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Weak Allee effect, grazing, and S-shaped bifurcation curves

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We study a one-dimensional reaction-diffusion model arising in population dynamics where the growth rate is a weak Allee type. In particular, we consider the effects of grazing on the steady states and discuss the complete evolution of the bifurcation curve of positive solutions as the grazing parameter varies. We obtain our results via the quadrature method and Mathematica computations. We establish that the bifurcation curve is S-shaped for certain ranges of the grazing parameter. We also prove this occurrence of an S-shaped bifurcation curve analytically.

1. Introduction

For a given population, a linear relationship between the population's size and its per capita growth rate is often assumed. This correlation is known as logistic type growth and reflects that as a given population grows, its per capita growth rate declines linearly. However, it has been observed that for small population densities, the per capita growth rate increases rather than declines. Logistic growth cannot account for this initial increase, and an alternate model, dubbed the Allee effect [Allee 1938] must be invoked.

The general idea behind the Allee effect is that for small population densities, a variety of factors (such as a shortage of mates or predator saturation) result in an initial increase in the per capita growth rate. There are two types of Allee phenomena: strong Allee effect, whose per capita growth rate begins negative, and weak Allee effect, whose per capita growth rate is initially positive; these are usually modeled in the literature by quadratic functions of the population size. Thus, the

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mathematical analysis of such models is considerably more challenging since the per capita growth rates are neither linear nor always nondecreasing.

When considering the long-term stability of a given population, it is insightful to study other factors affecting the population. By including an additional term that accounts for these natural phenomena, such as grazing, more accurate models can be obtained. Grazing is a type of predation in which an herbivore feeds from plant life. It is also similar to natural predation found in fish populations. The grazing term used in previous models, $cu^2/(1+u^2)$ (see [Van Nes and Scheffer 2005]), is known as the rate of grazing. Since the grazing population is constant, the term converges to c at high vegetation density levels. The effects of grazing have previously been studied with logistic growth, as in [Lee et al. 2011]. In the latter paper it was shown that, for certain ranges of the parameters involved (including c), the bifurcation curve of positive steady states is S-shaped.

Our primary motivation is to analyze the consequences of grazing on a weak Allee effect problem and on a strong Allee effect, in order to determine its effect on the steady state solutions.

Hence, we examine the structure of positive solutions of the steady state equations obtained from the reaction diffusion model,

$$u_t = \frac{1}{\lambda} u_{xx} + u\tilde{f}(u) - \frac{cu^2}{1+u^2} \quad \text{in } (0, 1),$$

with Dirichlet boundary conditions, namely

$$\begin{aligned} -u'' &= \lambda\left[u\tilde{f}(u) - \frac{cu^2}{1+u^2}\right] = \lambda f(u) \quad \text{in } (0, 1),\\ u(0) &= 0, \qquad u(1) = 0, \end{aligned}$$

where u is the population density, $\tilde{f}(u)$ is the per capita growth rate, $1/\lambda$ is the diffusion coefficient where $\lambda > 0$ is a constant, and $c \geq 0$ is also a constant.

Previous studies have analyzed positive solutions to Allee effect problems, both strong and weak (see [Shi and Shivaji 2006], for example), but to our knowledge no information is known about the combination of grazing with Allee effect. In this paper, we will analyze how the addition of a grazing term in combination with a weak Allee and also in combination with a strong Allee type affects the steady states in the one-dimensional problem. Our analysis is completed via the quadrature method [Brown et al. 1981; Laetsch 1970], which we will discuss in Section 2. In Sections 3–5, we provide a detailed analysis on the case when $\tilde{f}$ is weak Allee, i.e.,

$$\tilde{f}(u) = (u+1)(b-u), \quad b > 1.$$

In Section 3, we present some necessary analysis of the zeros of our nonlinearity, f, and in Section 4, we provide the complete evolution of the bifurcation curve of positive solutions via Mathematica computations. In particular, we obtain S-shaped bifurcation curves for certain ranges of parameters bifurcating from the nontrivial branch of solutions. We note here that for all parameter values $u \equiv 0$ is a solution of (1-1). In Section 5, we provide various analytical results, including a proof of the occurrence of such an S-shaped bifurcation curve. In Section 6, we study the case when $\tilde{f}$ represents a logistic growth rate, that is,

$$\tilde{f}(u) = (1 - bu)$$

and provide the evolution of the bifurcation curve as c varies. Next, Section 7 provides the evolution of the bifurcation curve for the case when $\tilde{f}$ is of strong Allee type. That is,

$$\tilde{f}(u) = (u-1)(b-u), \quad b > 1. \tag{1-1}$$

Unlike the weak Allee case and the logistic case, for the strong Allee case, we notice that the variation of c had little effect on the general structure of the bifurcation curve. In particular, no S-shaped bifurcation curve occurred for any parameter values.

Finally, in Section 8, we conclude the paper by considering the biological implications arising from our results. In particular, the ranges of conditional and unconditional persistence in terms of the diffusion coefficient as c varies will also be discussed. Interestingly, our analysis proves that for weak Allee effect growth models, when grazing is large, there exist no ranges of the diffusion coefficient for which conditional persistence exists.

2. Quadrature method

In this section, we recall results via the quadrature method developed by Laetsch [1970] and Brown, Ibrahim, and Shivaji [Brown et al. 1981] to analyze positive solutions to the boundary value problem

$$\begin{aligned} -u''(x) &= \lambda f(u(x)), \quad x \in (0,1), \\ u(0) &= 0, \\ u(1) &= 0, \end{aligned} \tag{2-1}$$

where $f : [0, \infty) \to (0, \infty)$ is a C^1 function and λ is a nonnegative parameter.

