involve

a journal of mathematics

Multiscale adaptively weighted least squares finite element methods for convection-dominated PDEs

Bridget Kraynik, Yifei Sun and Chad R. Westphal

mathematical sciences publishers

2012 vol. 5, no. 1

INVOLVE 5:1(2012)

msp

Multiscale adaptively weighted least squares finite element methods for convection-dominated PDEs

Bridget Kraynik, Yifei Sun and Chad R. Westphal

(Communicated by John Baxley)

We consider a weighted least squares finite element approach to solving convection-dominated elliptic partial differential equations, which are difficult to approximate numerically due to the formation of boundary layers. The new approach uses adaptive mesh refinement in conjunction with an iterative process that adaptively adjusts the least squares functional norm. Numerical results show improved convergence of our strategy over a standard nonweighted approach. We also apply our strategy to the steady Navier–Stokes equations.

1. Introduction

In this paper we consider numerically approximating solutions to the convection-diffusion partial differential equation

$$\begin{cases} -\varepsilon\Delta u + \boldsymbol{b}\cdot\nabla u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, $u = u(x, y)$ is the solution, ∇u and Δu are the gradient and Laplacian of u, $\partial\Omega$ is the boundary of domain Ω, f is a known data function, g is a known boundary function, and ε and $\boldsymbol{b}$ are coefficients for diffusion and convection, respectively. For $\varepsilon \ll |\boldsymbol{b}|$ we say that this represents a convection-dominated diffusion problem. In such cases, solutions tend to develop boundary layers, that is, components of the solution that have steep gradients near the boundary. To illustrate this, consider the following ordinary differential equation analogy:

$$\begin{cases} -\varepsilon u'' + bu' = 0 & \text{in } (0, 1), \\ u(0) = 1, \\ u(1) = 0, \end{cases} \tag{2}$$

MSC2010: 65N30, 65N50, 35J57.
Keywords: partial differential equations, finite element methods, convection, boundary layers.
This work was sponsored by the National Science Foundation under grant DMS-0755260 through the Wabash Summer Institute in Mathematics at Wabash College.

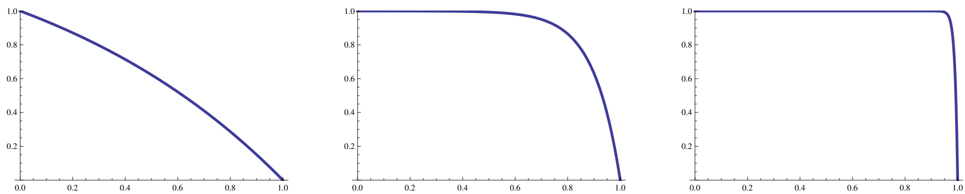

Figure 1. Solution of (2) for $\varepsilon = 1$ (left), $\varepsilon = 0.1$ (middle), and $\varepsilon = 0.01$ (right).

where bu' is the convection term and $-\varepsilon u''$ the diffusion term. We call the ODE convection-dominated when $\varepsilon \ll |b|$, and to illustrate this we set $b = 1$ and consider the following solution plots for different values of ε in Figure 1.

We can see that as $\varepsilon \to 0$, a boundary layer forms near $x = 1$. This behavior is difficult to approximate computationally and is also present in the solution of system (1) for regions of Ω near boundary points with $\boldsymbol{n} \cdot \boldsymbol{b} > 0$, where $\boldsymbol{n}$ is an outward unit normal to $\partial\Omega$. See [Brenner and Scott 1994; Braess 2001] for background on finite element methods for such problems.

The method we develop here is a generalization of a least squares finite element discretization for scalar elliptic equations. In general, a least squares approach to (1) tends to be an effective way to approximate solutions; however, convergence is degraded in the presence of dominant convection. We consider a least squares functional minimized with respect to a weighted L^2-norm, where the weights are chosen adaptively in the context of an adaptive mesh refinement routine. This idea is inspired by work of Westphal et al. [Lee et al. 2006; 2008; Cai and Westphal 2008], where a weighted functional is used to improve solutions to problems with singularities.

