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Induced subgraphs of Johnson graphs

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The Johnson graph $J(n, N)$ is defined as the graph whose vertices are the n-subsets of the set $\{1, 2, \ldots, N\}$, where two vertices are adjacent if they share exactly $n-1$ elements. Unlike Johnson graphs, induced subgraphs of Johnson graphs (JIS for short) do not seem to have been studied before. We give some necessary conditions and some sufficient conditions for a graph to be JIS, including: in a JIS graph, any two maximal cliques share at most two vertices; all trees, cycles, and complete graphs are JIS; disjoint unions and Cartesian products of JIS graphs are JIS; every JIS graph of order n is an induced subgraph of $J(m, 2n)$ for some $m \leq n$. This last result gives an algorithm for deciding if a graph is JIS. We also show that all JIS graphs are edge move distance graphs, but not vice versa.

1. Introduction

We work with finite, simple graphs. Let $F = \{S_1, \ldots, S_m\}$ be a family of finite sets. The *intersection graph* of F, denoted $\Omega(F)$, is the graph whose vertices are the elements of F, where two vertices S_i and S_j, $i \neq j$, are adjacent if they share at least one element. More generally, for a fixed positive integer p, the *p-intersection graph* of F, denoted $\Omega_p(F)$, is the graph whose vertices are the elements of F, where two vertices are adjacent if they share at least p elements. (Thus $\Omega_p(F)$ is a subgraph of $\Omega_1(F) = \Omega(F)$.) McKee and McMorris [1999] give an extensive and excellent survey of intersection graphs, which also includes a section on p-intersection graphs. Here we narrow attention to p-intersection graphs of families of $(p+1)$-sets, so that two vertices S_i and S_j are adjacent if $|S_i \cap S_j| = |S_i| - 1 = |S_j| - 1$, i.e., S_i and S_j differ by exactly one element.

Another way to view these graphs is as induced subgraphs of Johnson graphs. Given positive natural numbers $n \leq N$, the *Johnson graph* $J(n, N)$ is defined as the graph whose vertices are the n-subsets of the set $\{1, 2, \ldots, N\}$, where two vertices are adjacent if they share exactly $n-1$ elements. Hence a graph G is isomorphic

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to an induced subgraph of a Johnson graph if and only if it is possible to assign, for some fixed n, an n-set S_v to each vertex v of G such that distinct vertices have distinct corresponding sets, and vertices v and w are adjacent if and only if S_v and S_w share exactly $n-1$ elements. When this happens, we say the family of n-sets $F = \{S_v : v \in V(G)\}$ *realizes* G as an induced subgraph of a Johnson graph, which we abbreviate by saying G is *JIS*. Thus, F realizes G as a JIS graph if and only if G is isomorphic to $\Omega_{n-1}(F)$, which in turn is isomorphic to an induced subgraph of $J(n, N)$, where $N = \left|\bigcup_{S \in F} S\right|$.

Although there is a considerable amount of literature written on Johnson graphs, we have not been able to find any on their induced subgraphs. It would be desirable to obtain "nice" necessary and sufficient conditions for when a graph is JIS. In this paper, we only give some necessary conditions and some sufficient conditions.

A *clique* in a graph G is a complete subgraph of G. A clique L in G is called a *maximal clique*, or a *maxclique* for short, if there is no larger clique $L' \subseteq G$ that contains L. In Section 2 we describe how the maxcliques of a graph play a role in whether or not it is JIS. In particular, Proposition 2(1) states that any two distinct maxcliques in a JIS graph can share at most two vertices. It follows, for example, that the graph "K_5 minus one edge" is not JIS, since it contains two maximal 4-cliques that share three vertices.

The conditions given in Section 2 are necessary, but not sufficient, for a graph to be JIS. In Section 3 we show that the complete bipartite graph $K_{2,3}$, as well as a few other graphs, satisfy all these necessary conditions but are not JIS. In Section 3 we also give some sufficient conditions for a graph to be JIS, including the following:

- All complete graphs and all cycles are JIS.
- A graph is JIS if and only if all its connected components are JIS.
- The Cartesian product of two JIS graphs is JIS.

Despite not having a "nice" characterization of JIS graphs, for any graph G the question "Is G JIS?" is decidable; this follows from Theorem 10, which says that every JIS graph of order n is isomorphic, for some $m \le n$, to an induced subgraph of the Johnson graph $J(m, 2n)$. In other words, every JIS graph of order n can, for some $m \le n$, be realized by m-subsets of $\{1, 2, \dots, 2n\}$. This gives us a simple (albeit slow) algorithm for determining if a graph G is JIS: Do an exhaustive search among all n-families of m-subsets of $\{1, \dots, 2n\}$, where n is the order of G and $m \le n$, to see if any of them realizes G as a JIS graph.

