Elliptic curves, eta-quotients and hypergeometric functions

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The well-known fact that all elliptic curves are modular, proven by Wiles, Taylor, Breuil, Conrad and Diamond, leaves open the question whether there exists a nice representation of the modular form associated to each elliptic curve. Here we provide explicit representations of the modular forms associated to certain Legendre form elliptic curves $2E_1(\lambda)$ as linear combinations of quotients of Dedekind’s eta-function. We also give congruences for some of the modular forms’ coefficients in terms of Gaussian hypergeometric functions.

1. Introduction and statement of results

Wiles and Taylor [1995] proved that all semistable elliptic curves over $\mathbb{Q}$ are modular. Their result was later extended by Breuil, Conrad, Diamond and Taylor [Breuil et al. 2001] to all elliptic curves over $\mathbb{Q}$.

This correspondence allows facts about elliptic curves to be proven using modular forms, and vice versa. (See [Koblitz 1993] for more background on the theory of elliptic curves and modular forms.)

Let $E$ be an elliptic curve over $\mathbb{Q}$. If $q := e^{2\pi i z}$, $\text{GF}(p)$ is the finite field with $p$ elements, and $N(p)$ is the number of points on $E$ over $\text{GF}(p)$, then the modularity theorem implies that there exists a corresponding weight-2 newform $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ such that if $p$ is a prime of good reduction, then $a(p) = 1 + p - N(p)$.

For example, if $\eta(z)$ is Dedekind’s eta-function,

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

then the elliptic curves $y^2 = x^3 + 1$ and $y^2 = x^3 - x$ have the corresponding modular forms $\eta(6z)^4$ and $\eta(4z)^2 \eta(8z)^2$, respectively; see [Martin and Ono 1997].


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It is natural to ask which elliptic curves have corresponding modular forms that are quotients of eta-functions. Martin and Ono [1997] have answered this question by listing all such eta-quotients

$$f(z) = \prod_\delta \eta(\delta z)^{r_\delta} \quad (\delta, r_\delta \in \mathbb{Z})$$

which are weight-2 newforms, and they gave corresponding modular elliptic curves. (For more on the theory of eta-quotients, see [Ono 2004, Section 1.4].)

We show, for certain values of $\lambda \in \mathbb{Q} \setminus \{0, 1\}$, that the elliptic curves $2E_1(\lambda)$ defined by

$$2E_1(\lambda): y^2 = x(x-1)(x-\lambda) \quad (1-1)$$

correspond to modular forms which are linear combinations of eta-quotients.

**Remark.** The proof of Theorem 1.1 will make clear how one can generate many more such examples.

Let

$$f_\lambda(z) := \sum_{n=1}^{\infty} 2a_1(n; \lambda)q^n \quad (1-2)$$

be the weight-2 newform corresponding to the elliptic curve $2E_1(\lambda)$. It will be convenient to express eta-quotients using the notation

$$\left[ \prod_\delta \delta^{r_\delta} \right] := \prod_\delta \eta(\delta z)^{r_\delta}. \quad (1-3)$$

For example, in place of $\frac{\eta(2z)^2\eta(4z)^2\eta(5z)\eta(40z)}{\eta(z)\eta(8z)}$ we write $[1^{-1}2^24^25^18^{-1}40^1]$.

**Theorem 1.1.** If $\lambda \in \{\frac{27}{16}, 5, \frac{81}{49}, -\frac{7}{25}\}$, then $2E_1(\lambda)$ corresponds to the modular forms given here:

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>conductor $N$</th>
<th>eta-quotient $f_\lambda(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{27}{16}$</td>
<td>33</td>
<td>$[1^211^2] + 3 \cdot [3^233^2] + 3 \cdot [1^13^111^133^1]$</td>
</tr>
<tr>
<td>5</td>
<td>40</td>
<td>$[1^{-1}2^24^25^18^{-1}40^1] + [1^15^{-1}8^110^220^240^{-1}]$</td>
</tr>
<tr>
<td>$\frac{81}{49}$</td>
<td>42</td>
<td>$2 \cdot [1^{-1}2^23^17^214^{-1}42^1] - 3 \cdot [3^16^121^142^1]$ + $[2^13^26^{-1}7^121^{-1}42^2] + [1^13^{-1}6^214^121^242^{-1}]$</td>
</tr>
<tr>
<td>$-\frac{7}{25}$</td>
<td>70</td>
<td>$[1^{-1}2^25^27^{-1}10^{-1}14^235^270^{-1}] - [1^22^{-1}5^{-1}7^210^214^{-1}35^{-1}70^2]$</td>
</tr>
</tbody>
</table>

We show, for all $\lambda \in \mathbb{Q} \setminus \{0, 1\}$, that the Fourier coefficients of all $f_\lambda(z)$ satisfy an interesting hypergeometric congruence. For a prime $p$ and an integer $n$, define
ord$_p(n)$ to be the power of $p$ dividing $n$, and if $\alpha = \frac{a}{b} \in \mathbb{Q}$, then set ord$_p(\alpha) = \text{ord}_p(a) - \text{ord}_p(b)$. We show that with this notation, the numbers $2a_1(p; \lambda)$ satisfy the following congruences.

**Theorem 1.2.** Let $\lambda \not\in \{0, 1\}$ be rational and let $p = 2f + 1$ be an odd prime such that ord$_p(\lambda(\lambda - 1)) = 0$. Then

$$2a_1(p; \lambda) \equiv (-1)^{\frac{p+1}{2}}(p-1) \sum_{k=0}^{f} \binom{f+k}{k} \binom{f}{k} (-\lambda)^k \pmod{p}.$$ 

**Remarks.** In light of Theorem 1.1, this implies that the congruence in Theorem 1.2 holds for the coefficients of the linear combinations of eta-quotients given above.

- A well-known theorem of Hasse states that for every prime $p$,

$$|a(p)| < 2\sqrt{p}.$$ 

Theorem 1.2 therefore determines $2a_1(p; \lambda)$ uniquely for primes $p > 16$.

**Example.** Consider $\lambda = \frac{27}{16}$. Then $\lambda(\lambda - 1) = \frac{3^3 \cdot 11}{2^5}$ and so for $p \not\in \{2, 3, 11\}$ prime we observe the congruence by inspecting the coefficients of $2E_1(\frac{27}{16})$ for applicable primes $p < 30$, where $B(p; \lambda)$ is defined to be the right-hand side of the congruence in Theorem 1.2:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$2a_1(p; \frac{27}{16})$</th>
<th>$B(p; \frac{27}{16})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$-2 \equiv 3 \pmod{5}$</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>$4 \equiv 4 \pmod{7}$</td>
<td>4</td>
</tr>
<tr>
<td>13</td>
<td>$-2 \equiv 11 \pmod{13}$</td>
<td>11</td>
</tr>
<tr>
<td>17</td>
<td>$-2 \equiv 15 \pmod{17}$</td>
<td>15</td>
</tr>
<tr>
<td>19</td>
<td>$0 \equiv 0 \pmod{19}$</td>
<td>0</td>
</tr>
<tr>
<td>23</td>
<td>$8 \equiv 8 \pmod{23}$</td>
<td>8</td>
</tr>
<tr>
<td>29</td>
<td>$-6 \equiv 23 \pmod{29}$</td>
<td>23</td>
</tr>
</tbody>
</table>

2. Elliptic curves and modular forms

In this section we prove Theorem 1.1. If $E$ is an elliptic curve over $\mathbb{Q}$, then its conductor $N$ is a product of the primes $p$ of bad reduction for $E$, with exponents determined by the extent to which $E$ is singular over GF$(p)$. (An algorithm by Tate for computing conductors is given in [Cremona 1997].) Moreover, the modularity theorem implies that the modular form $f(z)$ corresponding to $E$ is an element of $S_2(\Gamma_0(N))$. In particular, for an elliptic curve $2E_1(\lambda)$, proving the correctness of any representation of $f_\lambda(z)$ in terms of eta-quotients amounts to checking that the given eta-quotients are elements of $S_2(\Gamma_0(N))$ and checking a finite number of coefficients of their Fourier expansions against those of $f_\lambda$. 

