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 mathematical sciences publishers

# Permutation notations for the exceptional Weyl group $F_4$

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(Communicated by Joseph Gallian)

This paper describes a permutation notation for the Weyl groups of type  $F_4$  and  $G_2$ . The image in the permutation group is presented as well as an analysis of the structure of the group. This description enables faster computations in these Weyl groups which will prove useful for a variety of applications.

## 1. Introduction

Weyl groups, or finite Coxeter groups, are widely used in mathematics and in applications (some examples are given in Section 2). They are most commonly represented by generators and relations. The disadvantage of that representation is that elements are not uniquely represented by strings or even minimal strings of generators. For the classical Weyl groups combinatorialists use one-line permutation notation, which corresponds to the orbit of the standard basis vectors under the Weyl group. This combinatorial representation provides unique representation, which makes it efficient for computation (see [Haas and Helminck 2012]). Many properties of the elements, such as length and order, can be quickly read from the combinatorial representation (see, for example, [Haas et al. 2007]). Further, the unique representation provides insight into more complex structures such as involution and twisted involution posets; see [Haas and Helminck 2011].

For the exceptional Weyl groups of type  $G_2$ ,  $F_4$ ,  $E_7$  and  $E_8$ , the orbit of the standard basis vectors includes not just the positive and negative axes but additional vectors, making description by permutation somewhat less obvious. Nonetheless,

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*MSC2010:* 20G15, 20G20, 22E15, 22E46, 43A85.

*Keywords:* Weyl groups, Coxeter groups, one-line notation, permutations.

This work was partially supported by NSF grant DMS-0532140 and NSA grant H98230-06-1-0098. Haas was partially supported by NSF grant DMS-0721661, and Helminck by NSF grant DMS-0532140.

similar representations can be made in these cases as well. In this paper we give a permutation representation for the Weyl group of type  $F_4$  and discuss a number of properties of this representation. We also give a similar presentation for  $G_2$ .

## 2. Motivation

Given a field  $k$ , symmetric  $k$ -varieties are the homogenous spaces  $G/H$ , where  $G$  is the set of  $k$ -rational points of reductive group  $\bar{G}$  defined over  $k$  and  $H$  the set of  $k$ -rational points of the set of fixed points of an automorphism  $\sigma$  (defined over  $k$ ) of the group  $\bar{G}$ . For  $k$  the real or  $p$ -adic numbers these are also known as reductive symmetric spaces. These symmetric  $k$ -varieties have a detailed fine structure of root systems and Weyl groups, similar to that of the group  $G$  itself. This fine structure involves 4 (restricted) root systems and Weyl groups. To study the structure of symmetric  $k$ -varieties one needs detailed descriptions of this fine structure and how they act on the various types of elements of these root systems and Weyl groups. For example, to study the representations associated with these symmetric  $k$ -varieties one needs a detailed description of the orbits of (minimal) parabolic  $k$ -subgroups acting on these symmetric  $k$ -varieties. A characterization of these orbits was given in [Helminck and Wang 1993]. They showed that these orbits can be characterized by  $\bigcup_{i \in I} W_G(A_i)/W_H(A_i)$ , where  $\{A_i \mid i \in I\}$  is a set of representatives of the  $H$ -conjugacy classes of the  $\sigma$ -stable maximal  $k$ -split tori,  $W_G(A_i)$  is the set of Weyl group elements that have a representative in  $N_G(A_i)$ , the normalizer of  $A_i$  in  $G$ , and  $W_H(A_i)$  is the set of Weyl group elements that have a representative in  $N_H(A_i)$ . To fully classify these orbits one needs to compute the subgroups  $W_H(A_i)$  of  $W_G(A_i)$ . This requires a detailed analysis of the structure of the Weyl groups and their subgroups.