The organization of this paper is as follows: in Section 2 we introduce a reformulated version of (1) and the adaptively weighted procedure; in Section 3 we provide several numerical tests to demonstrate the effectiveness of the method compared to a more standard approach; and in Section 4 we show the robustness of the idea by applying it analogously to a moderately high Reynolds number Navier–Stokes system for steady fluid flow.

2. Methodology

The $L^2(\Omega)$ norm of a function f is defined to be

$$\|f\| = \|f\|_{L^2(\Omega)} = \left(\int_\Omega |f|^2\right)^{1/2},$$

and $L^2(\Omega)$ is the space of functions in Ω that have finite $L^2(\Omega)$ norms. Likewise, we define $H^1(\Omega)$ as the subspace of $L^2(\Omega)$ where all first partial derivatives of functions are also in $L^2(\Omega)$.

With the substitution $\boldsymbol{\sigma} = -\varepsilon \nabla u$, we rewrite (1) as the first-order equations

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} \cdot \nabla u = 0 & \text{in } \Omega, \\ \boldsymbol{\sigma} + \varepsilon \nabla u = \mathbf{0} & \text{in } \Omega, \\ \nabla \times \boldsymbol{\sigma} = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \\ \hat{\tau} \cdot \boldsymbol{\sigma} = -\varepsilon \hat{\tau} \cdot \nabla g & \text{on } \partial\Omega. \end{cases} \tag{3}$$

The third equation holds because $\nabla \times \boldsymbol{\sigma} = \nabla \times (-\varepsilon \nabla u) = -\varepsilon (\nabla \times \nabla u) = 0$. In the fifth equation, $\hat{\tau}$ is a unit tangent vector to $\partial\Omega$ and this new boundary condition is simply a statement about the directional derivative of g along $\partial\Omega$.

We first consider what we refer to as the standard least squares approach. Since we seek a finite element solution, we partition Ω into an initial triangulation denoted as Ω^h. Here, h denotes the size, or width of the triangles and $(u^h, \boldsymbol{\sigma}^h)$ represents an approximate solution to $(u, \boldsymbol{\sigma})$, the exact solution of system (3). Define

$$\boldsymbol{V} = \{v \in H^1(\Omega) : v = g \text{ on } \partial\Omega\},$$
$$\boldsymbol{\Sigma} = \{\boldsymbol{s} \in H^1(\Omega)^2 : \tau \cdot \boldsymbol{s} = -\varepsilon \tau \cdot \nabla g \text{ on } \partial\Omega\}$$

as sets of admissible solutions, and let $\boldsymbol{V}^h \subseteq \boldsymbol{V}$ and $\boldsymbol{\Sigma}^h \subseteq \boldsymbol{\Sigma}$ be finite dimensional subsets in which we seek approximate solutions.

A standard LS approach seeks a pair of solutions $(u^h, \boldsymbol{\sigma}^h) \in \boldsymbol{V}^h \times \boldsymbol{\Sigma}^h$ which minimizes the functional

$$G(u^h, \boldsymbol{\sigma}^h; f) = \|\nabla \cdot \boldsymbol{\sigma}^h + \boldsymbol{b} \cdot \nabla u^h - f\|^2 + \|\boldsymbol{\sigma}^h + \varepsilon \nabla u^h\|^2 + \|\nabla \times \boldsymbol{\sigma}^h\|^2. \tag{4}$$

For elliptic problems that are diffusion dominated, minimizing (4) using standard finite element spaces results in good convergence. However, for convection-dominated problems, minimizing (4) results in slow convergence until h is very small (typically $h \approx \mathcal{O}(\varepsilon)$). Other finite element approaches tend to be unstable in convection-dominated regimes and solutions may exhibit oscillatory behavior; see, e.g., [Bochev and Gunzburger 2009; Strang and Fix 1973].