The *p-intersection number* of a graph G is defined as the smallest k such that G is isomorphic to the p-intersection graph of a family of subsets of $\{1, \dots, k\}$ ([McKee and McMorris 1999], p. 91). Thus, an immediate corollary of Theorem 10 is that every JIS graph of order n has, for some $m \le n$, $(m-1)$-intersection number at most $2n$.

In the final section of this paper we discuss edge move distance graphs and their relationship to JIS graphs.

2. Maxcliques in JIS Graphs

Given n-sets $S_1, \ldots, S_k$ with $n \geq 1$ and $k \geq 2$, we say they share an *immediate subset* if $\left|\bigcap_{i=1}^{k} S_i\right| = n-1$. Similarly, $S_1, \ldots, S_k$ share an *immediate superset* if $\left|\bigcup_{i=1}^{k} S_i\right| = n+1$. Observe that for $k = 2$, S_1 and S_2 share an immediate subset if and only if they share an immediate superset: $|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2| = 2n - |S_1 \cap S_2|$; hence $|S_1 \cup S_2| = n+1$ if and only if $|S_1 \cap S_2| = n-1$. We begin with the following elementary result on realizations of complete graphs as JIS graphs.

Lemma 1. *Let $S_1, \ldots, S_k$ be n-sets that pairwise share an immediate subset, where $n \geq 1$ and $k \geq 3$. Then $S_1, \ldots, S_k$ share an immediate subset or an immediate superset, but not both.*

Proof. We first show that for $k \geq 3$, if $S_1, \ldots, S_k$ share an immediate subset, then they do not share an immediate superset. Suppose $T = S_1 \cap \cdots \cap S_k$ has $n-1$ elements. Then, for each i, $S_i \setminus T$ has exactly one element, x_i. For all $j \neq i$, $x_i \notin S_j$ since $S_i \neq S_j$. It follows that $S_1 \cup \cdots \cup S_k$ has at least $n - 1 + k \geq n + 2$ elements, since $k \geq 3$. Thus $S_1, \ldots, S_k$ do not share an immediate superset.

Now suppose $S_1, \ldots, S_k$ pairwise share an immediate subset. We use induction on k to prove that they share an immediate subset or an immediate superset.

Assume $k = 3$. Let $T = S_1 \cap S_2$. If $T \subset S_3$, then $|S_1 \cap S_2 \cap S_3| = |T| = n - 1$, and we're done. So assume $T \not\subset S_3$. Note that $|S_1 \setminus T| = |S_2 \setminus T| = 1$. Hence, for S_3 to share $n - 1$ elements with each of S_1 and S_2, it must contain an $(n-2)$-subset of T, as well as $S_1 \setminus T$ and $S_2 \setminus T$, and no other elements. It follows that $|S_1 \cup S_2 \cup S_3| = n + 1$, as desired.

Now assume $k \geq 4$. Then, by our induction hypothesis, $S_1, \ldots, S_{k-1}$ share an immediate subset or an immediate superset; and similarly for $S_2, \ldots, S_k$. We have four cases:

Case 1: $S_1, \ldots, S_{k-1}$ share an immediate subset and $S_2, \ldots, S_k$ share an immediate subset. Then $S_1, \ldots, S_k$ share $S_2 \cap S_3$ as an immediate subset.

Case 2: $S_1, \ldots, S_{k-1}$ share an immediate superset and $S_2, \ldots, S_k$ share an immediate superset. Then $S_1, \ldots, S_k$ share $S_2 \cup S_3$ as an immediate superset.

Case 3: $S_1, \ldots, S_{k-1}$ share an immediate subset and $S_2, \ldots, S_k$ share an immediate superset. Let $T = S_1 \cap \cdots \cap S_{k-1}$. Then, for $1 \leq i \leq k-1$, $S_i \setminus T$ has exactly one element, x_i; and, for $1 \leq j \leq k-1$ with $j \neq i$, $x_i \notin S_j$ since $S_i \neq S_j$. Since $|S_2 \cup \cdots \cup S_k| = n + 1 = |S_2 \cup S_3|$, S_k is a proper subset of $S_2 \cup S_3 = T \cup \{x_2, x_3\}$. And since S_2, S_3, S_k share an immediate superset, they do not share an immediate subset; hence $T \not\subset S_k$. This implies that $x_2, x_3 \in S_k$ since S_k has n elements and

$T \cup \{x_2, x_3\}$ has $n+1$ elements. But $x_2, x_3 \notin S_1$, so $|S_1 \cap S_k| < n-1$, which contradicts the hypothesis of the lemma.

