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Trapping light rays aperiodically with mirrors

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(Communicated by Joseph O'Rourke)

We construct a configuration of disjoint segment mirrors in the plane that traps a single light ray aperiodically, providing a negative solution to a conjecture of O'Rourke and Petrovici. We expand this to show that any finite number of rays from a source can be trapped aperiodically.

1. Background and statement of results

We consider a point source of light together with a finite collection of disjoint, double-sided segment mirrors in the plane. Light rays travel from the source in fixed directions until they contact a mirror, at which point they reflect according to the laws of geometric optics: the angle of incidence equal to the angle of reflection. Rays that contact mirror endpoints are assumed to die there, and such rays are called *degenerate*. A ray is said to be *trapped* if it never escapes from the convex hull of the mirrors. A ray is said to be trapped *aperiodically* if it reflects at infinitely many distinct mirror points. Such a ray reflects forever but never retraces its own path.

O'Rourke and Petrovici [2001] asked whether the light rays emanating in every direction from a point source could be simultaneously trapped in such a system. The authors show that the set of rays from a source trapped periodically is countable; this can also be shown for the set of degenerate rays. The remaining rays either escape or are trapped aperiodically. Thus, if all the light rays from a point source are trapped by the mirrors, then there must be uncountably many aperiodically trapped rays. The authors' conjecture that mirrors cannot trap a light ray aperiodically would have implied that no mirror system could trap all light from a point source. We provide a counterexample to this conjecture:

Theorem 1. *There is a mirror configuration in the plane that traps a light ray aperiodically.*

David Milovich tells us this was proved independently in 2002 by Ben Stephens, then a graduate student at MIT, but left unpublished; see [O'Rourke 2005].

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We then build upon this construction inductively to show:

Theorem 2. *For any $n \geq 1$, there is a mirror configuration in the plane that traps n distinct aperiodic rays from a single source.*

2. Proofs

The bases for both arguments lie in a fundamental result from mathematical billiards: any billiard path in the square with irrational slope traces out an aperiodic trajectory [Tabachnikov 2005, pp. 25–26]. Our method is to recreate the dynamics of the billiard path in the square using disjoint segment mirrors.

Proof of Theorem 1. The construction will focus around the 2×2 square in the (x, y)-plane defined by $\max\{|x|, |y|\} = 1$, i.e., the square with the four vertices $(\pm 1, \pm 1)$. This will be referred to simply as *the square*. We begin with two segment mirrors, the top and bottom of the square. Fix a point p and an angle θ, measured counterclockwise from horizontal, so that the initial position and direction (p, θ) would define an aperiodic billiard trajectory in the square. Although this trajectory is aperiodic, it consists of only four distinct directions: θ, $-\theta$, $\pi - \theta$, and $\pi + \theta$.

We place six additional mirrors outside the opened square, four horizontal and two vertical (Figure 1), in such a way that a ray that exits the opened square in one of these four directions necessarily returns to the same point of the square where it exited. Such a ray always hits three mirrors (horizontal, vertical, then horizontal) before returning; thus its direction changes as if it reflects only off of a vertical mirror. Ignoring the path outside the square, the trajectory behaves as though it has reflected off the square's (absent) vertical edge (Figure 2).

A set of mirrors with this property is given explicitly at the end of this proof, as can be verified by simple trigonometry. Within the square, the light ray (p, θ) travels as though all four mirrors of the square were in place. In particular, this ray is trapped aperiodically.

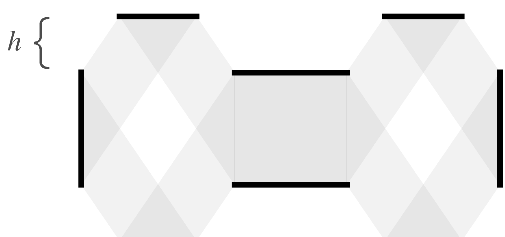

Figure 1. The mirror configuration, along with four strips of parallel rays, representing all four possible directions of escape, shown exiting the central square and returning.

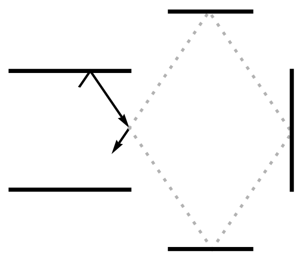

Figure 2. The ray appearing to reflect off the right vertical edge of the square.

We conclude with the mirror coordinates. Fix $h > 0$; this parameter gives the height difference between the top of the square and the height of two higher horizontal mirrors. "$P \sim Q$" will denote the closed segment mirror from P to Q. Our configuration is symmetric about both the x- and y-axes. Coordinates for the three mirrors that meet the upper right quadrant of the plane are given: (the remaining 5 can be obtained via symmetry)

$$(-1, 1) \sim (1, 1),$$
$$(1 + h\,|\cot\theta|, 1 + h) \sim (1 + (h+2)|\cot\theta|, 1 + h),$$
$$(1 + (2h+2)|\cot\theta|, -1) \sim (1 + (2h+2)|\cot\theta|, 1). \qquad \square$$

Proof of Theorem 2. Suppose $0 < \theta_1 < \pi/2$ and (p, θ_1) is trapped as described in Theorem 1 in a configuration with $h = h_1$. To trap an additional ray from the source, we choose another aperiodic direction θ_2 and height h_2 carefully to ensure that the new construction does not interact with the old.