Lemma 2.1 [Laetsch 1970]. *Let u be a positive solution to* (2-1) *with $\|u\|_\infty = u(\frac{1}{2}) = \rho > 0$. Such a solution exists if and only if*

$$G(\rho) := \int_0^\rho \frac{dt}{\sqrt{F(\rho) - F(t)}} = \sqrt{\frac{\lambda}{2}}, \tag{2-2}$$

where

$$F(u) = \int_0^u f(s)\,ds.$$

Proof. ($\Rightarrow$) Since (2-1) is an autonomous differential equation, if u is a positive solution to (2-1) such that $u'(x_0) = 0$ for some $x_0 \in (0, 1)$, then both $v(x) := u(x_0 + x)$ and $w(x) := u(x_0 - x)$ satisfy the initial value problem

$$-z''(x) = \lambda f(z(x)), \quad x \in [0, d),$$
$$z(0) = u(x_0), \quad z'(0) = 0,$$

where $d = \min\{x_0, 1 - x_0\}$. By Picard's existence and uniqueness theorem, we can infer that $u(x_0 + x) \equiv u(x_0 - x)$ for all $x \in [0, d)$. Thus, solutions of (2-1) must be symmetric around $x = \frac{1}{2}$, at which point u attains its maximum $\rho := u(\frac{1}{2})$. Multiplying the differential equation in (2-1) by $u'(x)$ gives

$$-\left(\frac{[u'(x)]^2}{2}\right)' = \lambda[F(u(x))]', \tag{2-3}$$

where $F(s) = \int_0^s f(z)\,dz$.

Integrating both sides, we derive

$$\frac{u'(x)}{\sqrt{F(\rho) - F(u(x))}} = \sqrt{2\lambda}, \quad x \in \left[0, \tfrac{1}{2}\right). \tag{2-4}$$

Integrating again, we obtain

$$\int_0^{u(x)} \frac{dt}{\sqrt{F(\rho) - F(t)}} = \sqrt{2\lambda}x, \quad x \in \left[0, \tfrac{1}{2}\right]. \tag{2-5}$$

Substituting $x = \frac{1}{2}$ into (2-5) and using $u(\frac{1}{2}) = \rho$, we now have

$$G(\rho) := \int_0^{\rho} \frac{dt}{\sqrt{F(\rho) - F(t)}} = \sqrt{\frac{\lambda}{2}}. \tag{2-6}$$

Thus, if u is a solution of (2-1) with $\|u\|_\infty = \rho$, then ρ must satisfy the equation $G(\rho) = \sqrt{\lambda/2}$.

($\Leftarrow$) Now, if we have such a value for ρ, we can define our solution u through the equation

$$\int_0^{u(x)} \frac{dt}{\sqrt{F(\rho) - F(t)}} = \sqrt{2\lambda}x, \quad x \in \left[0, \tfrac{1}{2}\right].$$

By the implicit function theorem, u is differentiable; therefore,

$$u'(x) = \sqrt{2\lambda[F(\rho) - F(u(x))]}.$$

Differentiating again gives us

$$-u''(x) = \lambda f(u(x)).$$

Also, it is easy to see that $u(0) = 0$. Finally, defining $u(x)$ to be a symmetric solution, we have that u is a positive solution to (2-1) with $\|u\|_\infty = \rho$ if and only if $\sqrt{\lambda/2} = G(\rho)$. □

Remark 2.1. For values of ρ satisfying the following two conditions, the improper integral in (2-2) will be well-defined and convergent:

$$f(\rho) > 0,$$
$$F(\rho) > F(s) \quad \text{for all } s \in [0, \rho).$$

The following lemma is taken from [Brown et al. 1981], and the proof makes critical use of Lebesgue's dominated convergence theorem to prove the existence of the integral in (2-7).

Lemma 2.2. *$G(\rho)$ is continuous and differentiable on the set*

$$S := \{\rho > 0 \mid f(\rho) > 0 \text{ and } F(\rho) - F(s) > 0 \text{ for all } s \in [0, \rho)\};$$

moreover,

$$G'(\rho) = \int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{\frac{3}{2}}} dv, \tag{2-7}$$

where

$$H(s) := F(s) - \frac{s}{2} f(s). \tag{2-8}$$

3. Preliminaries

We consider the following reaction term, which combines weak Allee effect and grazing:

$$f(u) = u(u+1)(b-u) - \frac{cu^2}{1+u^2} = \frac{u(u+1)(b-u)(1+u^2) - cu^2}{1+u^2},$$

where $b > 1$ and $c \geq 0$. The numerator of $f(u)$ is a fifth degree polynomial. Study of the roots of $f(u)$ reveals the existence of one negative root and one root at $u = 0$, regardless of the values chosen for b and c. On the contrary, the three remaining roots are dependent on the value of the constant c. These three roots fluctuate between real and imaginary values as c changes. Let σ represent the smallest positive root of $f(u)$ in all cases, and let σ_0 and σ_1 denote the two remaining roots. We must also note that a special case occurs for small values of b: for $b \in (1, b_0)$, (for some $b_0 > 1$), there is always exactly one positive real root of $f(u)$, σ. In