Case 4: $S_1, \ldots, S_{k-1}$ share an immediate superset and $S_2, \ldots, S_k$ share an immediate subset. This case is similar to Case 3. □

We now use Lemma 1 to establish restrictions on how maxcliques in a JIS graph can intersect or connect to each other by edges.

Proposition 2. *Suppose G is JIS and L and L' are distinct maxcliques in G.*

(1) *L and L' share at most two vertices.*

(2) *If L and L' share exactly two vertices, then no vertex in $V(L) \setminus V(L')$ is adjacent to a vertex in $V(L') \setminus V(L)$.*

(3) *If L and L' share exactly one vertex, then each vertex in either of the two sets $V(L) \setminus V(L')$ and $V(L') \setminus V(L)$ is adjacent to at most one vertex in the other set.*

Proof. Let $\{S_v : v \in V(G)\}$ be a family of n-sets that realizes G as a JIS graph.

(1) Suppose towards contradiction that L and L' are distinct maxcliques that share three (or more) vertices, u, v, and w. Let x be a vertex of L not in L', and x' a vertex of L' not in L; x and x' exist since L and L' are distinct and maximal. Then, by Lemma 1, the sets S_x, S_u, S_v, and S_w share an immediate subset or an immediate superset. Similarly for $S_{x'}$, S_u, S_v, and S_w. But S_u, S_v, and S_w cannot share both an immediate subset and an immediate superset. It follows that S_x and $S_{x'}$ share an immediate subset or an immediate superset, which implies that x and x' are adjacent. Hence every vertex of L is adjacent to every vertex of L', but this contradicts the assumption that L is a maxclique in G.

(2) Let L and L' be distinct maxcliques that share exactly two vertices, v and w. Suppose towards contradiction that there exist adjacent vertices $x \in V(L) \setminus V(L')$ and $x' \in V(L') \setminus V(L)$. Then the induced subgraph of G containing $\{x, x', v, w\}$ is a 4-clique. Let L'' be the maxclique that contains this 4-clique. Then L'' is distinct from L and shares at least three vertices with it. This contradicts (1).

(3) The proof is similar to the proof of (2). Let L and L' be distinct maxcliques that share exactly one vertex, v. Suppose towards contradiction that there exist vertices $x \in V(L) \setminus V(L')$ and $x', y' \in V(L') \setminus V(L)$ with x adjacent to x' and y'. Then the induced subgraph of G containing $\{x, x', y', v\}$ is a 4-clique, and the maxclique that contains this 4-clique is distinct from L' and shares at least three vertices with it. This contradicts (1). □

Proposition 3. *Suppose $L_1, \ldots, L_k$, where k is odd and at least 3, are distinct maxcliques in a graph G such that L_i shares exactly two vertices with L_{i+1} for $1 \le i \le k-1$, and L_k shares exactly two vertices with L_1; then G is not JIS.*

Proof. In the following, L_{i+1} refers to L_1 whenever $i = k$. Suppose towards contradiction that G is realized as a JIS graph by a family of n-sets. Note that each L_i has at least three vertices, since otherwise it would not be distinct from L_{i+1}. Hence, by Lemma 1, we can label each L_i as either "sub" or "super" according to whether the n-sets assigned to its vertices share an immediate subset or an immediate superset. Then, since k is odd, there exists a j such that L_j and L_{j+1} have the same label. Now, L_j and L_{j+1} share two vertices; therefore the n-sets assigned to their vertices must all share the same immediate subset or immediate superset, which makes all vertices in L_j adjacent to those in L_{j+1}, giving a contradiction. □

An equivalent way of stating the above result is: One can label every maxclique in a JIS graph with a $+$ or $-$ (or any two symbols) in such a way that any two maxcliques that share two vertices have distinct labels.

3. Miscellaneous JIS and non-JIS graphs

In this section we give some sufficient conditions for when a graph is JIS. We also describe some graphs that satisfy all the conditions listed in the results of the previous section as necessary for a graph to be JIS, but are not JIS.

Proposition 4. *All complete graphs and all cycles are JIS.*

Proof. For each n, K_n is realized as a JIS graph by the 1-sets $\{1\}, \{2\}, \dots, \{n\}$. For each $n \geq 3$, the n-cycle is realized as a JIS graph by the 2-sets $\{1, 2\}$, $\{2, 3\}$, $\dots$, $\{n-1, n\}$, $\{n, 1\}$. □

We define the *n-core* of a graph G as the graph obtained by recursively removing all vertices of degree less than n until there are none left.