We first provide a formula for the dimension of the space of cusp forms of weight 2 and level \( N \), \( S_2(\Gamma_0(N)) \). We then show that the eta-quotients making up the linear combinations are elements of \( S_2(\Gamma_0(N)) \) and use the dimension formula to show that equality of two elements of \( S_2(\Gamma_0(N)) \) always depends only on some finite set of coefficients.

The linear combinations of eta-quotients in this paper were generated by the following algorithm:

1. Given a rational number \( \lambda / \in \{0, 1\} \), compute the conductor \( N \) of \( 2E_1(\lambda) \). (The modular form corresponding to \( 2E_1(\lambda) \) will be an element of \( S_2(\Gamma_0(N)) \).)
2. Compute \( \dim \mathbb{C} S_2(\Gamma_0(N)) \).
3. Generate eta-quotients which are elements of \( S_2(\Gamma_0(N)) \).
4. Attempt to construct a basis for \( S_2(\Gamma_0(N)) \) using these eta-quotients.

Of course, once one is armed with a basis of eta-quotients for \( S_2(\Gamma_0(N)) \), it is simple to express \( f_\lambda(z) \) in terms of this basis.

**Dimension of \( S_2(\Gamma_0(N)) \).** It will be useful to know not only that \( S_2(\Gamma_0(N)) \) is finite-dimensional for every positive integer \( N \), but also its exact dimension \( d_N := \dim \mathbb{C} S_2(\Gamma_0(N)) \).

The following formula for \( d_N \) is a simplification of [Ono 2004, Theorem 1.34], which gives a formula for the quantity \( \dim \mathbb{C} S_k(\Gamma_0(N), \chi) - \dim \mathbb{C} M_{2-k}(\Gamma_0(N), \chi) \), in the case where \( k = 2 \) and \( \chi = \epsilon \) is the trivial character modulo \( N \).

**Proposition 2.1.** If \( N \) is a fixed positive integer and \( r_p := \text{ord}_p(N) \), define

\[
\lambda_p := \begin{cases} p^{r_p \over 2} + p^{r_p \over 2} - 1 & \text{if } r_p \equiv 0 \pmod{2}, \\ 2p^{r_p - \over 2} & \text{if } r_p \equiv 1 \pmod{2}. \end{cases}
\]

With this notation,

\[
d_N = 1 + \frac{N}{12} \prod_{p | N} (1 + p^{-1}) - \frac{1}{2} \prod_{p | N} \lambda_p - \frac{1}{4} \sum_{x \pmod{N}} 1 - \frac{1}{3} \sum_{x \pmod{N}} 1.
\]

**Proof.** This follows from [Ono 2004, Theorem 1.34], noting that the conductor of the trivial character is 1 and that \( M_0(\Gamma_0(N), \epsilon) \) is the space of constant functions and hence has dimension 1.

**Proof of Theorem 1.1.** Let \( N \) be the conductor of \( E = 2E_1(\lambda) \) and let \( d_N = \dim \mathbb{C} S_2(\Gamma_0(N)) \) as before. Conditions under which an eta-quotient is an element of \( S_2(\Gamma_0(N)) \) are provided in [Ono 2004, Theorems 1.64 and 1.65]: If \( f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta} \) is an eta-quotient which vanishes at each cusp of \( \Gamma_0(N) \), such that the pairs \( (\delta, r_\delta) \) satisfy \( \sum_{\delta | N} r_\delta = 4, \sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24}, \) and \( \sum_{\delta | N} {N \over \delta} r_\delta \equiv 0 \pmod{24}, \)
then \( f(z) \in S_2(\Gamma_0(N)) \). The order of vanishing of such an \( f(z) \) at the cusp \( \frac{c}{d} \) is given by [Ono 2004, Theorem 1.65] as

\[
\frac{N}{24} \sum_{d|N} \frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d, \frac{N}{d})d\delta}.
\]