One undesirable aspect of the standard least squares approach is that there is not only significant error near boundary layers, but that the error may remain large even in regions of the domain where the solution is smooth. To reduce this "pollution effect", we introduce weight functions into the functional (4) to redefine the metric of the approximation space. By doing this, we are able to force the least squares functional to choose a better solution globally (i.e., in the regions of the domain where the solution is smooth) and segregate errors to a small region near boundary layers. Thus, we want to choose the weight function, w, to be large (a value at or near 1) where the error is small, and small (a value near 0) where the error is large.

In this paper we use what we call a sigma-based weighting strategy that uses the approximate solution for $\boldsymbol{\sigma}$ to construct the weight function. An alternative, one we refer to as functional based weighting, uses locally evaluated functional values to generate weights. Though both strategies have merits, we focus here on sigma-based weights. Consider an approximate solution $\boldsymbol{\sigma}^h$ evaluated on a single finite element triangle, T:

$$\|\boldsymbol{\sigma}^h\|_T = \left(\int_T |\boldsymbol{\sigma}^h|^2\right)^{1/2},$$

which we may use as a local indicator of where the solution is likely to have steep gradients (recall the definition of $\boldsymbol{\sigma}$). We thus choose a weight function, w, on each T by the procedure illustrated in Figure 2.

We choose $w_{min} = e^{-h/\varepsilon}$. For coarse meshes, where weighting is most needed, w_{min} is very near zero. For increasingly fine meshes, where the weight procedure is needed less, we have $\lim_{h\to 0} w_{min} \to 1$. Thus our algorithm remains robust for a wide range of convection-diffusion regimes.

With such an appropriate weight function chosen we find an improved approximate solution by choosing u^h and $\boldsymbol{\sigma}^h$ that minimize the weighted least squares functional

$$\begin{aligned} &G(u^h, \boldsymbol{\sigma}^h; f) \\ &\quad = \|w(\nabla\cdot\boldsymbol{\sigma}^h + \boldsymbol{b}\cdot\nabla u^h - f)\|^2 + \|w(\boldsymbol{\sigma}^h + \varepsilon\nabla u^h)\|^2 + \|w(\nabla\times\boldsymbol{\sigma}^h)\|^2, \end{aligned} \tag{5}$$

where we note that setting $w = 1$ corresponds to the original least squares functional (4). Since this approach obviously requires an initial approximate solution to choose w, it makes sense to conduct this in a nested iteration approach where the initial approximation is found cheaply on a coarse mesh and the improved approximation is found on refined mesh. In other words, our approach is to incorporate

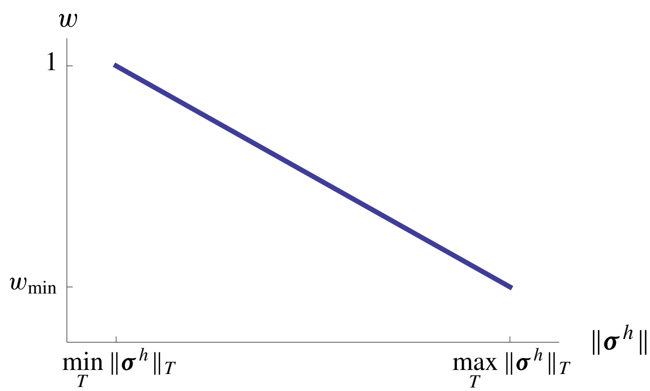

Figure 2. The relationship between $\|\boldsymbol{\sigma}^h\|_T$ and the weight function.

refining the weight function in (5) into an adaptive mesh refinement routine for finding increasingly accurate approximations on a sequence of refined meshes.