Proposition 5. *A graph is JIS if and only if its 2-core is JIS.*

Proof. Suppose G is obtained from a graph G' by removing exactly one vertex, w, which has degree 0 or 1. By induction, it is enough to show that G is JIS if and only if G' is JIS. Clearly, if G' is JIS, then so is G, since any induced subgraph of a JIS graph is JIS. To prove the converse, suppose G is JIS. Let $\{S_x : x \in V(G)\}$ be n-sets that realize G as a JIS graph. Pick distinct a and b that are not in any of the sets S_x. For each $x \in V(G)$, let $S'_x = S_x \cup \{a\}$. Let $S'_w = S_v \cup \{b\}$, where $v \in V(G')$ is arbitrary if w has degree 0, and v is adjacent to w if w has degree 1. Then $\{S'_x : x \in V(G')\}$ are $(n+1)$-sets that realize G' as a JIS graph, as desired. □

It follows as a trivial corollary that all trees are JIS.

Proposition 6. *A graph is JIS if and only if all its connected components are JIS.*

Proof. One direction is trivial: every induced subgraph of a JIS graph, and in particular every connected component of it, is JIS. We prove the converse by induction on the number of components of G.

Base step: Suppose that G has two components, G_i, $i = 1, 2$, each realized as a JIS graph by a family of sets F_i. We can assume without loss of generality that each set in F_1 is disjoint from each set in F_2.

We would like each set in F_1 to have the same size as each set in F_2, in order to obtain $F_1 \cup F_2$ as a family that realizes G as a JIS graph. If this is not already so, we proceed as follows. Let m_i denote the number of elements in each set in F_i. We can assume $n_1 > n_2$. Now add the first $n_1 - n_2$ elements of the first set in F_1 to every set in F_2.

Once the sets in the two families all have the same size, we must make sure that sets corresponding to vertices in different components of G do not share immediate subsets. This will automatically be true for sets that had two or more elements before any extra elements were added to them (since we started with the sets in F_1 disjoint from those in F_2), but not for singletons. We remedy this by adding, for each i, an element e_i to every set in F_i, where e_1 and e_2 are distinct elements not already in any set in any F_i. It is now easy to verify that $F_1 \cup F_2$ realizes G as a JIS graph.

The inductive step follows trivially from the base step. □

Proposition 7. *The Cartesian product of two JIS graphs is JIS.*

Proof. Let G and G' be JIS graphs that are realized, respectively, by sets $\{S_x : x \in V(G)\}$ and $\{S'_{x'} : x' \in V(G')\}$. We can assume without loss of generality that every S_x is disjoint from every $S'_{x'}$.

For each vertex $v = (x, x') \in V(G \times G')$, let $T_v = S_x \cup S'_{x'}$. By definition, two vertices $v = (x, x')$ and $w = (y, y')$ of $G \times G'$ are adjacent if and only if $x = x'$ and y is adjacent to y' or $y = y'$ and x is adjacent to x'. Thus, T_v and T_w share an immediate subset if and only if v and w are adjacent. Hence the sets $\{T_v : v \in G \times G'\}$ realize $G \times G'$ as a JIS graph. □

Proposition 8. *The complete bipartite graph $K_{2,3}$ is not JIS.*

Proof. Label the two degree-3 vertices of $K_{2,3}$ as v and w, and the three degree-2 vertices as x, y, and z, as in Figure 1. Suppose towards contradiction that there exists a family of n-sets $\{S_u : u \in V(K_{2,3})\}$ that realizes $K_{2,3}$ as a JIS graph. Since v and w have distance two (where *distance* is the number of edges in the shortest

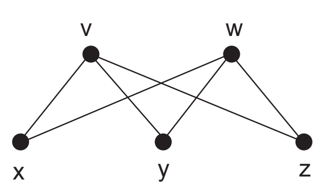

Figure 1. $K_{2,3}$ with labeled vertices.

path joining the two vertices), S_v and S_w must share exactly $n-2$ elements (this does not work for distance ≥ 3; it works only for distance ≤ 2). Let $T = S_v \cap S_w$. Then, since each of x, y, and z is adjacent to both v and w, S_x, S_y, and S_z must each contain T as a subset. Therefore, by subtracting T from every S_u, $u \in V(K_{2,3})$, we get a family of 2-sets that realizes $K_{2,3}$. Hence we will assume that every S_u has exactly two elements. It follows that S_v and S_w are disjoint; and S_x, S_y, and S_z are pairwise disjoint and each shares exactly one element with each of S_v and S_w.