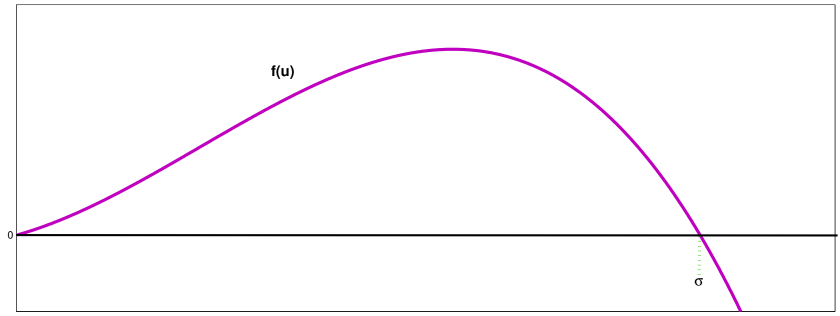

Figure 1. Graph of $f(u)$ with root at σ.

order for $G(\rho)$ to be defined, this variance demands that further analysis of $f(u)$ be completed case-wise (see Remark 2.1).

Remark 3.1. Based on our computations and aid from Mathematica, we conjecture that $b_0 \approx 2.852$.

If $b \in (b_0, \infty)$, the characteristic shape of $f(u)$ varies as the value of c changes. There exists $c_0 > 0$ so that for $c \in (0, c_0)$, $f(u)$ has only one real root, σ. In this case, $f(u)$ resembles Figure 1.

Correspondingly, in this case, $F(u)$ will take the form exemplified in Figure 2. As you can see, $G(\rho)$ will be well-defined for $\rho \in (0, \sigma)$. As c increases, the shape of $f(u)$ changes. There exists $c_1 > c_0$ so that for $c \in (c_0, c_1)$, $f(u)$ has exactly 3 real positive roots, (σ, σ_0, and σ_1). The shape of $f(u)$ is illustrated in Figure 3.

There exists $\hat{c}_1 < c_1$ such that for $c \in (c_0, \hat{c}_1)$, the graph of $F(u)$ resembles Figure 4. We let $\gamma \in (\sigma_0, \sigma_1)$, so that $F(\gamma) = F(\sigma)$. Recall from Remark 2.1 that, to guarantee that $G(\rho)$ is well-defined, we need $f(\rho) > 0$ as well as $F(\rho) > F(u)$ whenever $0 \le u < \rho$. Thus, in this case, $G(\rho)$ will be viable only for $\rho \in (0, \sigma)$ and $\rho \in (\gamma, \sigma_1)$. (The boxed region in Figure 4 has been magnified in Figure 5.)

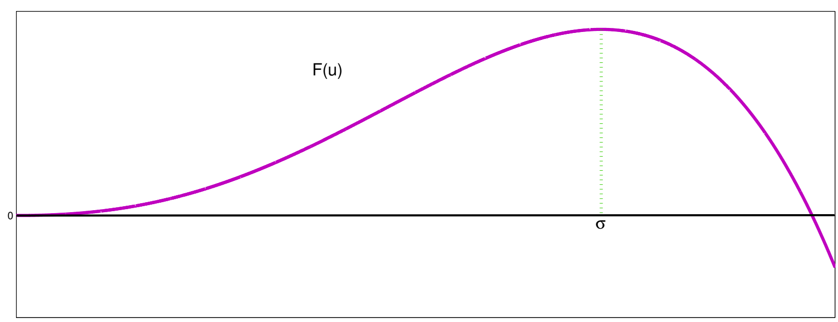

Figure 2. Graph of $F(u)$ for $c \in (0, c_0)$.

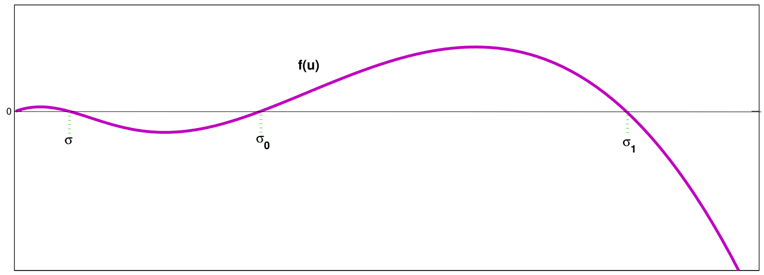

Figure 3. Graph of $f(u)$ with roots at σ, σ_0, and σ_1.

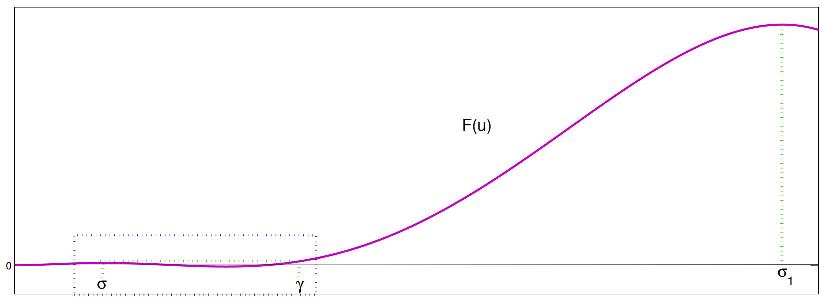

Figure 4. Graph of $F(u)$ for $c \in (c_0, \hat{c}_1)$.