We describe the solution process in the following algorithm:

- **Start:** Consider minimizing (5) on an initial coarse triangulation Ω^H, where H is the mesh size. Initially set $w = 1$.
- **Coarse solve:** Minimize (5) to find $(u^H, \boldsymbol{\sigma}^H) \in \boldsymbol{V}^H \times \boldsymbol{\Sigma}^H$.
- **Construct weights:** Using the rule illustrated in Figure 2, choose w to be a piecewise linear function on each element in Ω^H.
- **Refine mesh:** The locally evaluated least squares functional is used to determine triangles in Ω^H with the highest concentration of error, which are refined by splitting each into four smaller triangles. Let $h = H/2$ represent the mesh size of the refined mesh, Ω^h.
- **Fine solve:** Minimize (5) to find $(u^h, \boldsymbol{\sigma}^h) \in \boldsymbol{V}^h \times \boldsymbol{\Sigma}^h$. Set $H \leftarrow h$ as the coarse scale for the problem and repeat the procedure.

Figure 3 illustrates the multilevel iterative algorithm.

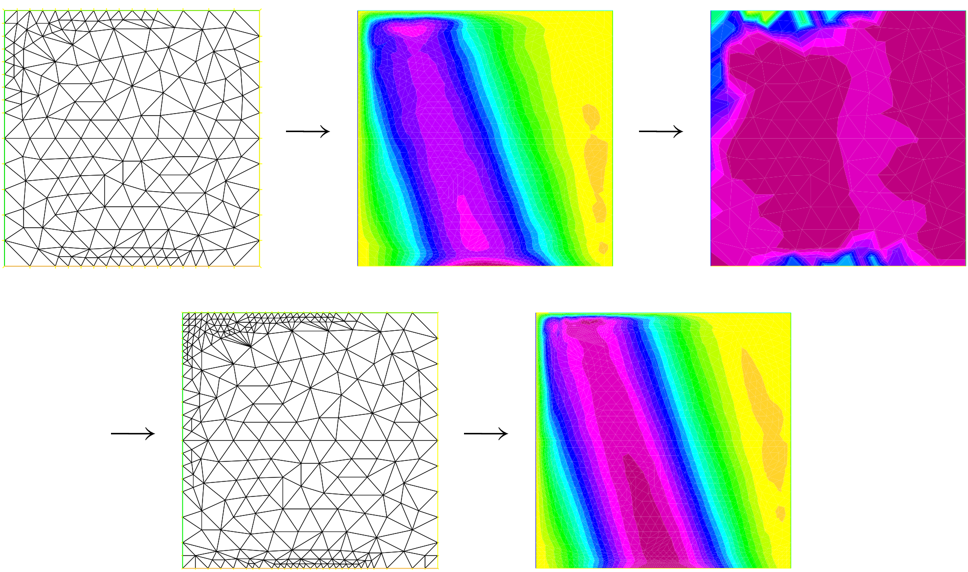

Figure 3. Iterative process for computing approximate solutions: coarse mesh, coarse solution, weight function, refined mesh, fine solution.

3. Testing and results

We test several problems with various levels of difficulty. We compare our approximate solution $(u^h, \boldsymbol{\sigma}^h)$ to a control solution to get the associated error. This control solution is obtained by computing the solution on a superfine scale mesh using the standard LS approach over several iterations. We assume it to be sufficiently accurate for our purpose of comparison. We compute the L^2 norm of this error as a measure of accuracy of the approximated solution. In all cases we use conforming piecewise quadratic finite elements for each unknown. In the computational tests in this section, we choose $\Omega = (0, 1)^2$ and zero Dirichlet boundary conditions on the north, east and west boundaries, and define a nonzero $g(x)$ on the south boundary.

The following four examples compare the efficiency of the standard LS approach and our sigma-based weighting strategy. Both axes in all graphs are on a $\log_{10}$ scale. The points that are higher have larger errors than the lower ones.

Example 1. We solve the system (1) with a constant $\boldsymbol{b} = \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$, a smooth $g = 16x^2(1-x)^2$, and a relatively large $\varepsilon = 0.005$. The results are shown in Figure 4; it can be seen that our sigma-based weighting method yields a more accurate solution (by a factor of 3 approximately) than the standard LS approach.