So, without loss of generality, $S_v = \{1, 2\}$, and $S_w = \{3, 4\}$. Therefore, again without loss of generality, $S_x = \{1, 3\}$, and $S_y = \{2, 4\}$. And there is nothing left for S_z. □

The graph $K_{2,3}$ can be thought of as two 4-cycles that share three vertices. So one may wonder whether the graph θ_n consisting of two n-cycles that share $n-1$ vertices is also not JIS. It turns out that θ_n is not JIS only for $n = 4$ and $n = 5$. The proof that θ_5 is not JIS is very similar to the proof that $K_{2,3}$ is not JIS, and we therefore omit it. The proof that θ_n is JIS for $n \geq 6$ is a straightforward construction, which we also omit.

One may also wonder whether $K_{2,3}$ becomes JIS if an edge is added to it. There are, up to isomorphism, two ways to add an edge to $K_{2,3}$: add an edge that connects the two degree-3 vertices; or add an edge that connects two of the three degree-2 vertices. It turns out that neither of these two graphs is JIS. The proof that the former graph is not JIS follows immediately from Proposition 3. The proof that the latter graph (which we call Δ_2) is not JIS is given below in Proposition 9.

The graphs Δ_i depicted in Figure 2 have the following pattern (ignore the vertex labels and the $+$ and $-$ signs for now; they are used later): Δ_i consists of a chain of i "consecutively adjacent" triangles, plus one vertex which is connected to the two vertices of degree 2 in the triangle chain. It turns out that, like $K_{2,3}$, Δ_2, Δ_4, and Δ_6 satisfy the necessary conditions in the results of the previous sections for being JIS, but are not JIS; Δ_3 and Δ_5, however, are JIS. We prove these claims below, except for Δ_6: its proof is similar to that of Δ_2 and Δ_4, but is more tedious, and in our opinion not worth being included here. We did not check which Δ_i are JIS for $i \geq 7$, but, from the pattern for $i \leq 6$, it seems that:

Conjecture. Δ_i is JIS if and only if i is odd.

Proposition 9. (*i*) *The graphs* Δ_2 *and* Δ_4 *are not JIS.* (*ii*) *The graphs* Δ_3 *and* Δ_5 *are JIS.*

Remark. As mentioned above, Δ_2 is isomorphic to $K_{2,3}$ plus an edge that connects two of its three degree-2 vertices. Because of this, the proof that $K_{2,3}$ is not JIS can be easily modified to prove that Δ_2 is not JIS. However, we give a different proof below, one that can be naturally extended to also prove that Δ_4 (and Δ_6) is not JIS.

Proof. Label the vertices of Δ_2 as $v_1, \ldots, v_5$, as in Figure 2. The $+$ and $-$ signs will be explained shortly. Suppose, towards contradiction, that Δ_2 can be realized as a JIS graph by sets $S_1, \ldots, S_5$ (for simplicity, we write S_i instead of S_{v_i}). Each of the two triangles in Δ_2 is a maxclique. Thus, by Lemma 1, S_1, S_2, and S_3 must share an immediate subset or an immediate superset; similarly for S_2, S_3, and S_4. Furthermore, S_1, S_2, and S_3 share an immediate subset if and only if S_2, S_3, and S_4 share an immediate superset, because: if S_1, S_2, and S_3 share an immediate subset and S_2, S_3, and S_4 also share an immediate subset, then S_1 and S_4 must share $S_2 \cap S_3$ as an immediate subset, but this contradicts the fact that v_1 and v_4 are not adjacent; and if S_1, S_2, and S_3 share an immediate superset and S_2, S_3, and S_4 also share an immediate superset, then S_1 and S_4 must share $S_2 \cup S_3$ as an immediate superset, which implies that they also share an immediate subset, again contradicting the fact that v_1 and v_4 are not adjacent.

Thus, without loss of generality, we will assume that S_1, S_2, and S_3 share an immediate subset. This is indicated in Figure 2 by the $-$ sign; the $+$ signs indicate immediate supersets. So we will assume that $S_1 = \{1, 2, 3, 4\}$, $S_2 = \{1, 2, 3, 5\}$, and $S_3 = \{1, 2, 3, 6\}$; we explain in the next paragraph why there is no loss of generality in assuming that S_i are 4-sets (as opposed to larger sets). To make the notation more compact, we will drop the commas and the braces from each set; e.g., $S_1 = 1234$. Then S_4 must be a 4-subset of $S_2 \cup S_3 = 12356$. Since S_1 and S_4 have no immediate subset, we can without loss of generality assume that $S_4 = 2356$. Now, S_5 must differ by exactly one element from each of S_1 and S_4. The only possibilities are 1235, 1236, 2345, and 2346. But the first two are equal to S_2 and S_3 respectively; and the last two differ from S_2 and S_3 respectively by exactly one element, which is not allowed since v_5 is adjacent to neither v_2 nor v_3. Thus we have a contradiction, as desired.