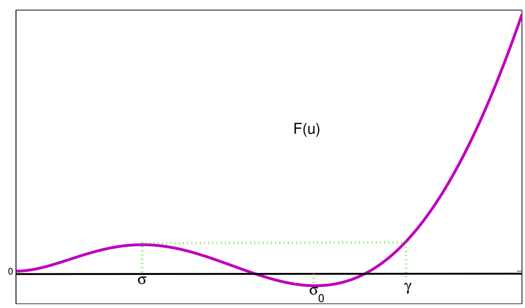

Figure 5. Magnified picture of the boxed region in Figure 4.

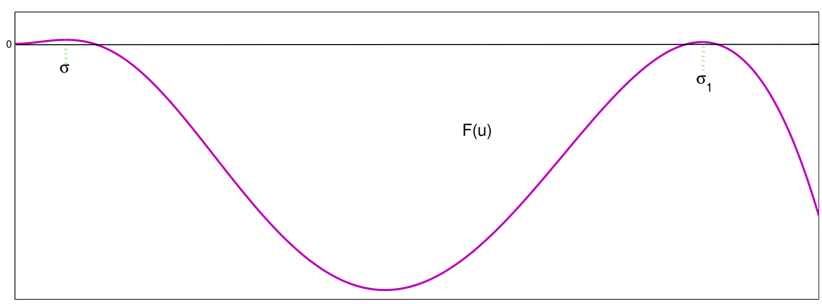

Figure 6. Graph of $F(u)$ for $c \in (\hat{c}_1, c_1)$.

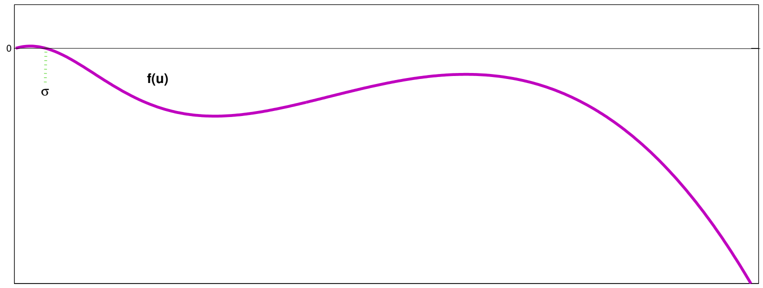

Figure 7. Graph of $f(u)$ with root at σ.

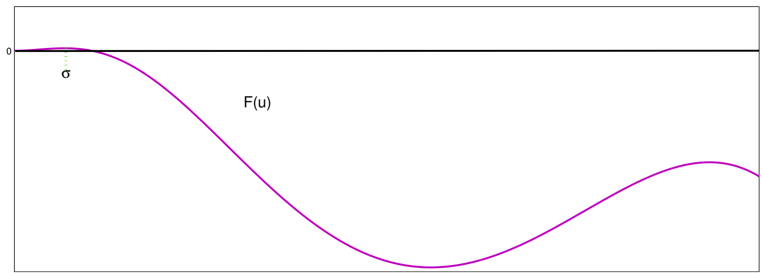

Figure 8. $F(u)$ for $c > c_1$.

The graph of $F(u)$ for $c \in (\hat{c}_1, c_1)$ is illustrated in Figure 6. Clearly, $G(\rho)$ in this instance is only well-defined for $\rho \in (0, \sigma)$. When c exceeds c_1, $f(u)$ is pulled downward and once again has only one real positive root. We will denote this root as σ while σ_0 and σ_1 are imaginary in this case. This is portrayed in Figure 7. Again, $G(\rho)$ is well-defined only for $\rho \in (0, \sigma)$. For $c > c_1$, $F(u)$ will take the form illustrated in Figure 8.

For each of these cases, the structure of positive solutions for (1-1) changes; thus, distinct bifurcation diagrams are obtained, as we now explain.

4. Computational results

In this section, we present the bifurcation diagrams for the weak Allee effect. Recalling Lemma 2.1, we obtained these results via Mathematica by plotting (2-2) for a fixed b-value over a range of c-values.

If $b \in (b_0, \infty)$, then there exist $c_0^*, c_1^*, c_2^* > 0$ such that:

1. If $c \in [0, c_0^*)$, there exist $\lambda_0 > 0$ and $\Lambda = \pi^2/f'(0)$ such that (2-1) has
   - no positive solution for $\lambda \in (0, \lambda_0)$,
   - exactly 1 positive solution for $\lambda = \lambda_0$,
   - exactly 2 positive solutions for $\lambda \in (\lambda_0, \Lambda)$, and
   - exactly 1 positive solution for $\lambda \in [\Lambda, \infty)$.

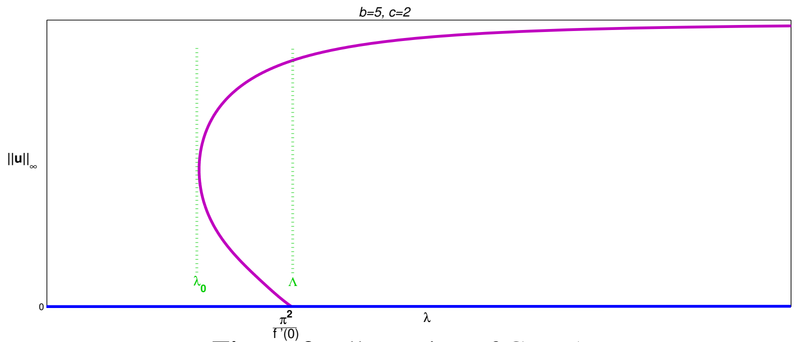

Figure 9. Illustration of Case 1.