Another example is that the classification of Cartan subspaces can be reduced to a classification of  $W_H(A)$ -conjugacy classes of  $\sigma$ -singular involutions. The  $W_G(A)$ -conjugacy classes of involutions were classified in [Helminck 1991]. A detailed analysis of the Weyl groups and their subgroups will enable one to determine how a  $W_G(A)$ -conjugacy class breaks up in  $W_H(A)$ -conjugacy classes. There are many other problems related to symmetric  $k$ -varieties for which one needs a detailed description of the various Weyl groups and their subgroups. The detailed combinatorial analysis of the structure of the Weyl groups of types  $F_4$  and  $G_2$  in this paper enables us to compute the necessary data to solve those problems for those symmetric  $k$ -varieties that have a restricted Weyl group of type  $F_4$  and  $G_2$ .

The classical text on Weyl groups is [Bourbaki 2002], while a good modern treatment to Weyl groups and their uses in Lie theory can be found in [Humphreys 1972]. The Weyl groups of type  $F$  and  $G$  are two of the exceptional Coxeter groups; see [Humphreys 1990] for a basic treatment of these groups.

### 3. The Weyl group of type $F_4$

The root system of type  $F_4$  has the following characteristics. There are  $n = 48$  roots. The usual basis is the set

$$\{\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\}.$$

The complete set of roots is  $\{\pm e_i, \pm e_i \pm e_j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}$ . The positive roots are  $\{e_i, e_i \pm e_j, \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$ . Recall the associated Weyl group is generated by the reflections over the hyperplanes orthogonal to the basis roots. These are usually denoted  $s_{\alpha_i}$ , which we abbreviate to  $s_i$ . Here we label the short positive roots with the numbers 1- 12 and describe how the Weyl group of type  $F_4$  is associated with a subgroup of the permutation group on  $[-12, \dots, 12]$ . I.e., each element in  $W(F_4)$  will be associated with a signed permutation on  $\{1, \dots, 12\}$ .

To begin compute the images of the short roots under the basis. These are given in Table 1. Each column in the table describes where each root goes under each basis reflection, so when read from top to bottom, the column gives one-line notation for the generators of the Weyl group. These generators are

$$\begin{aligned} s_1 &= (1, 3, 2, 4, 5, 7, 6, 8, 9, 11, 10, 12), \\ s_2 &= (1, 2, 4, 3, 5, 6, 8, 7, 9, 10, 12, 11), \\ s_3 &= (1, 2, 3, -4, 12, 11, 10, 9, 8, 7, 6, 5), \\ s_4 &= (9, 10, 11, 12, -5, 6, 7, 8, 1, 2, 3, 4). \end{aligned}$$

	root $r$	$s_{\alpha_1}(r)$	$s_{\alpha_2}(r)$	$s_{\alpha_3}(r)$	$s_{\alpha_4}(r)$
1	$e_1$	1	1	1	9
2	$e_2$	3	2	2	10
3	$e_3$	2	4	3	11
4	$e_4$	4	3	-4	12
5	$\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$	5	5	12	-5
6	$\frac{1}{2}(e_1 - e_2 + e_3 + e_4)$	7	6	11	6
7	$\frac{1}{2}(e_1 + e_2 - e_3 + e_4)$	6	8	10	7
8	$\frac{1}{2}(e_1 + e_2 + e_3 - e_4)$	8	7	9	8
9	$\frac{1}{2}(e_1 + e_2 + e_3 + e_4)$	9	9	8	1
10	$\frac{1}{2}(e_1 + e_2 - e_3 - e_4)$	11	10	7	2
11	$\frac{1}{2}(e_1 - e_2 + e_3 - e_4)$	10	12	6	3
12	$\frac{1}{2}(e_1 - e_2 - e_3 + e_4)$	12	11	5	4

**Table 1.** Generators of  $W(F_4)$ .

In cycle notation, they can be expressed as products of transpositions as follows:

$$\begin{aligned} s_1 &= (2, 3)(6, 7)(10, 11), & s_3 &= (8, 9)(7, 10)(6, 11)(5, 12)(4, -4), \\ s_2 &= (3, 4)(7, 8)(11, 12), & s_4 &= (1, 9)(2, 10)(3, 11)(4, 12)(5, -5). \end{aligned}$$

Note that the elements of  $W(F_4)$  are in one-to-one correspondence with only a subset of signed permutations on  $[1, \dots, 12]$ . In particular, since the first 4 elements give the image of the standard basis of  $\mathbb{R}^4$ , they determine the other 8 positions uniquely. In Section 3.10 we will see that there are further restrictions on what can occur in the first four places.