Note that by assuming that all S_i are 4-sets, we ended up with all of them sharing the two elements 2 and 3. If we instead assumed that S_i were n-sets with $n \geq 5$,

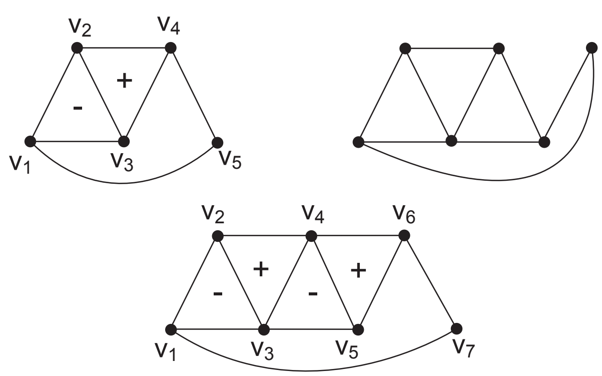

Figure 2. Δ_2, Δ_3, and Δ_4, with vertices labeled in Δ_2 and Δ_4.

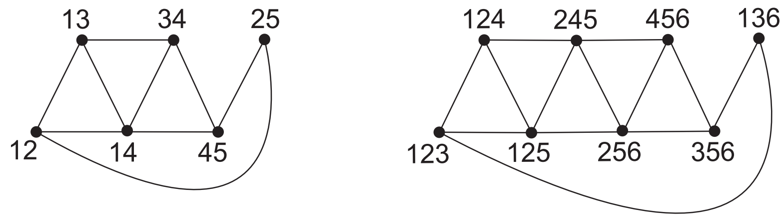

Figure 3. Δ_3 (left) and Δ_5 (right) realized as JIS graphs.

the proof would remain the same except that we would end up with all S_i sharing more than two elements. Hence there is no loss of generality in assuming that S_i are 4-sets (in fact, this shows that we could even assume they are 2-sets).

To prove that Δ_4 is not JIS, we start with the same assumptions that S_1, S_2, and S_3 share an immediate subset, S_2, S_3, and S_4 share an immediate superset, and $S_1 = 1234$, $S_2 = 1235$, $S_3 = 1236$, and $S_4 = 2356$. Now, S_3, S_4, and S_5 must share an immediate subset. So S_5 must contain $S_3 \cap S_4 = 236$. Since v_5 is adjacent to neither v_1 nor v_2, S_5 can contain neither 1 nor 4 nor 5. Hence, without loss of generality, $S_5 = 2367$. Continuing, S_4, S_5, and S_6 must share an immediate superset. So S_6 must be a 4-subset of $S_4 \cup S_5 = 23567$; i.e., we must drop one element from 23567 to get S_6. Dropping 5 or 7 gets us back to S_4 and S_5; hence we must drop 2, 3, or 6. The roles of 2 and 3 have been identical so far; so, without loss of generality, we must drop 2 or 6; so $S_6 = 2357$ or 3567. The former is not possible since v_6 and v_2 are not adjacent. And the latter is ruled out by noticing that 3567 differs from $S_1 = 1234$ by three elements, which contradicts the fact that v_6 and v_1 have distance two[1]. Thus we have reached a contradiction, as desired.

Part (ii) of the proposition is proved in Figure 3, which shows sets that realize Δ_3 and Δ_5 as JIS graphs. For the sake of compactness, braces and commas are omitted from the sets. □

We end this section with the following definition and question. Let G be a JIS graph, and suppose $F = \{S_u : u \in V(G)\}$ realizes G as a JIS graph. We define the *F-distance* between two vertices v and w of G to be $d_F(v, w) = |S_v \setminus S_w|$. It is easy to show this distance function is indeed a metric. The *JIS-diameter* of G is defined as

$$\max_{v,w \in V(G)} \min_F \{d_F(v, w)\}$$

where the minimum is taken over all families F that realize G as a JIS graph.

Question. Do there exist JIS graphs with arbitrarily large JIS-diameter?

[1]Note that $\Delta_4 - v_7$ *is* JIS, with S_1 and S_6 differing in three elements. We will refer back to this point at the very end of this section.

From the proof of Proposition 9 and the footnote in it, it follows that Δ_4 minus the degree-2 vertex v_7 has JIS-diameter 3: $S_1 = 1234$, $S_2 = 1235$, $S_3 = 1236$, $S_4 = 2356$, $S_5 = 2367$, and $S_6 = 3567$, i.e., v_1 and v_6 have F-distance 3.

4. An algorithm for recognizing JIS graphs

As mentioned in the introduction, the following theorem provides for an algorithm for deciding if a graph is JIS by doing a bounded exhaustive search.