(2-1)

It is straightforward to check that the formula above gives a positive order of vanishing for each eta-quotient at each cusp, that each eta-quotient satisfies the given congruence conditions, and that the \( r_\delta \) of each eta-quotient sum to 4. These conditions guarantee that each eta-quotient appearing in the table above lies in \( S_2(\Gamma_0(N)) \).

The eta-quotients given for \( \lambda = \frac{27}{16} \) form a basis for \( S_2(\Gamma_0(33)) \). Similarly, for \( \lambda = 5 \), the given eta-quotients along with \([2^210^2]\) form a basis; for \( \lambda = \frac{81}{39} \) the given eta-quotients along with \([1^{-1}2^23^26^{-1}7^{-1}14^221^242^{-1}]\) form a basis; and for \( \lambda = -\frac{7}{35} \) a complete basis is

\[
\{[5^27^2], [1^{-1}2^27^210^114^{-1}35^1], [10^214^2], [1^22^15^17^{-1}14^270^1], [1^22^15^{-1}2^110^214^{-1}35^{-1}70^2], [1^15^17^135^1], [1^15^210^{-1}14^{-1}35^{-1}70^2], [5^110^135^170^1], [1^{-1}2^22^17^135^{-1}70^2]\}.
\]

To see this, let \( g_{i,j} \) be the \( j \)-th Fourier coefficient of the \( i \)-th basis vector \( g_i \) and define \( t_1 < \cdots < t_{d_N} \) to be the first ascending set of indices for which the vectors \( \{g_{i,j}\}_{i=1}^{d_N} \) are linearly independent. One can find such a sequence by direct computation of the Fourier coefficients and inspection of the matrices \( \{g_{i,j}\}_{i=1}^{d_N} \) for various choices of small \( t_1 < \cdots < t_{d_N} \).

Now let \( v_i = (g_{i,t_1}, \ldots, g_{i,t_{d_N}}) \) and let \( b_1, \ldots, b_{d_N} \) be a basis for \( S_2(\Gamma_0(N)) \). If we have \( h_1, h_2 \in S_2(\Gamma_0(N)) \) with equal \( t_i \)-th coefficients, then these coefficients are zero in the difference \( h_1 - h_2 \). But \( h_1 - h_2 \) can be written as a linear combination \( \sum c_i b_i \) of basis elements, for constants \( c_i \). Hence \( \sum c_i v_i = 0 \) in \( \mathbb{R}^{d_N} \), so by linear independence all \( c_i = 0 \), and thus \( h_1 - h_2 = 0 \). It therefore suffices to check that the coefficients of \( f_\lambda \) on \( q^{t_1}, \ldots, q^{t_{d_N}} \) match the coefficients that result from the linear combination of eta-quotients.

\( \square \)

**Remark.** In practice, these computations can be done using a computer algebra system such as SAGE.

**Example.** We show that the modular form corresponding to \( 2E_1(\frac{27}{16}) \) is

\[
g(z) := [1^211^2] + 3 \cdot [3^233^2] + 3 \cdot [1^13^111^133^1].
\]

For convenience, let \( G = \{[1^211^2], [3^233^2], [1^13^111^133^1]\} \) be the set of eta-quotients making up the linear combination \( g(z) \). The conductor of \( 2E_1(\frac{27}{16}) \) is 33 and so the corresponding modular form \( f_{\frac{27}{16}}(z) \) is an element of \( S_2(\Gamma_0(33)) \).
To show that $g(z)$ is also an element of $S_2(\Gamma_0(33))$, it suffices to show that $G \subset S_2(\Gamma_0(33))$. Take $g_1(z) \in G$. By [Ono 2004, Theorem 1.64], $g_i(z)$ is a modular form of weight 2 for $\Gamma_0(33)$. By [Ono 2004, Theorem 1.65], $g_i(z)$ vanishes at all cusps of $\Gamma_0(33)$, and thus $g_i(z) \in S_2(\Gamma_0(33))$.

