  are eventually positive, there exists some  $s_0, t_0 \in \mathbb{Z}$ , with  $s_0, t_0 > 0$ , such that for all  $s \geq s_0$  and  $t \geq t_0$ ,  $A^s > 0$  and  $B^t > 0$ . Set  $k_0 = \max\{s_0, t_0\}$ . Then for all  $k \geq k_0$ ,  $(A \otimes B)^k = A^k \otimes B^k > 0$ . □

**Corollary 3.2.** *If  $\mathcal{A}$  and  $\mathcal{B}$  are PEP sign patterns, then  $\mathcal{A} \otimes \mathcal{B}$  is PEP.*

If either  $A$  or  $B$  is a reducible matrix, then  $A \otimes B$  is reducible since, without loss of generality, if

$$PAP^T = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

then

$$(P \otimes I)(A \otimes B)(P \otimes I)^T = \begin{bmatrix} A_{11} \otimes B & 0 \\ A_{21} \otimes B & A_{22} \otimes B \end{bmatrix}.$$

Thus [Corollary 3.2](#) provides another way to construct PEP sign patterns having reducible positive part.

**Example 3.3.** Let

$$B = \frac{1}{100} \begin{bmatrix} 130 & -30 & 0 \\ 130 & 0 & -30 \\ -31 & 30 & 101 \end{bmatrix}.$$

In [[Berman et al. 2010](#)] it was shown that  $B$  is eventually positive, and in fact  $B^k > 0$  for  $k \geq 10$ .

Let  $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ . Then  $A^k > 0$  for  $k \geq 2$ , hence  $A$  is eventually positive.

Then

$$B \otimes A = \frac{1}{100} \begin{bmatrix} 260 & 390 & -60 & -90 & 0 & 0 \\ 130 & 0 & -30 & 0 & 0 & 0 \\ 260 & 390 & 0 & 0 & -60 & -90 \\ 130 & 0 & 0 & 0 & -30 & 0 \\ -62 & -93 & 60 & 90 & 202 & 303 \\ -31 & 0 & 30 & 0 & 101 & 0 \end{bmatrix}.$$

Moreover  $(B \otimes A)^{10} > 0$  and  $(B \otimes A)^{11} > 0$ , so  $B \otimes A$  is eventually positive and  $\text{sgn}(B \otimes A)$  is a PEP sign pattern with reducible positive part.

Any 0 in  $\text{sgn}(B \otimes A)$  from [Example 3.3](#) may be changed to  $-$  to get yet another PEP sign pattern with reducible positive part.

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