Example 2. Next we take a nonconstant convection coefficient,

$$\boldsymbol{b} = \left(\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right),$$

with g and ε as in Example 1. The results, shown in Figure 5, show that our approach still outperforms the standard one, though by a lesser factor than before.

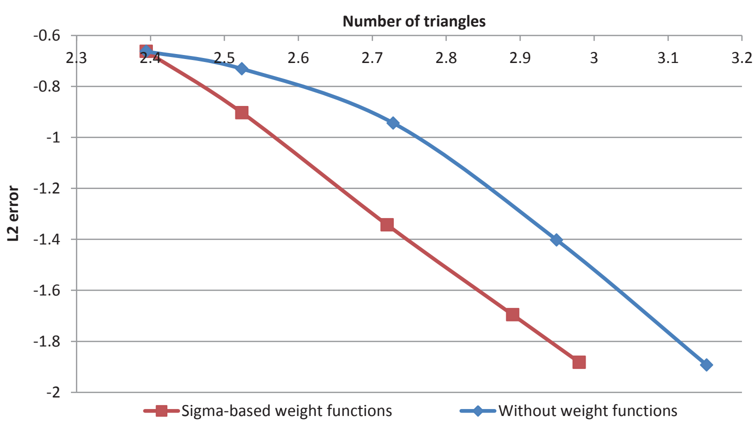

Figure 4. Log-log plot of L^2 norm of error (with respect to control solution) as a function of the number of triangles, for $\boldsymbol{b} = \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$, $g = 16x^2(1-x)^2$, $\varepsilon = 0.005$ (Example 1).

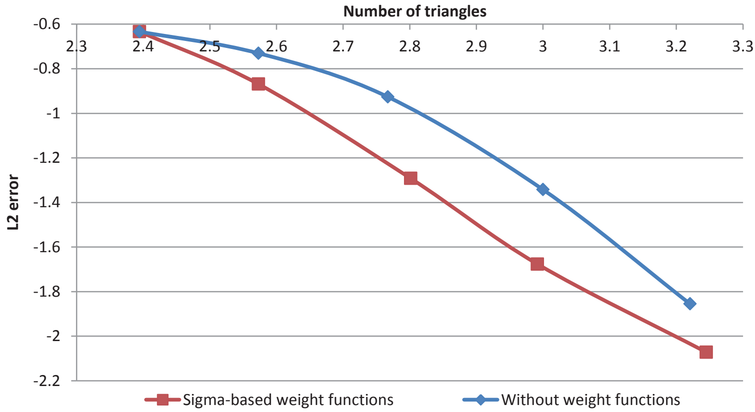

Figure 5. Like Figure 4, with $\boldsymbol{b} = \left(-y/\sqrt{x^2+y^2}, x/\sqrt{x^2+y^2}\right)$, $g = 16x^2(1-x)^2$, $\varepsilon = 0.005$ (Example 2).

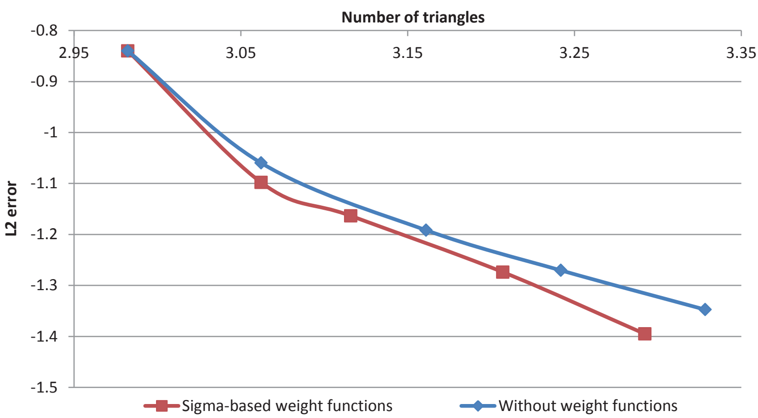

Figure 6. Like Figure 4, with g discontinuous (Example 3).