Since $\text{ord}_3(33) = \text{ord}_{11}(33) = 1$, we have $\lambda_3 = \lambda_{11} = 2$ and evaluation of the dimension formula in Proposition 2.1 gives

$$\dim_{\mathbb{C}} S_2(\Gamma_0(33)) = 1 + \frac{33}{12} \prod_{p \mid 33} (1 + p^{-1}) - \frac{1}{2} \prod_{p \mid 33} \lambda_p - \frac{1}{4} \sum_{x \pmod{33}} 1 - \frac{1}{3} \sum_{x \pmod{33}} 1$$

$$= 1 + \frac{33}{12} \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{11}\right) - \frac{1}{2}(\lambda_3)(\lambda_{11}) - \frac{1}{4}(0) - \frac{1}{3}(0)$$

$$= 3.$$

It remains to show that $G$ is a basis for $S_2(\Gamma_0(33))$. Any dependence relation satisfied by the elements of $G$ would imply a dependence relation among their coefficients. It thus suffices to find a set of indices $t_1 < t_2 < t_3$ such that the $3 \times 3$ matrix formed by the $t_i$-th coefficients of these eta-quotients is nonsingular. For this particular $\lambda$, the first three coefficients suffice.

This implies that any two elements of $S_2(\Gamma_0(33))$ which agree on the first three coefficients are equal. In fact, we observe that the first three coefficients of the modular form corresponding to $2E_1\left(\frac{27}{16}\right)$ are the same as the first three coefficients of $g(z)$. That is, the coefficients of $g(z) = q + q^2 - q^3 - q^4 + \cdots$ agree with the coefficients of $f_{\frac{27}{16}}(z)$.

3. Gaussian hypergeometric functions and proof of Theorem 1.2

We recall some facts about Gaussian hypergeometric functions over finite fields of prime order and use the Gaussian hypergeometric function $\text{2F1}_1\left(\phi, \phi \mid \lambda\right)$ to prove Theorem 1.2.

**Gaussian hypergeometric functions.** Greene [1987] defined Gaussian hypergeometric functions over arbitrary finite fields and showed that they have properties analogous to those of classical hypergeometric functions. We recall some definitions and notation from [Ono 1998] in the case of fields of prime order.

**Definition 3.1.** If $p$ is an odd prime, $\text{GF}(p)$ is the field with $p$ elements, and $A$ and $B$ are characters of $\text{GF}(p)$, define

$$\left(\begin{array}{c} A \\ B \end{array}\right) := \frac{B(-1)}{p} J(A, B) = \frac{B(-1)}{p} \sum_{x \in \text{GF}(p)} A(x) \bar{B}(1-x).$$
Furthermore, if \( A_0, \ldots, A_n \) and \( B_1, \ldots, B_n \) are characters of \( \text{GF}(p) \), define the Gaussian hypergeometric series \( n+1 \, F_n \left( \frac{A_0, A_1, \ldots, A_n}{B_1, \ldots, B_n} \mid x \right) \) by the following sum over all characters \( \chi \) of \( \text{GF}(p) \):

\[
n+1 \, F_n \left( \frac{A_0, A_1, \ldots, A_n}{B_1, \ldots, B_n} \mid x \right) := \frac{p}{p-1} \sum_{\chi} \left( \frac{A_0 \chi}{\chi} \right) \left( \frac{A_1 \chi}{B_1 \chi} \right) \cdots \left( \frac{A_n \chi}{B_n \chi} \right) \chi(x)
\]