**3.1. A minimal word algorithm.** We develop a method for the important task of converting from this one-line notation to the standard representation of an element as a minimal word. For  $x \in W(F_4)$ , recall that the length of  $x$ ,  $l(x)$ , is the number of letters in the minimal word of  $x$ . It is well-known that the length of  $x$  equals the number of positive roots mapped to negative roots by  $x$ .

**Lemma 3.2.** *Any nontrivial element of  $W(F_4)$  maps at least one of  $e_4$ ,  $e_2 - e_3$ ,  $e_3 - e_4$ , and  $\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$  to a negative root.*

*Proof.* This set of roots is exactly the set of roots which get mapped to negative roots under the basis reflections.  $\square$

**Lemma 3.3.** *Let  $x \in W(F_4)$ . Then  $x$  maps  $\alpha_i$  to a negative root if and only if  $l(xs_i) < l(x)$ .*

*Proof.* This follows directly from the definitions.  $\square$

**Algorithm 3.4.** Given an element  $x = (a_1, a_2, \dots, a_{12}) \in W(F_4)$ , the following algorithm will output a minimal word for  $x$ .

1. If all  $a_i > 0$ , go to step 6. Otherwise, go to step 2.
2. If  $a_4 < 0$ , right multiply by  $s_3$  and go to step 1. Otherwise, go to step 3.
3. If  $a_5 < 0$ , right multiply by  $s_4$  and go to step 1. Otherwise, go to step 4.
4. If  $a_3 < 0$ , right multiply by  $s_2$  and go to step 1. Otherwise, go to step 5.
5. Right multiply by  $s_1$  and go to step 1.
6. If the resulting element is not the identity, compare it to the following list in order to determine the final step(s).
  - (a)  $\{1, 3, 2, 4, 5, 7, 6, 8, 9, 11, 10, 12\} = s_1$ .
  - (b)  $\{1, 2, 4, 3, 5, 6, 8, 7, 9, 10, 12, 11\} = s_2$ .
  - (c)  $\{1, 3, 4, 2, 5, 7, 8, 6, 9, 11, 12, 10\} = s_2s_1$ .
  - (d)  $\{1, 4, 2, 3, 5, 8, 6, 7, 9, 12, 10, 11\} = s_1s_2$ .
  - (e)  $\{1, 4, 3, 2, 5, 8, 7, 6, 9, 12, 11, 10\} = s_1s_2s_1$ .

**Theorem 3.5.** *Algorithm 3.4 produces a minimal word for  $x$ .*

*Proof.* Note that even if  $a_i > 0$  for all  $i$ , the length of  $x$  may not be zero because not all positive roots are represented in the list of twelve roots. In particular, there are six elements of  $W(F_4)$  such that  $a_i > 0$  for all  $i$ . They are precisely those listed in Step 6. of the algorithm together with the identity. Clearly steps 2 and 3 reduce the length of  $x$ . If we arrive at step 4, i.e.,  $a_4 > 0$ , and  $a_3 < 0$ , then one can check that  $e_3 - e_4$  maps to a negative root under  $x$ , so multiplying by  $s_2$  will reduce the length of  $x$ .