Example 3. We return to $\boldsymbol{b}$ and ε as in Example 1, and choose a discontinuous boundary function,

$$g = \begin{cases} 1 & \text{if } x \in (0.2, 0.8), \\ 0 & \text{else}. \end{cases}$$

Here the two curves (Figure 6) come even closer than in the previous example, but the solution with sigma-based weights is still the more accurate one. With discontinuous data on the boundary, the solution here is much more difficult to approximate numerically, so the overall error is larger than the previous examples.

Example 4. For our final example in this section, we decrease ε by an order of magnitude, that is, $\varepsilon = 0.0005$, while $\boldsymbol{b} = \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$ and $g = 16x^2(1-x)^2$ stay

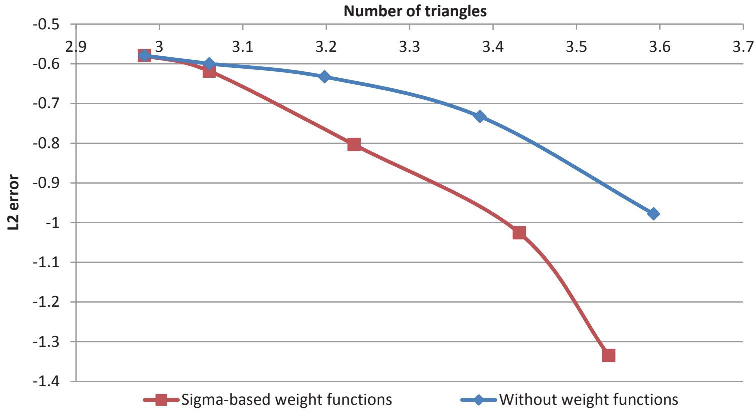

Figure 7. Like Figure 4, with $\boldsymbol{b} = \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$, $g = 16x^2(1-x)^2$, $\varepsilon = 0.0005$.

the same as in Example 1. Here again, our method shows an improvement over the standard approach (Figure 7).

4. Results for Navier–Stokes equations

The preceding examples suggest that the sigma-based weighting method is generally more efficient than the standard LS approach. As a further case study, we consider a more complicated system of equations that retains the same set of challenges as the convection-dominated diffusion problem.

In this section we directly apply the adaptively weighted norm minimization strategy to a more difficult system of equations. We consider the stationary incompressible Navier–Stokes equations in the form

$$\begin{cases} -\dfrac{1}{\mathrm{Re}}\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\boldsymbol{u}$ denotes the velocity of fluid flow in the x and y direction, p the pressure of fluid flow, $\boldsymbol{f}$ a given body force, and Re denotes the Reynolds number, a measure of the potential turbulence of the fluid. With the two substitutions

$$\varepsilon = \frac{1}{\mathrm{Re}} \quad \text{and} \quad \boldsymbol{U} = -\varepsilon\nabla\boldsymbol{u},$$

the first equation in (6) becomes

$$\nabla\cdot\boldsymbol{U} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f}. \tag{7}$$

Utilizing a Newton linearization, we have the following approximation

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} \approx \boldsymbol{u}_{\text{old}} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}_{\text{old}} - \boldsymbol{u}_{\text{old}} \cdot \nabla \boldsymbol{u}_{\text{old}},$$

where $\boldsymbol{u}_{\text{old}} \approx \boldsymbol{u}$ is a known approximation to $\boldsymbol{u}$. This current solution, $\boldsymbol{u}_{\text{old}}$, is initially set to be $(0, 0)$. During the iteration process, each time we obtain a new approximate solution to $\boldsymbol{u}$, we assign its value to $\boldsymbol{u}_{\text{old}}$. Therefore, the older $\boldsymbol{u}_{\text{old}}$ in (6) will be replaced by the new one to better approximate the left-hand side of (6). After the substitution, (6) is reformulated as

$$\begin{cases} \nabla \cdot \boldsymbol{U} + \boldsymbol{u}_{\text{old}} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}_{\text{old}} + \nabla p = \boldsymbol{f} + \boldsymbol{u}_{\text{old}} \cdot \nabla \boldsymbol{u}_{\text{old}} & \text{in } \Omega, \\ \boldsymbol{U} + \varepsilon \nabla \boldsymbol{u} = \boldsymbol{0} & \text{in } \Omega, \\ \nabla \times \boldsymbol{U} = \boldsymbol{0} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \partial\Omega. \end{cases}$$

