Constructions of potentially eventually positive sign patterns with reducible positive part

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Potentially eventually positive (PEP) sign patterns were introduced by Berman et al. (Electron. J. Linear Algebra 19 (2010), 108–120), where it was noted that a matrix is PEP if its positive part is primitive, and an example was given of a $3 \times 3$ PEP sign pattern with reducible positive part. We extend these results by constructing $n \times n$ PEP sign patterns with reducible positive part, for every $n \geq 3$.

1. Introduction

A sign pattern matrix (or sign pattern) is a matrix having entries in \{+,-,0\}. For a real matrix $A$, $\text{sgn}(A)$ is the sign pattern having entries that correspond to the signs of the entries in $A$. If $\mathcal{A}$ is an $n \times n$ sign pattern, the qualitative class of $\mathcal{A}$, denoted $Q(\mathcal{A})$, is the set of all $A \in \mathbb{R}^{n \times n}$ such that $\text{sgn}(A) = \mathcal{A}$, where $\text{sgn}(A) = [\text{sgn}(a_{ij})]$; such a matrix $A$ is called a realization of $\mathcal{A}$. Qualitative matrix problems were introduced by Samuelson [1947] in the mathematical modeling of problems from economics. Sign pattern matrices have useful applications in economics, population biology, chemistry and sociology. If $P$ is a property of a real matrix, then a sign pattern $\mathcal{A}$ is potentially $P$ (or allows $P$) if there is some $A \in Q(\mathcal{A})$ that has property $P$.

The spectrum of a square matrix $A$, denoted $\sigma(A)$, is the multiset of the eigenvalues of $A$, and the spectral radius of $A$ is defined as $\rho(A) = \max\{|\lambda|: \lambda \in \sigma(A)\}$. Matrix $A$ has the strong Perron–Frobenius property if $\rho(A) > 0$ is a simple strictly dominant eigenvalue of $A$ that has a positive eigenvector. A matrix $A \in \mathbb{R}^{n \times n}$ is eventually positive if there exists a $k_0 \in \mathbb{Z}^+$ such that for all $k \geq k_0$, $A^k > 0$, where the inequality is entrywise. Handelman developed the following test for eventual positivity in [Handelman 1981]: a matrix $A$ is eventually positive if and only if both $A$ and $A^T$ satisfy the strong Perron–Frobenius property. If there exists a $k$ such


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that $A^k > 0$ and $A^{k+1} > 0$, then $A$ is eventually positive [Johnson and Tarazaga 2004]. A sign pattern $\mathcal{A}$ is potentially eventually positive (PEP) if there exists an eventually positive realization $A \in Q(\mathcal{A})$.

For a sign pattern $\mathcal{A} = [\alpha_{ij}]$, define the positive part of $\mathcal{A}$ to be $\mathcal{A}^+ = [\alpha^+_{ij}]$ and the negative part of $\mathcal{A}$ to be $\mathcal{A}^- = [\alpha^-_{ij}]$, where

$$
\alpha^+_{ij} = \begin{cases} 
+ & \text{if } \alpha_{ij} = +, \\
0 & \text{if } \alpha_{ij} = 0 \text{ or } \alpha_{ij} = -.
\end{cases}
\quad \alpha^-_{ij} = \begin{cases} 
- & \text{if } \alpha_{ij} = -, \\
0 & \text{if } \alpha_{ij} = 0 \text{ or } \alpha_{ij} = +.
\end{cases}
$$

Clearly $\mathcal{A} = \mathcal{A}^+ + \mathcal{A}^-$. For a matrix $A \in \mathbb{R}^{n \times n}$, the positive part $A^+$ of $A$ and negative part $A^-$ of $A$ are defined analogously, and $A = A^+ + A^-$. A digraph $\Gamma = (V, E)$ consists of a finite, nonempty set $V$ of vertices, together with a set $E \subseteq V \times V$ of arcs. Note that a digraph allows loops (arcs of the form $(v, v)$) and may have both arcs $(v, w)$ and $(w, v)$ but not multiple copies of the same arc. Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$. The digraph of $A$, denoted $\Gamma(A)$, has vertex set $\{1, \ldots, n\}$ and arc set $\{(i, j) : a_{ij} \neq 0\}$. If $\mathcal{A}$ is a sign pattern, then $\Gamma(\mathcal{A}) = \Gamma(A)$ where $A \in Q(\mathcal{A})$. A digraph $\Gamma$ is strongly connected if for any two distinct vertices $v$ and $w$ of $\Gamma$, there is a path in $\Gamma$ from $v$ to $w$.