If we arrive at step 5, i.e.,  $a_3, a_4, a_5 > 0$ , but some other  $a_i$  is negative, then we show that  $a_2$  must be negative. Suppose instead that  $a_2 > 0$ . Let  $\langle i \rangle + \langle j \rangle$  denote the root which is the vector sum of roots  $i$  and  $j$ . E.g.,  $\langle 5 \rangle = \frac{1}{2}(\langle 1 \rangle - \langle 2 \rangle - \langle 3 \rangle - \langle 4 \rangle)$  and since  $x$  is a linear map this implies  $\langle a_5 \rangle = \frac{1}{2}(\langle a_1 \rangle - \langle a_2 \rangle - \langle a_3 \rangle - \langle a_4 \rangle)$ . Rearranging gives  $\langle a_1 \rangle = \langle a_2 \rangle + \langle a_3 \rangle + \langle a_4 \rangle + 2\langle a_5 \rangle$ , with all terms on the right positive by assumption. Therefore,  $a_1 > 0$ . Similar calculations done in the correct order show that all other  $a_i$  must be positive. Explicitly:  $\langle a_{10} \rangle = \langle a_5 \rangle + \langle a_2 \rangle$ ;  $\langle a_{12} \rangle = \langle a_5 \rangle + \langle a_4 \rangle$ ;  $\langle a_{11} \rangle = \langle a_5 \rangle + \langle a_3 \rangle$ ;  $\langle a_6 \rangle = \langle a_{12} \rangle + \langle a_3 \rangle$ ;  $\langle a_7 \rangle = \langle a_{12} \rangle + \langle a_2 \rangle$ ;  $\langle a_8 \rangle = \langle a_{10} \rangle + \langle a_3 \rangle$ ;  $\langle a_9 \rangle = \frac{1}{2}(\langle a_1 \rangle + \langle a_2 \rangle + \langle a_3 \rangle + \langle a_4 \rangle)$ .

Thus  $a_2 < 0$ . In this case  $e_2 - e_3$  will be mapped to a negative root, so right multiplication by  $s_1$  will reduce the length.

If we arrive at step 6 then all  $a_i > 0$ . Clearly these must be products of  $s_1$  and  $s_2$  only. The 5 elements listed above plus the identity are all the possibilities.  $\square$

One can determine the length of any  $x \in W(F_4)$  by finding a reduced word as above. In what follows we give a combinatorial description of length. Partition the short roots of  $F_4$  into the three sets

$$\alpha = \{\pm 1, \pm 2, \pm 3, \pm 4\}, \quad \beta = \{\pm 5, \pm 6, \pm 7, \pm 8\}, \quad \gamma = \{\pm 9, \pm 10, \pm 11, \pm 12\}.$$

**Lemma 3.6.** *For all  $x \in W(F_4)$ ,  $\{x(\alpha), x(\beta), x(\gamma)\} = \{\alpha, \beta, \gamma\}$ . In other words,  $x$  permutes the sets  $\alpha, \beta$  and  $\gamma$ .*

**Theorem 3.7.** *For an element  $x = (a_1, a_2, \dots, a_{12})$ , define*

$$N(x) = |\{i : a_i < 0\}|$$

and

$$p(a_i, a_j) = \begin{cases} 0 & \text{if } |a_i| < |a_j| \text{ and } a_i > 0, \\ 2 & \text{if } |a_i| < |a_j| \text{ and } a_i < 0, \\ 1 & \text{if } |a_i| > |a_j|. \end{cases}$$

Find  $k$  such that  $\{\pm a_{4k+1}, \pm a_{4k+2}, \pm a_{4k+3}, \pm a_{4k+4}\} = \alpha$ . If  $k = 1$ ,

$$l(x) = \sum_{i>j} p(a_{4k+i}, a_{4k+j}) + N(x).$$

Otherwise,

$$l(x) = \sum_{i<j} p(a_{4k+i}, a_{4k+j}) + N(x).$$

*Proof.* The length counts the number of positive roots mapped to negative roots under  $x$ . The function  $N(x)$  counts the number of short roots mapped to negative roots, while the  $p(a_{4k+i}, a_{4k+j})$  terms account for the number of long roots mapped to negative roots. There are three cases depending on which set is mapped to  $\alpha$ .