Notice the similarity between this system and (3), which gives us confidence that the weighted norm procedure can improve a least squares solution method for this system. For large Re, turbulent flow characteristics, including boundary layers, may develop, which is similar to the behavior of convection-dominated PDEs. Therefore, we define our weighted, linearized LS functional to be

$$G(\mathbf{u}, \mathbf{U}, p; \boldsymbol{f}) = \|w(\nabla \cdot \boldsymbol{U} + \boldsymbol{u}_{\text{old}} \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}_{\text{old}} + \nabla p) - (\boldsymbol{f} + \boldsymbol{u}_{\text{old}} \cdot \nabla \boldsymbol{u}_{\text{old}})\|^2 \\ + \|w(\boldsymbol{U} + \varepsilon \nabla \boldsymbol{u})\|^2 + \|w(\nabla \times \boldsymbol{U})\|^2 + \|w(\nabla \cdot \boldsymbol{u})\|^2,$$

where w denotes our weight function. On each mesh we carry out several steps of Newton linearization, and then adaptively refine our mesh. Weight functions are now constructed based on $\boldsymbol{U}$ (which is analogous to $\boldsymbol{\sigma}$ in the convection-dominated diffusion system).

To test our weighting strategy, we choose our domain Ω to be $(0, 1)^2 \setminus (0, 0.5]^2$. We set $\varepsilon = 1/200$ and $\boldsymbol{u} = ((1 - e^{-(y-0.5)/\varepsilon})(1 - e^{-(1-y)/\varepsilon}), 0)$ on the upper west boundary and $\boldsymbol{u} = (0, -(1 - e^{-(x-0.5)/\varepsilon})(1 - e^{-(1-x)/\varepsilon}))$ on the south boundary. We again set $\boldsymbol{f}$ to be 0 for simplicity. Figure 8 shows the control solution for our test problem.

We set the number of Newton linearization steps on each mesh to 3. Figure 9 compares the accuracy of both approaches to solving Navier–Stokes equations. The x- and y-axis are on a $\log_{10}$ scale.

The result shows again that our sigma-based weighting method is more efficient at solving the Navier–Stokes equations than having no weight functions.

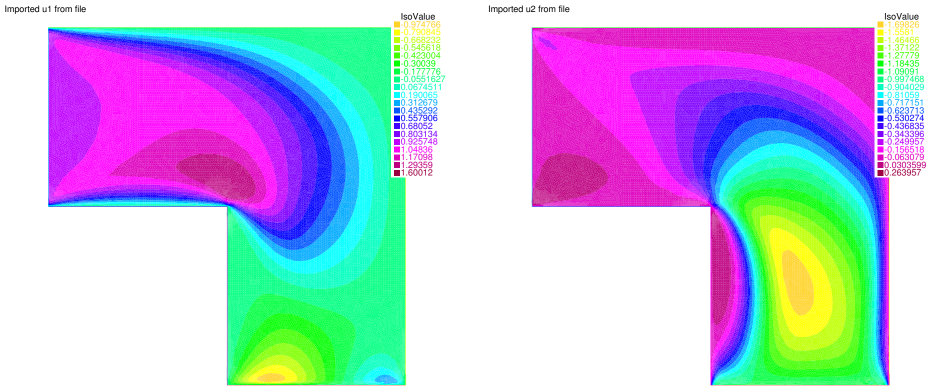

Figure 8. The control solution for u_1 (left) and u_2 (right), obtained on a fine mesh and presumed very accurate.