A square matrix $A$ is reducible if there exists a permutation matrix $P$ such that

$$
PAP^T = \begin{bmatrix} A_{11} & 0 \\
A_{21} & A_{22} \end{bmatrix}
$$

where $A_{11}$ and $A_{22}$ are nonempty square matrices and 0 is a (possibly rectangular) block consisting entirely of zero entries, or $A$ is the $1 \times 1$ zero matrix. If $A$ is not reducible, then $A$ is called irreducible. It is well known that for $n \geq 2$, $A$ is irreducible if and only if $\Gamma(A)$ is strongly connected. For a strongly connected digraph $\Gamma$, the index of imprimitivity is the greatest common divisor of the lengths of the cycles in $\Gamma$. A strongly connected digraph is primitive if its index of imprimitivity is one; otherwise it is imprimitive. The index of imprimitivity of a nonnegative sign pattern $\mathcal{A}$ is the index of imprimitivity of $\Gamma(\mathcal{A})$ and $\mathcal{A} \geq 0$ is primitive if $\Gamma(\mathcal{A})$ is primitive, or equivalently, if the index of imprimitivity of $\mathcal{A}$ is one.

The study of PEP sign patterns was introduced in [Berman et al. 2010], where it was shown that if $\mathcal{A}^+$ is primitive, then $\mathcal{A}$ is PEP, and where the first example of a PEP sign pattern with reducible positive part was given: the $3 \times 3$ pattern

$$
\mathcal{B} = \begin{bmatrix} + & - & 0 \\
+ & 0 & - \\
- & + & + \end{bmatrix}.
$$

In Section 2 we extend the results of [Berman et al. 2010] by generalizing the $3 \times 3$ pattern $\mathcal{B}$ given there to a family of PEP sign patterns having reducible positive part for every order $n \geq 3$. 

In Section 3 we examine the effect of the Kronecker product on PEP sign patterns and obtain another method of constructing PEP sign patterns with reducible positive part.

2. A family of sign patterns generalizing \( \mathcal{B} \)

The sign pattern \( \mathcal{B} \) from [Berman et al. 2010] was the first PEP sign pattern with a reducible positive part. This sign pattern may be generalized by defining the \( n \times n \) sign pattern

\[
\mathcal{B}_n = \begin{bmatrix}
+ & - & \cdots & - & 0 \\
+ & 0 & \cdots & 0 & - \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
+ & 0 & \cdots & 0 & - \\
- & + & \cdots & + & + \\
\end{bmatrix}.
\]

The following result, which is a special case of the Schur–Cohn criterion (see, e.g., [Marden 1949]), will be used in the proof that \( \mathcal{B}_n \) is PEP.

**Lemma 2.1.** If the polynomial \( f(x) = x^2 - \beta x + \alpha \) satisfies \(|\beta| < 1 + \alpha < 2\), then all zeros of \( f(x) \) lie strictly inside the unit circle.

It is well known that if the characteristic polynomial of \( A \) is \( p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \) then \( a_{n-k} = (-1)^k E_k(A) \), where \( E_k(A) \) is the sum of the \( k \times k \) principal minors of \( A \) (see, e.g., [Horn and Johnson 1985]).

**Theorem 2.2.** For \( n \geq 3 \) the \( n \times n \) sign pattern \( \mathcal{B}_n \) is PEP.

**Proof.** For \( t > 0 \), let \( B_n(t) \) be the \( n \times n \) matrix

\[
B_n(t) = \begin{bmatrix}
1 + (n-2)t & -t & \cdots & -t & 0 \\
1 + t & 0 & \cdots & 0 & -t \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 + t & 0 & \cdots & 0 & -t \\
-(n-2)t - \frac{1}{2}t^2 & t & \cdots & t & 1 + \frac{1}{2}t^2 \\
\end{bmatrix}.
\]

Then \( B_n(t) \in Q(\mathcal{B}_n) \), and 1 is an eigenvalue of \( B_n(t) \) with positive right eigenvector \( \mathbb{1} \) (the all ones vector) and positive left eigenvector

\[
w = \begin{bmatrix}
\frac{2n-5}{t} & 1 & \cdots & 1 & \frac{2n-4}{t}
\end{bmatrix}^T.
\]

We show that for some choice of \( t > 0 \), 1 is a simple strictly dominant eigenvalue of \( B_n(t) \) and hence \( B_n(t) \) is eventually positive. Since \( 1 \in \sigma(B_n(t)) \) and rank \( B_n(t) \leq 3 \), the characteristic polynomial \( p_{B_n(t)}(x) \) of \( B_n(t) \) is of the form

\[
p_{B_n(t)}(x) = x^{n-3} (x-1)(x^2 - \beta x + \alpha) = x^n - (1+\beta)x^{n-1} + (\alpha+\beta)x^{n-2} - \alpha x^{n-3}.
\]
Computing $\alpha$ and $\beta$ using the sums of principal minors to evaluate the characteristic polynomial gives $\beta = \frac{1}{2} t^2 + (n - 2) t + 1$ and $\alpha = (n - 2) t \left( 1 + 2 t + \frac{1}{2} t^2 \right)$. For $n > 3$, setting $t = 1/(2(n-2))$ gives $|\beta| < 1 + \alpha < 2$, which, using Lemma 2.1, guarantees that the two nonzero eigenvalues of $B_n$ other than 1 have modulus strictly less than 1 (recall that a $3 \times 3$ eventually positive matrix $B_3 \in Q(\mathcal{B}_3)$ was given in [Berman et al. 2010] so we have not been concerned with this case in choosing $t$).

We illustrate this theorem with an example.