(See illustration in Figure 9.)

2. If $c \in [c_0^*, c_1^*)$, there exist $\lambda_0, \lambda_1, \lambda_1, \lambda_2, \Lambda > 0$ such that (2-1) has
   - no positive solution for $\lambda \in (0, \lambda_0)$,
   - exactly 1 positive solution for $\lambda = \lambda_0$,
   - exactly 2 positive solutions for $\lambda \in (\lambda_0, \lambda_1)$,
   - exactly 3 positive solutions for $\lambda = \lambda_1$,
   - exactly 4 positive solutions for $\lambda \in (\lambda_1, \lambda_2)$,
   - exactly 3 positive solutions for $\lambda = \lambda_2$,
   - exactly 2 positive solutions for $\lambda \in (\lambda_2, \Lambda)$, and
   - exactly 1 positive solution for $\lambda \in [\Lambda, \infty)$.

   (See illustration in Figure 10.)

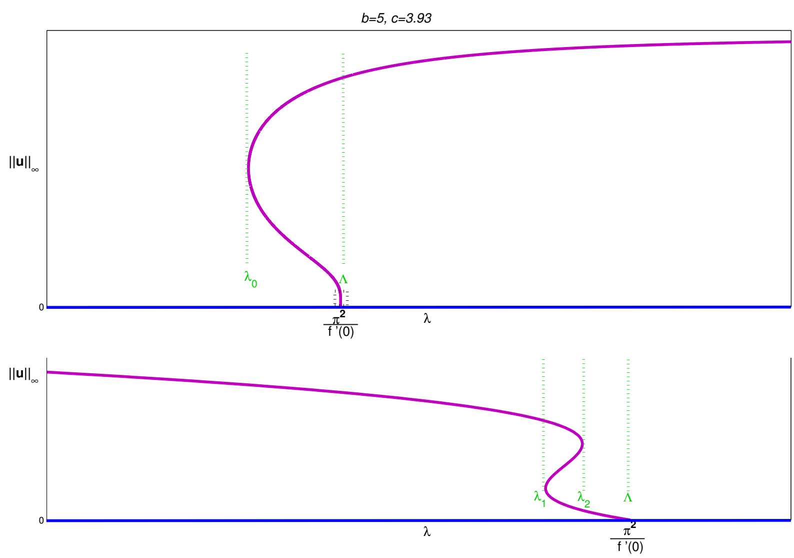

Figure 10. Illustration of Case 2. The bottom diagram shows the contents of the small dotted box under magnification.

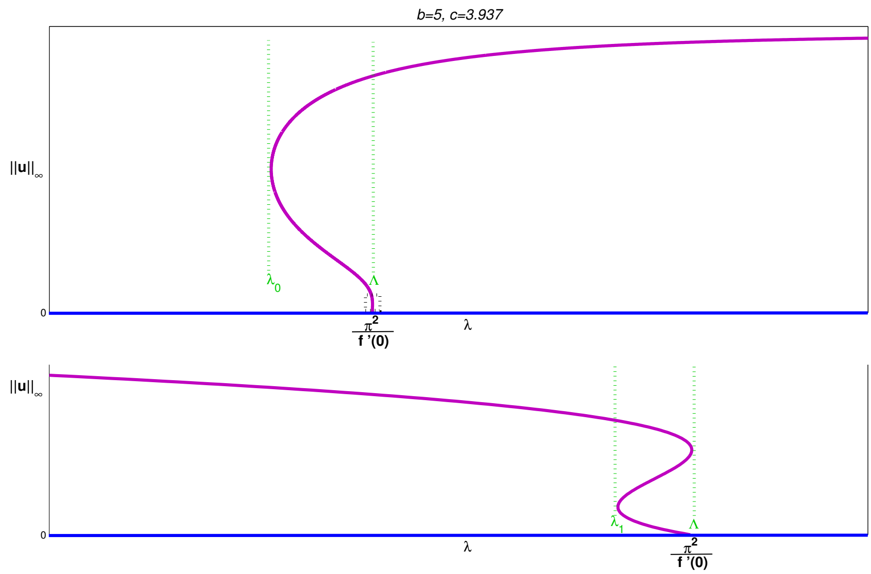

Figure 11. Illustration of Case 3, including magnified detail.

3. If $c = c_1^*$, there exist $\lambda_0, \lambda_1, \Lambda > 0$ such that (2-1) has
   - no positive solution for $\lambda \in (0, \lambda_0)$,
   - exactly 1 positive solution for $\lambda = \lambda_0$,
   - exactly 2 positive solutions for $\lambda \in (\lambda_0, \lambda_1)$,
   - exactly 3 positive solutions for $\lambda = \lambda_1$,
   - exactly 4 positive solutions for $\lambda \in (\lambda_1, \Lambda)$,
   - exactly 2 positive solutions for $\lambda = \Lambda$, and
   - exactly 1 positive solution for $\lambda \in (\Lambda, \infty)$.

   (See illustration in Figure 11.)
4. If $c = (c_1^*, c_2^* = b - 1)$, there exist $\lambda_0, \lambda_1, \lambda_2, \Lambda > 0$ such that (2-1) has
   - no positive solution for $\lambda \in (0, \lambda_0)$,
   - exactly 1 positive solutions for $\lambda = \lambda_0$,
   - exactly 2 positive solutions for $\lambda \in (\lambda_0, \lambda_1)$,
   - exactly 3 positive solutions for $\lambda = \lambda_1$,
   - exactly 4 positive solutions for $\lambda \in (\lambda_1, \Lambda)$,
   - exactly 3 positive solutions for $\lambda \in [\Lambda, \lambda_2)$,
   - exactly 2 positive solutions for $\lambda = \lambda_2$, and
   - exactly 1 positive solution for $\lambda \in (\lambda_2, \infty)$.