Theorem 10. *Every JIS graph of order n is isomorphic, for some $m \le n$, to an induced subgraph of the Johnson graph $J(m, 2n)$.*

Proof. Let G be a JIS graph of order n with c connected components.

Case 1. Assume $c = 1$, i.e., G is connected. In this case we will prove a slightly stronger result, which we will use in the proof of Case 2:

G is isomorphic, for some $m \le n$, to an induced subgraph of $J(m, 2n-1)$.

The case $n = 1$ is trivial; so we assume $n \ge 2$. Since G is connected, there exists an ordering $v_1, v_2, \dots, v_n$ of the vertices of G such that for each $i \ge 2$, v_i is adjacent to at least one of $v_1, \dots, v_{i-1}$. Since G is JIS, for some $k \ge 1$ there exist k-sets $\{S_1, \dots, S_n\}$ that realize G as a JIS graph, where S_i corresponds to the vertex v_i. Since v_1 and v_2 are adjacent, $|S_1 \cap S_2| = k - 1$. Since v_3 is adjacent to at least one of v_1 and v_2, $|S_1 \cap S_2 \cap S_3| \ge k - 2$. Continuing this way, we see that $|S_1 \cap \dots \cap S_n| \ge k - (n - 1)$. Let

$$S'_i = S_i \setminus (S_1 \cap \dots \cap S_n)$$

for $1 \le i \le n$. Then for all i, $|S'_i| = m$ where $m \le k - (k - (n - 1)) = n - 1$, and it is easily verified that the family of sets $\{S'_1, \dots, S'_n\}$ realizes G as a JIS graph.

Now, since v_1 and v_2 are adjacent, $|S'_1 \cup S'_2| = m + 1$. Since v_3 is adjacent to at least one of v_1 and v_2, $|S'_1 \cup S'_2 \cup S'_3| \le m + 2$. Continuing this way, we see that $|S'_1 \cup \dots \cup S'_n| \le m + n - 1 \le 2n - 2$, which implies G is an induced subgraph of $J(m, 2n-1)$, $m \le n-1$. (Note: we proved the inequalities $|S'_1 \cup \dots \cup S'_n| \le 2n-2$ and $m \le n - 1$ only for $n \ge 2$, not for $n = 1$.)

Case 2. Assume $c \ge 2$. Let n_i be the order of the ith component of G. Then, by Case 1 above, for each i there is a family F_i of m_i-sets, $m_i \le n_i$, that realizes the ith component of G as a JIS graph, such that the union of the sets in F_i has at most $2n_i - 1$ elements. Thus $\bigcup F_i$ has at most $2n - c$ elements.

We can assume $m_1 \ge m_i$ for all i. We can also assume that for all $i \ne j$, every set in the family F_i is disjoint from every set in F_j. To make all sets in all the families have the same size, for each i such that $m_1 > m_i$ we add the first $m_1 - m_i$ elements of the first set in F_1 to every set in F_i. After adding these extra elements, we must make sure that sets corresponding to vertices in different components of G do not share immediate subsets. This will automatically be true for sets that had two or more elements before the extra elements were added, but not for singletons. We remedy this by adding, for each i, an element e_i to every set in F_i, where $e_1, \ldots, e_c$ are distinct elements not already in any set in any F_i. Let $F = \bigcup F_i$. Then G is realized as a JIS graph by F, which is a family of (m_1+1)-sets whose union has at most $2n-c+c=2n$ elements, where $m_1+1 \le n_1+1 \le n$. Thus G is an induced subgraph of $J(m, 2n)$ where $m = m_1+1 \le n$. □

Remark. It is not difficult to modify the above proof in Case 1 to show that if G is connected, then it is an induced subgraph of $J(n, 2n)$. It would be interesting to see for which graphs the bounds n and $2n$ can be lowered. Note that if G consists of exactly $n \ge 2$ vertices of degree zero, then the bound $2n$ is optimal.

5. Edge move distance graphs and JIS graphs

Since the 1970s many authors have written on various metrics defined on sets of graphs; see, for instance, [Benadé et al. 1991; Chartrand et al. 1997; 1990; Deza and Deza 2009; Johnson 1987; Kaden 1983; Zelinka 1985]. Among them are edge move, edge rotation, edge jump, and edge slide distances. In general, given a metric d on a set of graphs $S = \{G_1, \ldots, G_k\}$, the *distance graph* of S, denoted $D_d(S)$, has S as its vertex set, where two vertices G_i and G_j are adjacent if $d(G_i, G_j) = 1$. We will see shortly that distance graphs associated with the edge move metric are closely related to JIS graphs.