**Example 2.3.** Let $n = 5$. Following the proof of Theorem 2.2, we choose $t = \frac{1}{6}$ and define

$$B_5 = B_5 \left( \frac{1}{6} \right) = \frac{1}{6} \begin{bmatrix} 9 & -1 & -1 & 1 & 0 \\ 7 & 0 & 0 & 0 & -1 \\ 7 & 0 & 0 & 0 & -1 \\ 7 & 0 & 0 & 0 & -1 \\ \frac{37}{12} & 1 & 1 & 1 & \frac{73}{12} \end{bmatrix}.$$ 

Moreover, we have

$$\sigma(B_5) = \{ 1, \frac{1}{144} (109 + i \sqrt{2087}), \frac{1}{144} (109 - i \sqrt{2087}), 0, 0 \} \approx \{ 1, 0.7569 + 0.3172 i, 0.7569 - 0.3172 i, 0, 0 \},$$

and $[1 \ 1 \ 1 \ 1 \ 1]^T$ and $[\frac{5}{6} \ \frac{1}{36} \ \frac{1}{36} \ \frac{1}{36} \ 1]^T$ are right and left eigenvectors, respectively, corresponding to $\rho(B_5) = 1$. Therefore $B_5$ and $B_5^T$ have the strong Perron–Frobenius property, so $B_5$ is eventually positive by Handelman’s criterion.

In [Berman et al. 2010] it was shown that if the sign pattern $\mathcal{A}$ is PEP, then any sign pattern achieved by changing one or more zero entries of $\mathcal{A}$ to be nonzero is also PEP. Applying this to $\mathcal{B}_n$ yields a variety of additional PEP sign patterns having reducible positive part.

### 3. Kronecker products

The Kronecker product (sometimes called the tensor product) is a useful tool for generating larger eventually positive matrices and thus PEP sign patterns. The **Kronecker product** of $A = [a_{ij}]$ and $B = [b_{ij}]$ is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}.$$ 

It is clear that if $A > 0$ and $B > 0$, then $A \otimes B > 0$. The following facts can be found in many linear algebra books; see [Reams 2006], for example. For $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$, $(A \otimes B)^k = A^k \otimes B^k$. For $A$, $C$, $B$, $D$ of appropriate dimensions,
we have \((A \otimes B)(C \otimes D) = (AC) \otimes (BD)\). There exists a permutation matrix \(P\) such that \(B \otimes A = P(A \otimes B)P^T\).

**Proposition 3.1.** If \(A\) and \(B\) are eventually positive matrices, then \(A \otimes B\) is eventually positive.

**Proof.** Assume that \(A\) and \(B\) are eventually positive matrices. Since \(A\) and \(B\) are eventually positive, there exists some \(s_0, t_0 \in \mathbb{Z}\), with \(s_0, t_0 > 0\), such that for all \(s \geq s_0\) and \(t \geq t_0\), \(A^s > 0\) and \(B^t > 0\). Set \(k_0 = \max\{s_0, t_0\}\). Then for all \(k \geq k_0\), \((A \otimes B)^k = A^k \otimes B^k > 0\). \(\square\)

**Corollary 3.2.** If \(A\) and \(B\) are PEP sign patterns, then \(A \otimes B\) is PEP.

If either \(A\) or \(B\) is a reducible matrix, then \(A \otimes B\) is reducible since, without loss of generality, if

\[
PAP^T = \begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix}
\]

then

\[
(P \otimes I)(A \otimes B)(P \otimes I)^T = \begin{bmatrix}
A_{11} \otimes B & 0 \\
A_{21} \otimes B & A_{22} \otimes B
\end{bmatrix}.
\]

Thus Corollary 3.2 provides another way to construct PEP sign patterns having reducible positive part.

**Example 3.3.** Let

\[
B = \frac{1}{100} \begin{bmatrix}
130 & -30 & 0 \\
130 & 0 & -30 \\
-31 & 30 & 101
\end{bmatrix}.
\]

In [Berman et al. 2010] it was shown that \(B\) is eventually positive, and in fact \(B^k > 0\) for \(k \geq 10\).

Let \(A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\). Then \(A^k > 0\) for \(k \geq 2\), hence \(A\) is eventually positive. Then

\[
B \otimes A = \frac{1}{100} \begin{bmatrix}
260 & 390 & -60 & -90 & 0 & 0 \\
130 & 0 & -30 & 0 & 0 & 0 \\
260 & 390 & 0 & 0 & -60 & -90 \\
130 & 0 & 0 & 0 & -30 & 0 \\
-62 & -93 & 60 & 90 & 202 & 303 \\
-31 & 0 & 30 & 0 & 101 & 0
\end{bmatrix}.
\]

Moreover \((B \otimes A)^{10} > 0\) and \((B \otimes A)^{11} > 0\), so \(B \otimes A\) is eventually positive and \(\text{sgn}(B \otimes A)\) is a PEP sign pattern with reducible positive part.

Any 0 in \(\text{sgn}(B \otimes A)\) from Example 3.3 may be changed to \(-\) to get yet another PEP sign pattern with reducible positive part.
References


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