Suppose  $x(\alpha) = \alpha$ . Each of the positive long roots  $e_i \pm e_j$ ,  $i < j$  is the sum or difference of the roots  $\langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 4 \rangle$ ; where the difference is taken as  $\langle i \rangle - \langle j \rangle$  where  $i < j$ . Thus to determine which of these is mapped to a negative long root, we need only consider the sum and difference of  $\langle a_i \rangle$  for  $i = 1, \dots, 4$ . It is easy to check that  $\langle a_i \rangle + \langle a_j \rangle$  is a negative root exactly when either  $|a_i| > |a_j|$  and  $a_j < 0$  or when  $|a_i| < |a_j|$  and  $a_i < 0$ . As well,  $\langle a_i \rangle - \langle a_j \rangle$  is negative exactly when  $|a_i| < |a_j|$  and  $a_i < 0$  or when  $|a_i| > |a_j|$  and  $a_j > 0$ .

Suppose  $x(\beta) = \alpha$ . Each of the positive long roots  $e_i \pm e_j$ ,  $j < i$  is the sum or difference of the roots  $\langle 5 \rangle, \langle 6 \rangle, \langle 7 \rangle, \langle 8 \rangle$  where the difference is taken as  $\langle i \rangle - \langle j \rangle$  where  $j < i$ . With this reversed order the same conditions for when  $\langle a_i \rangle + \langle a_j \rangle$  and  $\langle a_i \rangle - \langle a_j \rangle$  are negative will still hold.

Suppose  $x(\gamma) = \alpha$ . Each of the positive long roots  $e_i \pm e_j$ ,  $i < j$  is the sum or difference of the roots  $\langle 9 \rangle, \langle 10 \rangle, \langle 11 \rangle, \langle 12 \rangle$ ; where the difference is taken as  $\langle i \rangle - \langle j \rangle$  where  $i < j$ . Again the same conditions hold.  $\square$

**3.8. Group structure and notation properties.** It is also useful to consider the images of the three sets of short roots as permutations in signed  $S_4$ . Refer to the elements in positions 1-4 as set  $A$ , the elements in positions 5-8 as set  $B$ , and the elements in positions 9-12 as set  $C$ . Formally, let  $f \in F_4$  such that  $f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12})$ . For  $1 \leq i \leq 4$  let  $a_i = f_i \pmod{4}$ , using the representatives  $\{1, 2, 3, 4\}$  for  $\mathbb{Z}_4$ . Similarly,  $b_i = f_{4+i} \pmod{4}$ , and  $c_i = f_{8+i} \pmod{4}$ , for  $1 \leq i \leq 4$  again using the representatives  $\{1, 2, 3, 4\}$  for  $\mathbb{Z}_4$ . Let  $A = (a_1, a_2, a_3, a_4)$ ,  $B = (b_1, b_2, b_3, b_4)$ , and  $C = (c_1, c_2, c_3, c_4)$ . Denote  $(|a_1|, |a_2|, |a_3|, |a_4|)$  by  $|A|$ , and define  $|B|$  and  $|C|$  analogously. For example, if  $f = (6, -8, 5, -7, 9, 11, -10, 12, -2, 4, 1, 3)$ , then  $|A| = (2, 4, 1, 3)$ ,  $|B| = (1, 3, 2, 4)$ , and  $|C| = (2, 4, 1, 3)$ .

**Theorem 3.9.** *The parity of the negations in each block, given the order of the sets  $\alpha$ ,  $\beta$ , and  $\gamma$ , is the following:*

set order	block A (1–4)	block B (5–8)	block C (9–12)
$\alpha\beta\gamma$	even	even	even
$\alpha\gamma\beta$	odd	even	even
$\beta\alpha\gamma$	odd	odd	odd
$\beta\gamma\alpha$	even	odd	odd
$\gamma\alpha\beta$	odd	even	odd
$\gamma\beta\alpha$	even	odd	even

*Proof.* Note that generators  $s_1$  and  $s_2$  do not change the parity of negations in any set, nor do they change the order of the sets. Therefore it suffices to inductively show that this table holds after operating by generators  $s_3$  and  $s_4$  on the right. It is simple to compute using the following rules.  $s_3$  swaps the second and third blocks, and adds or subtracts one negative from the first block.  $s_4$  swaps the first and third blocks, and adds or subtracts one negative from the second block.  $\square$

**3.10. Restrictions on the values of  $|a_i|$ .** Let  $\mathbb{V}$  be the subset of  $S_4$  generated by (12)(34) and (13)(24), and  $K$  be the subset of  $S_4$  generated by (23) and (34).