Figure 9. Comparison of weighted and nonweighted approaches for the Navier–Stokes example (log-log plot).

5. Conclusion

We find that defining and adaptively modifying weight functions in a least squares functional can improve the efficiency of the method for convection-dominated problems. Our approach uses approximate solutions on coarse meshes to adapt the metric of the approximation space so that the error is reduced with respect to a better scaled norm than a standard approach. The procedure is easily adapted to more difficult convection-dominated problems, such as the steady Navier–Stokes equations.

References

[Bochev and Gunzburger 2009] P. B. Bochev and M. D. Gunzburger, *Least-squares finite element methods*, Applied Math. Sci. **166**, Springer, New York, 2009. MR 2010b:65004 Zbl 1168.65067

[Braess 2001] D. Braess, *Finite elements: theory, fast solvers, and applications in solid mechanics*, 2nd ed., Cambridge University Press, Cambridge, 2001. MR 2001k:65002 Zbl 0976.65099

[Brenner and Scott 1994] S. C. Brenner and L. R. Scott, *The mathematical theory of finite element methods*, Texts in Applied Mathematics **15**, Springer, New York, 1994. MR 95f:65001 Zbl 0804.65101

[Cai and Westphal 2008] Z. Cai and C. R. Westphal, "A weighted $H(\mathrm{div})$ least-squares method for second-order elliptic problems", *SIAM J. Numer. Anal.* **46**:3 (2008), 1640–1651. MR 2008m:65314 Zbl 1168.65069

[Lee et al. 2006] E. Lee, T. A. Manteuffel, and C. R. Westphal, "Weighted-norm first-order system least squares (FOSLS) for problems with corner singularities", *SIAM J. Numer. Anal.* **44**:5 (2006), 1974–1996. MR 2008a:65221 Zbl 1129.65087

[Lee et al. 2008] E. Lee, T. A. Manteuffel, and C. R. Westphal, "Weighted-norm first-order system least-squares (FOSLS) for div/curl systems with three dimensional edge singularities", *SIAM J. Numer. Anal.* **46**:3 (2008), 1619–1639. MR 2009c:65316 Zbl 1170.65095

[Strang and Fix 1973] G. Strang and G. J. Fix, *An analysis of the finite element method*, Prentice-Hall, Englewood Cliffs, NJ, 1973. MR 56 #1747 Zbl 0356.65096

Received: 2010-12-20 Accepted: 2011-09-04

bkraynik11@alumnimail.wooster.edu
Department of Mathematics and Computer Science, College of Wooster, Wooster, OH 44691, United States

ysun13@wabash.edu *Department of Mathematics and Computer Science, Wabash College, Crawfordsville, IN 47933, United States*

westphac@wabash.edu *Department of Mathematics and Computer Science, Wabash College, Crawfordsville, IN 47933, United States*

involve

msp.berkeley.edu/involve

EDITORS

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Pietro Cerone	Victoria University, Australia pietro.cerone@vu.edu.au	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tobrie1@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Joseph Gallian	University of Minnesota Duluth, USA jgallian@d.umn.edu	Robert J. Plemmons	Wake Forest University, USA plemmons@wfu.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	Józeph H. Przytycki	George Washington University, USA przytyck@gwu.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Karen Kafadar	University of Colorado, USA karen.kafadar@cudenver.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
David Larson	Texas A&M University, USA larson@math.tamu.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu	Michael E. Zieve	University of Michigan, USA zieve@umich.edu

PRODUCTION

Silvio Levy, Scientific Editor Sheila Newbery, Senior Production Editor Cover design: ©2008 Alex Scorpan

See inside back cover or http://msp.berkeley.edu/involve for submission instructions.

The subscription price for 2012 is US \$105/year for the electronic version, and \$145/year (+\$35 shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94704-3840, USA.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

http://msp.org/

A NON-PROFIT CORPORATION

Typeset in LaTeX

Copyright ©2012 by Mathematical Sciences Publishers

involve

2012 vol. 5 no. 1