   (See illustration in Figure 12.)
5. If $c = c_2^* = b - 1$, there exist $\lambda_0, \Lambda > 0$ such that (2-1) has
   - no positive solution for $\lambda \in (0, \lambda_0)$,
   - exactly 1 positive solutions for $\lambda = \lambda_0$,

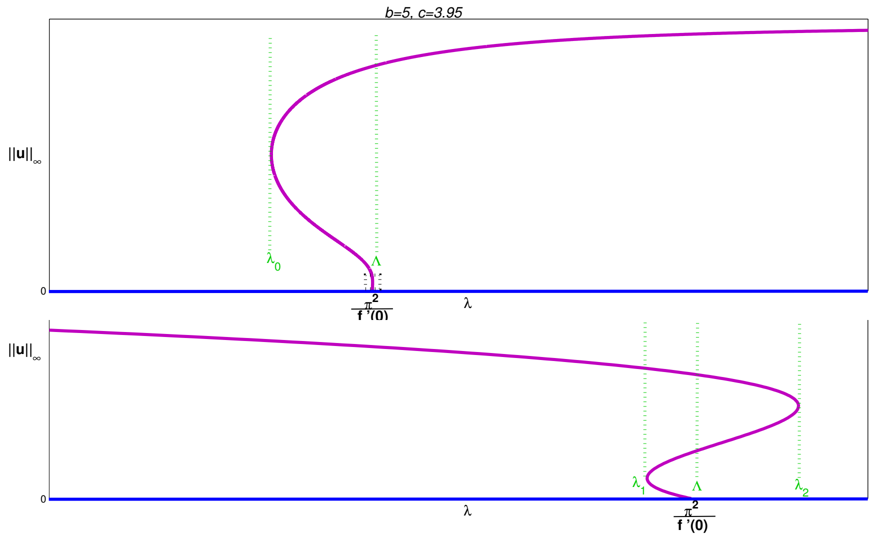

Figure 12. Illustration of Case 4, including magnified detail.

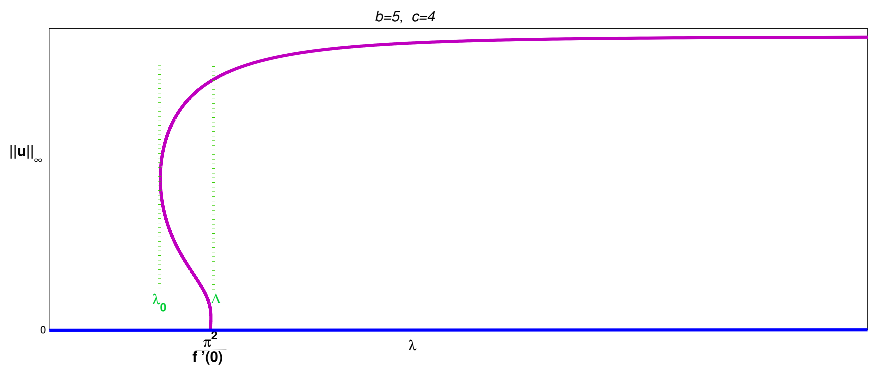

Figure 13. Illustration of Case 5.

- exactly 2 positive solutions for $\lambda \in (\lambda_0, \Lambda)$, and
- exactly 1 positive solutions for $\lambda \in [\Lambda, \infty)$.

(See illustration in Figure 13.)

6. If $c \in (c_2^* = b - 1, c_3^*)$, there exist $\lambda_0, \Lambda, \lambda_2 > 0$ such that (2-1) has
   - no positive solution for $\lambda \in (0, \lambda_0)$,
   - exactly 1 positive solution for $\lambda = \lambda_0$,
   - exactly 2 positive solutions for $\lambda \in (\lambda_0, \Lambda]$,
   - exactly 3 positive solutions for $\lambda \in (\Lambda, \lambda_2)$,
   - exactly 2 positive solutions for $\lambda = \lambda_2$, and
   - exactly 1 positive solution for $\lambda \in (\lambda_2, \infty)$.

   (See illustration in Figure 14.)

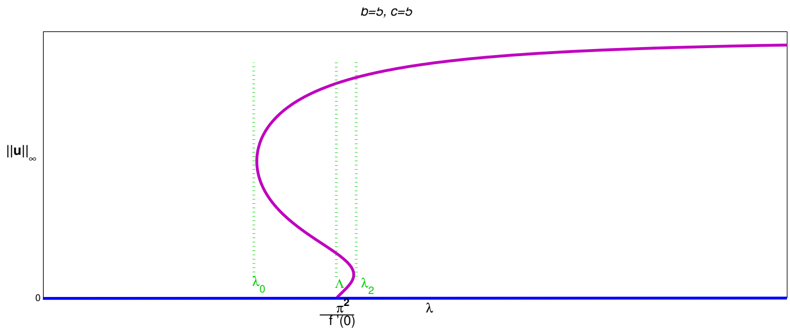

Figure 14. Illustration of Case 6.