For  $X \in S_4$  define  $v(X)$  to be the unique element of  $\mathbb{V}$  in the coset  $KX$ .

**Theorem 3.11.** *Let  $f \in W(F_4)$  with sets  $A_f$ ,  $B_f$  and  $C_f$  as defined above. Then  $|C_f| = |B_f|v(|A_f|)$ ,  $|B_f| = |A_f|v(|C_f|)$ , and  $|A_f| = |C_f|v(|B_f|)$ .*

*Alternative statement: Let  $f \in W(F_4)$  with sets  $f_\alpha$ ,  $f_\beta$ , and  $f_\gamma$  as defined above. Then  $f_\gamma = f_\beta v(f_\alpha)$ ,  $f_\beta = f_\alpha v(f_\gamma)$ , and  $f_\alpha = f_\gamma v(f_\beta)$ .*

*Proof.* We proceed by induction. The statement is true for  $f = \text{identity}$ . Assume its true for  $f$ , we show its true for  $s_i f$  for each  $s_i$ . Note that  $v((s_i f)_\mu) = v(f_\mu)$  when  $i = 1, 2, 4$  and  $\mu = \alpha, \beta, \gamma$ ;  $v((s_3 f)_\alpha) = v(f_\alpha)$ ,  $v((s_3 f)_\beta) = (14)(23)v(f_\gamma)$  and  $v((s_3 f)_\gamma) = (14)(23)v(f_\beta)$ . The cases for  $s_i$  where  $i \neq 3$  are straightforward.

Furthermore,  $(s_3 f)_\beta = (14)(23)f_\gamma$  and  $(s_3 f)_\gamma = (14)(23)f_\beta$ . These equations provide all of the required components for the proof. For example assume  $f_\beta = f_\gamma v(f_\alpha)$ . Since  $v((s_3 f)_\alpha) = v(f_\alpha)$  and  $(s_3 f)_\gamma = (14)(23)f_\beta$  and  $(s_3 f)_\beta = (14)(23)f_\gamma$  it follows that  $(s_3 f)_\gamma = (s_3 f)_\beta v((s_3 f)_\alpha)$ .  $\square$

**3.12.  $W(F_4)$  as a semidirect product.** Let  $F_D$  denote the subgroup of  $W(F_4)$  containing all  $d \in F_D$  where  $d(\alpha) = \alpha$ ,  $d(\beta) = \beta$ , and  $d(\gamma) = \gamma$ , and let  $F_S$  be the subgroup of  $W(F_4)$  generated by the generators  $s_3$  and  $s_4$ . Let  $T$  be the group representing the order of the sets  $\alpha$ ,  $\beta$  and  $\gamma$ . Define  $\tau : W(F_4) \mapsto T$  in the obvious way. Note that  $\tau(f) = id$  if and only if  $f \in F_D$ . Now by Theorem 3.9, the sets  $\alpha$ ,  $\beta$  and  $\gamma$  occur in order  $\alpha\beta\gamma$  in the bottom row notation of  $f$  if and only if the permutation  $A_f$  contains an even number of negative signs.

**Lemma 3.13.**  $F_D$  is isomorphic to  $D_4$ .

*Proof.* The map  $\psi : F_D \rightarrow D_4$  such that  $\psi(f) = A_f$  for  $f \in F_D$  provides the isomorphism.  $\square$

**Theorem 3.14.**  $W(F_4) = F_D \rtimes F_S$ .

*Proof.* We can represent  $f \in W(F_4)$  by a pair  $(d, s)$  where  $f = ds$  and  $s$  is the unique element of  $F_S$  such that  $\tau(s) = \tau(f)$ . Define  $\phi_s : F_D \mapsto F_D$  where  $\phi_s(d) = sds^{-1}$  for  $d \in F_D$  and  $s \in F_S$ . One can check that if  $f_1 = d_1 s_1$ , represented by the pair  $(d_1, s_1)$ , and  $f_2 = d_2 s_2$ , represented by the pair  $(d_2, s_2)$ , then  $f_1 f_2 = d_1 \phi_{s_1}(d_2) s_1 s_2$ , represented by the pair  $(d_1 \cdot \phi_{s_1}(d_2), s_1 \cdot s_2)$ .  $\square$

One might hope that this semidirect product would provide an efficient notation for computation in  $W(F_4)$ . A road block to this seems to be finding a combinatorial description of the multiplication.