An *edge move* on a graph G consists of removing one edge from and adding a new edge to G, without changing its vertex set $V(G)$; i.e., one edge is "moved to a new position." The *edge move distance* $d_m(G, H)$ between two graphs G and H is defined as the fewest number of edge moves necessary to transform G into H, up to isomorphism. Note that for $d_m(G, H)$ to be defined, G and H must have the same order and the same size. It is easy to verify that d_m is a metric on any set of graphs of given order and size. Given a set S of graphs of the same order and size, the *edge move distance graph* of S, $D_m(S)$, is the graph whose vertices are the elements of S, where two vertices are adjacent if their edge move distance is one. When we say a graph is an edge move distance graph we mean it is isomorphic to one.

The connection between JIS graphs and edge move distance graphs can be seen by focusing on edge sets. Let G and H be graphs of the same order and size, with n edges each. If the edge sets $E(G)$ and $E(H)$ share exactly $n-1$ elements, then G and H have edge move distance one. Conversely, if G and H have edge move distance one, then their vertices can be labeled such that $E(G)$ and $E(H)$ share exactly $n-1$ elements. At first glance, this might seem to suggest that a graph is JIS if and only if it is isomorphic to an edge move distance graphs. We will show, however, that only half (one direction) of this statement is true.

Proposition 11. *Every JIS graph is an edge move distance graph.*

Proof. Let G be realized as a JIS graph by a family of n-sets $\{S_v : v \in V(G)\}$. We will construct a graph G_v for each $v \in V(G)$ such that $d_m(G_v, G_w) = 1$ if and only if S_v and S_w share an immediate subset.

We can assume that each S_v consists of positive integers. Let

$$k = 1 + \max\{i \in S_v : v \in V(G)\},$$

and let P be a path of length $2k$. Denote the vertices of P by $p_0, p_1, \ldots, p_{2k}$. For each $v \in V(G)$, we let G_v be the graph consisting of P plus the edges $p_i p_{2k-i}$ for all $i \in S_v$. Then it is easily verified that for $v \neq w$, G_v is not isomorphic to G_w, and $d_m(G_v, G_w) = 1$ if and only if S_v and S_w share an immediate subset. Therefore G is isomorphic to the edge move distance graph $D_m(\{G_v : v \in V(G)\})$. □

The converse is not true. The reason is that the number of edges shared by the edge sets of two graphs depends on how their vertices are labeled, whereas edge move distance is measured up to graph isomorphism.

Proposition 12. *The graph obtained by removing one edge from the complete graph K_n, where $n \geq 5$, is an edge move distance graph but is not JIS.*

Proof. Fix $n \geq 5$, and let H be the graph obtained by removing one edge from K_n. Then H contains two maximal $(n-1)$-cliques which share $n-2$ vertices. Hence, by Proposition 2(1), H is not JIS.

To show that H is an edge move distance graph, we construct a set of graphs $S = \{Q_1, Q_2, \ldots, Q_n\}$ such that $H \simeq D_m(S)$. For $1 \leq i \leq n$, Q_i has $n+2$ vertices: $V(Q_i) = \{v_1, v_2, \ldots, v_{n+2}\}$. For $1 \leq i \leq n-1$, we have

$$E(Q_i) = \{v_k v_{k+1} : 1 \leq k \leq n\} \cup \{v_{n-1} v_{n+1}, v_i v_{n+2}\};$$

and $E(Q_n) = (E(Q_1) \cup \{v_1 v_{n-2}\}) \setminus \{v_{n-2} v_{n-1}\}$.

Then one readily verifies for all $i \neq j$ except when $\{i, j\} = \{n-1, n\}$ that Q_i and Q_j have edge move distance one. Thus H is an edge move distance graph. □

Figures 4 and 5 show some of the Q_i in the case $n = 6$.

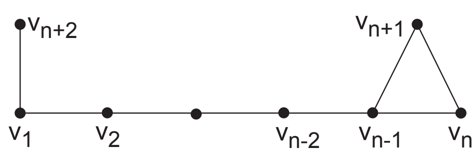

Figure 4. Q_1 for $n = 6$.

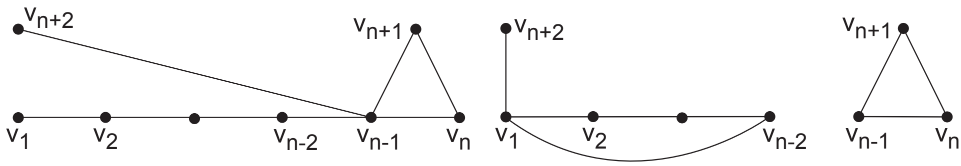

Figure 5. Q_{n-1} (left) and Q_n (right) for $n = 6$.

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