In particular, we are concerned with the Gaussian hypergeometric series \( 2 \, F_1(\lambda) \) defined by

\[
2 \, F_1(\lambda) := 2 \, F_1 \left( \frac{\phi, \phi}{\epsilon, \epsilon} \mid \lambda \right) = \frac{p}{p-1} \sum_{\chi} \left( \frac{\phi \chi}{\chi} \right)^2 \chi(\lambda)
\]

where \( \phi \) is the quadratic character of \( \text{GF}(p) \). It is shown in [Ono 1998] that if \( \lambda \in \mathbb{Q} \setminus \{0, 1\} \), then

\[
2 \, F_1(\lambda) = -\frac{\phi(-1) \, a_1(p; \lambda)}{p} \quad (3-1)
\]

for every odd prime \( p \) such that \( \text{ord}_p (\lambda(\lambda - 1)) = 0 \).

In addition, define the generalized Apéry number \( D(n; m, l, r) \) for every \( r \in \mathbb{Q} \) and every pair of nonnegative integers \( m \) and \( l \) by

\[
D(n; m, l, r) := \sum_{k=0}^{n} \binom{n+k}{k} m^l k^r.
\]

Ono also shows (ibid.) that if \( p = 2f + 1 \) is an odd prime and \( w = l + m \), then

\[
D(f; m, l, r) \equiv \left( \frac{p}{p-1} \right)^{w-1} w \, F_{w-1} \left( \frac{\phi, \phi, \ldots, \phi}{\epsilon, \epsilon, \ldots, \epsilon} \mid (-r)^l \right) \pmod{p}. \quad (3-2)
\]

**Proof of Theorem 1.2.** By (3-1) and the fact that \( \phi(-1) = (-1)^{\frac{p+1}{2}} \), we have that

\[
\frac{p}{p-1} \, 2 \, F_1(\lambda) = \frac{(-1)^{\frac{p+1}{2}} \, a_1(p; \lambda)}{p-1}.
\]

By (3-2), letting \( l = m = 1 \) (and thus \( w = 2 \)) and \( r = -\lambda \), we have

\[
\frac{p}{p-1} \, 2 \, F_1(\lambda) \equiv D(f; 1, 1, -\lambda) \pmod{p}.
\]

Combining these two equations and rearranging, we get

\[
2 \, a_1(p; \lambda) \equiv (-1)^{\frac{p+1}{2}} (p-1) D(f; 1, 1, -\lambda) \pmod{p}.
\]

Since

\[
D(f; 1, 1, -\lambda) = \sum_{k=0}^{n} \binom{f+k}{k} \binom{f}{k} (-\lambda)^k,
\]
we have
\[ 2a_1(p; \lambda) \equiv (-1)^{p+1} (p-1) \sum_{k=0}^{f} \binom{f+k}{k} \binom{f}{k} (-\lambda)^k \pmod{p}. \]

\[ \boxed{} \]

Remark. The binomial product \( \binom{f+k}{k} \binom{f}{k} \) can be combined into the multinomial coefficient \( \binom{f+k}{k, k, f-k} \) and so the congruence in Theorem 1.2 can also be written as
\[ 2a_1(p; \lambda) \equiv (-1)^{p+1} (p-1) \sum_{k=0}^{f} \binom{f+k}{k, k, f-k} (-\lambda)^k \pmod{p}. \]

References


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Zachary Mitchell, Grégory Simon and Xueying Zhao

A generalization of modular forms
Adam Haque

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Ramin Naimi and Jeffrey Shaw

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Bridget Kraynik, Yifei Sun and Chad R. Westphal

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Blake Allen, Erin Martin, Eric New and Dane Skabelund

Total positivity of a shuffle matrix
Audra McMillan

Betti numbers of order-preserving graph homomorphisms
Lauren Guerra and Steven Klee

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Patricia Cahn, Ruth Haas, Aloysius G. Helminick, Juan Li and Jeremy Schwartz

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Eric Larson and Larry Rolen

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Keenan Monks