#### 4. The Weyl group of type $G_2$

The root system of type  $G_2$  has the following characteristics. There are  $n = 12$  roots. The usual basis is the set  $\{\alpha_1 = e_1 - e_2, \alpha_2 = -2e_1 + e_2 + e_3\}$ . The complete set of roots is  $\{\pm(e_i - e_j)\}$ , where  $i < j$  and  $i, j \in \{1, 2, 3\}$ , and  $\{\pm(2e_i - e_j - e_k)\}$ , where  $\{i, j, k\} = \{1, 2, 3\}$ . The positive roots are  $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$ . Again we let  $s_i$  denote the reflection over the hyperplane orthogonal to  $\alpha_i$ . We label the short positive roots  $2\alpha_1 + \alpha_2$ ,  $\alpha_1 + \alpha_2$ , and  $\alpha_1$ , with the numbers 1-3 respectively, and describe how the Weyl group of type  $G_2$  is associated with a subgroup of the permutation group on  $[-3, \dots, 3]$ . Here are the images of roots 1, 2, and 3 under the generators of  $W(G_2)$ :

	root $r$	$s_{\alpha_1}(r)$	$s_{\alpha_2}(r)$
1	$-e_2 + e_3$	2	1
2	$-e_1 + e_3$	1	3
3	$e_1 - e_2$	-3	2

Reading from top to bottom in each column gives one-line notation for the generators, namely  $s_1 = (2, 1, -3)$  and  $s_2 = (1, 3, 2)$ .

As with  $W(F_4)$  we can give a simple combinatorial length formula for  $W(G_2)$ .

**Theorem 4.1.** *The length of an element  $x = (a_1, a_2, a_3)$  in  $W(G_2)$  is given by  $l(x) = \sum_{i < j} p(a_i, a_j)$  where  $p(a_i, a_j)$  is defined as follows:*

$$p(a_i, a_j) = \begin{cases} 0 & \text{if } |a_i| < |a_j| \text{ and } a_i > 0, \\ 2 & \text{if } |a_i| < |a_j| \text{ and } a_i < 0, \\ 1 & \text{if } |a_i| > |a_j|. \end{cases}$$

*Proof.* The length counts the number of positive roots mapped to negative roots under  $x$ . One can check that the positive roots in  $W(G_2)$  are of the form  $\langle i \rangle \pm \langle j \rangle$  where  $i < j$  and  $i, j \in \{1, 2, 3\}$ . To determine which of  $\langle i \rangle \pm \langle j \rangle$  are mapped to negative roots, we need to determine when  $\langle a_i \rangle \pm \langle a_j \rangle$  is a negative root. One can check that  $\langle a_i \rangle + \langle a_j \rangle$  is negative when  $|a_i| < |a_j|$  and  $a_i < 0$ , or when  $|a_i| > |a_j|$  and  $a_j < 0$ . Similarly  $\langle a_i \rangle - \langle a_j \rangle$  is negative when  $|a_i| < |a_j|$  and  $a_i < 0$ , or when  $|a_i| > |a_j|$  and  $a_j > 0$ .  $\square$

## Acknowledgements

The results presented here were obtained as part of the NCSU summer REU program. We would like to thank NSF and NSA for their support of this program.

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Received: 2011-06-25    Revised: 2011-08-22    Accepted: 2011-08-28

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The subscription price for 2012 is US \$105/year for the electronic version, and \$145/year (+\$35 shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94704-3840, USA.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY



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