Intuitively, we choose θ_2 to be very steep, thus when the ray (p, θ_2) escapes from the central square, it will also escape through the gaps of the initial construction. So for θ_2, we require that the initial position and direction (p, θ_2) follows an aperiodic trajectory in the square, that $\theta_1 < \theta_2 < \pi/2$, and that the ray starting at the bottom right corner of the square and traveling in the direction θ_2 escapes from the original mirror system without reflections. By symmetry, this guarantees that *any* ray that exits the opened square in one of the four possible directions $\theta_2, -\theta_2, \pi - \theta_2$, or $\pi + \theta_2$ will not reflect off any of the original mirrors. By choosing the height h_2 for the horizontal mirrors to be sufficiently large, we can ensure that after the ray (p, θ_2) exits the central square, its path will completely surround the original mirrors; formally, this is achieved when $h_2 > (h_1+1)\cot\theta_1 \tan\theta_2$, as can be verified with straightforward trigonometry.

This will guarantee that the new ray will not hit the original mirrors and that the new mirrors will not interfere with the original ray. (Compare Figure 1 with

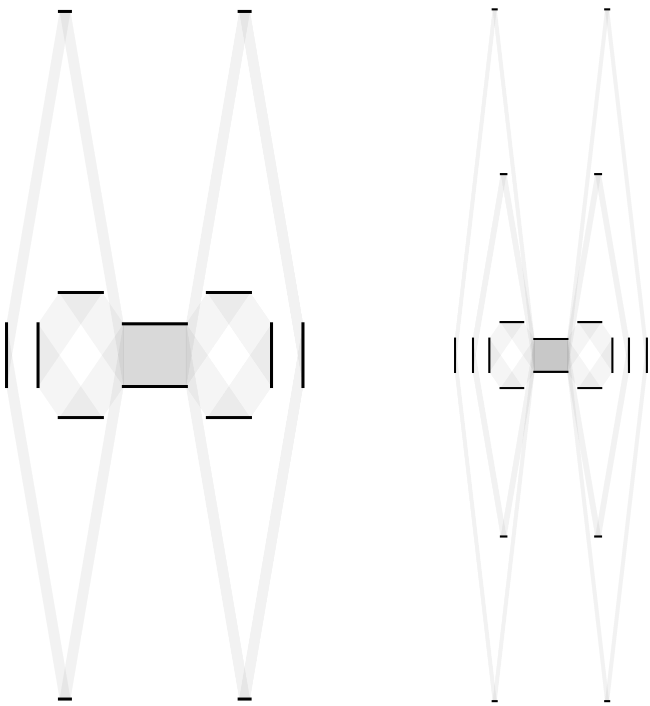

Figure 3. Examples for the cases $n = 2$ (left) and $n = 3$ (right).

the left part of Figure 3). In this way, the two rays are simultaneously trapped aperiodically.

This process can be continued. If θ_i and h_i (for $1 \le i < n$) have been chosen in this way, then choose θ_n such that (p, θ_n) follows an aperiodic billiard path in the square, that $\theta_{n-1} < \theta_n < \pi/2$, and that the ray from $(1, -1)$ in the direction θ_n escapes from the system without reflection. Choose h_n large enough to ensure that the path of the new ray will completely surround the original system — again, this is accomplished when $h_n > (h_{n-1} + 1) \cot \theta_{n-1} \tan \theta_n$. The ray (p, θ_n) is now trapped aperiodically, as are the previous $n - 1$ rays. Inductively, we can trap any finite number of rays aperiodically. $\square$

3. Further remarks

It may be of interest to strengthen the second theorem. In its original form, we had to choose particular rays which avoided mirrors already in place. This allowed for an easy inductive proof but can be avoided by a direct approach. The finite collection of directions (to be trapped aperiodically) can be arbitrary:

Any finite collection of rays from a source can be trapped aperiodically with finitely many disjoint segment mirrors.

We omit a formal proof but offer an outline. By rotating the plane about the point source, we show that we may assume that each ray's direction is irrational — that is, that they have irrational slope. Suppose $\{\theta_i : 1 \le i \le n\}$ is a finite collection of angles, which we can identify with rays eminating from the source. For $1 \le i \le n$, let R_i denote the set of rotations that rotate the plane in such a way that the angle θ_i becomes a rational direction after rotation. The set of rational directions is countable, and given a rational direction ρ, there is a unique rotation sending θ_i to ρ. So R_i is in bijection with the set of rational directions, hence R_i is countable. Thus, the set of all rotations that send *any* θ_i to a rational direction is countable, since it is the finite union $\bigcup_{i=1}^{n} R_i$. Because there are uncountably many rotations, some rotation does not send any θ_i to a rational direction — so this rotation sends each θ_i to an irrational direction. After applying such a rotation, we may assume that all of the given rays have irrational slopes. Hence (after rotating in such a way) each follows an aperiodic billiard path in the square.

We then proceed as in Theorem 2, creating n copies of the construction of Theorem 1. In this case, however, the angles θ_i are predetermined, so only the heights h_i can be adjusted. This degree of freedom is enough. Intuitively, as h_i increases, the mirror configuration expands. Provided each h_i is sufficiently large and they differ from one another by a sufficiently large amount, the mirror configurations will not interfere with one another (as in Figure 3). In such a system, each ray is aperiodically trapped.

These constructions do not answer the larger questions of trapping light (namely, if all light from a source can be trapped), but they do bring to the forefront some additional lines of inquiry. We've shown that the cardinality of aperiodically trapped rays can be any finite number, but must this cardinality be finite? Or, a weaker statement, must this cardinality be countable? These questions were originally posed in [O'Rourke and Petrovici 2001]. A positive answer to either would resolve the larger question of whether all light from a point source can be trapped with segment mirrors.

We would also like to mention that the approach used to prove Theorem 1 can be applied to polygons other than the square; our original proof was based on a quadrilateral with a nonperiodic billiard path constructed in [Galperin 1983].

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