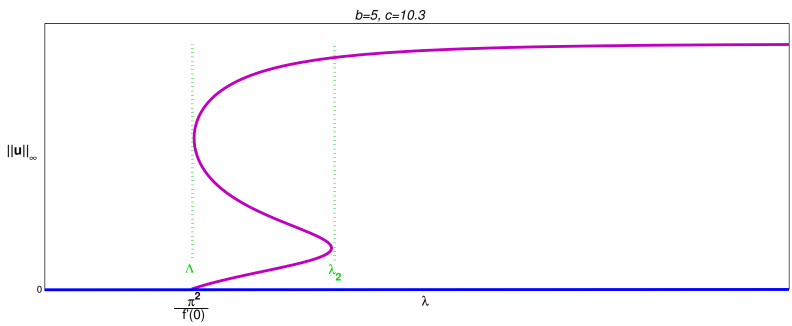

Figure 15. Illustration of Case 7.

7. If $c = c_3^*$, there exist $\Lambda, \lambda_2 > 0$ such that (2-1) has
   - no positive solution for $\lambda \in (0, \Lambda)$,
   - exactly 1 positive solution for $\lambda = \Lambda$,
   - exactly 3 positive solutions for $\lambda \in (\Lambda, \lambda_2)$,
   - exactly 2 positive solutions for $\lambda = \lambda_2$, and
   - exactly 1 positive solution for $\lambda \in (\lambda_2, \infty)$.

   (See illustration in Figure 15.)
8. If $c \in (c_3^*, c_4^*)$, there exist $\Lambda, \lambda_0, \lambda_2 > 0$ such that (2-1) has

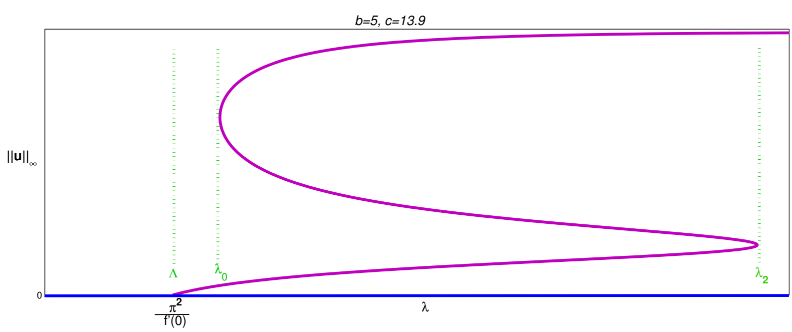

Figure 16. Illustration of Case 8.

- no positive solution for $\lambda \in (0, \Lambda]$,
- exactly 1 positive solution for $\lambda \in (\Lambda, \lambda_0)$,
- exactly 2 positive solutions for $\lambda = \lambda_0$,
- exactly 3 positive solutions for $\lambda \in (\lambda_0, \lambda_2)$,
- exactly 2 positive solutions for $\lambda = \lambda_2$, and
- exactly 1 positive solution for $\lambda \in (\lambda_2, \infty)$.

(See illustration in Figure 16.)

9. If $c \in [c_4^*, c_5^*)$, there exist $\Lambda, \lambda_0 > 0$ such that (2-1) has
    - no positive solution for $\lambda \in (0, \Lambda]$,
    - exactly 1 positive solution for $\lambda \in (\Lambda, \lambda_0)$,
    - exactly 2 positive solutions for $\lambda = \lambda_0$, and
    - exactly 3 positive solutions for $\lambda \in (\lambda_0, \infty)$.

    (See illustration in Figure 17.)

10. If $c \in [c_5^*, \infty)$, there exists $\Lambda > 0$ such that (1-1) has
    - no positive solution for $\lambda \in (0, \Lambda]$, and
    - exactly 1 positive solution for $\lambda \in (\Lambda, \infty)$.

    (See illustration in Figure 18.)

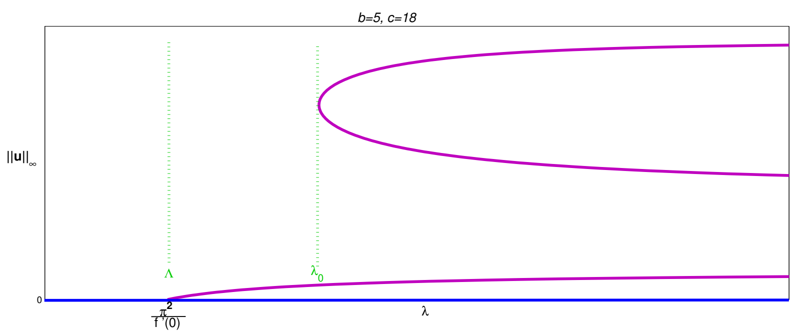

Figure 17. Illustration of Case 9.

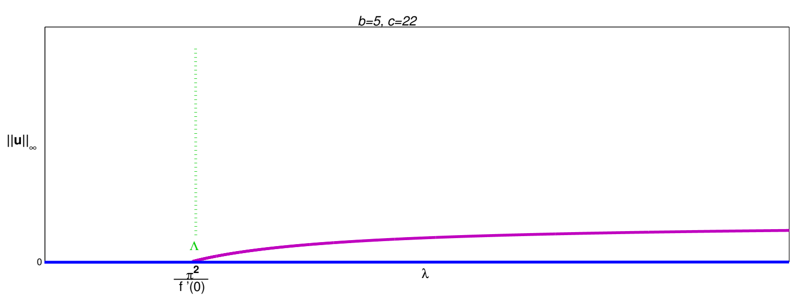

Figure 18. Illustration of Case 10.

5. Analytical results

In this section, we provide analytical proofs of several results that help detail the global behavior of bifurcation curves and further corroborate our computational results presented in the previous section. First we state two results on the behavior of $G(\rho)$ when $\rho \to 0^+$ and when $\rho \to \sigma^-$, where σ is the smallest positive root of $f(u)$. The proofs of these results are provided in the Appendix. One may also refer to [Laetsch 1970], where such results were discussed.

Lemma 5.1. $\lim_{\rho\to 0^+} G(\rho) = \pi/\sqrt{2b}$.

Lemma 5.2. $\lim_{\rho\to\sigma^-} G(\rho) = \infty$.

Next we establish a precise value of c such that for c smaller than this value, the bifurcation curve bifurcates to the left near $(\pi/\sqrt{2b}, 0)$ while for c greater than this value, the bifurcation curve bifurcates to the right near $(\pi/\sqrt{2b}, 0)$. Namely, we establish the following:

Theorem 5.1. *Let* $c_2^* = b - 1$. *If* $0 \le c < c_2^*$, *then* $G'(\rho) < 0$ *on some interval* $(0, \rho_1)$ *and if* $c > c_2^*$, *then* $G'(\rho) > 0$ *on some interval* $(0, \rho_1)$.

Proof. Recall

$$G'(\rho) = \int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{\frac{3}{2}}} dv \tag{5-1}$$

where

$$H(s) = F(s) - \frac{s}{2} f(s). \tag{5-2}$$

Note that $H(0) = 0$. Thus, showing $H'(s) > 0$ for some $0 < s < s_0$ with $s_0 \approx 0$ implies $G'(\rho) > 0$ on some interval $(0, \rho_1)$. We have

$$H'(s) = \tfrac{1}{2}[f(s) - sf'(s)].$$

$H'(0) = 0$ also. Therefore, we differentiate again.

$$H''(s) = -\frac{s}{2} f''(s).$$

Clearly, the sign of $H''(s)$ is dependent only on the sign of $f''(s)$. We know

$$f(s) = s(s+1)(b-s) - c\frac{s^2}{1+s^2} \tag{5-3}$$

$$= -s^3 + (b-1)s^2 + bs - \frac{cs^2}{1+s^2}. \tag{5-4}$$

By taking the first derivative and simplifying, we get

$$f'(s) = -3s^2 + 2(b-1)s + b - c\frac{2s}{(1+s^2)^2}.$$

Then, taking the second derivative and simplifying, we obtain

$$f''(s) = -6s + 2(b-1) - c\frac{2-6s^2}{(1+s^2)^3}.$$

Evaluating $f''(s)$ when $s = 0$ gives

$$f''(0) = 2(b-1) - 2c. \tag{5-5}$$

By analysis of (5-5), we see that $c < (b-1) \Rightarrow f''(0) > 0 \Rightarrow H''(s) < 0 \Rightarrow H'(s) < 0$ for $s \in (0, s_0)$ for some $s_0 > 0 \Rightarrow G'(\rho) < 0$ for $\rho \approx 0$. Conversely, we have $c > (b-1) \Rightarrow f''(0) < 0 \Rightarrow H''(s) > 0 \Rightarrow H'(s) > 0$ for $s \in (0, s_1)$ for some $s_1 > 0 \Rightarrow G'(\rho) > 0$ for $\rho \approx 0$. □

Now we establish our main result of this section, the occurrence of an S-shaped bifurcation curve.

Theorem 5.2. *Let* $b > 4$ *and* $c \in (b-1, \frac{3}{2}b - 3)$. *Then the bifurcation curve for* (1-1) *is guaranteed to be at least S-shaped.* (See Figure 19.)

Proof. The proof is divided into three steps. In Step 1, we establish that if $b > 2$ and $c \in (\max\{0, b-5\}, \frac{3}{2}b - 3)$, then $2 \in (0, \sigma)$, for which we must recall that σ is the smallest positive root of $f(u)$. In Step 2, we prove that if $b > 4$ and $c \in (\max\{0, b-5\}, \frac{3}{2}b-3)$, then $H(2) < 0$. In Step 3, we prove that the bifurcation curve is at least S-shaped.

Step 1. Consider the functions

$$f(u) = u(u+1)(b-u) - \frac{cu^2}{u^2+1}$$

and

$$k(u) = u(u+1)(b-u) - cu^2.$$

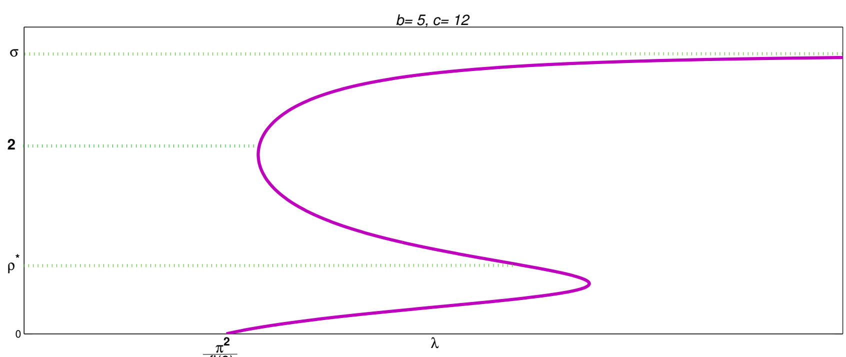

Figure 19

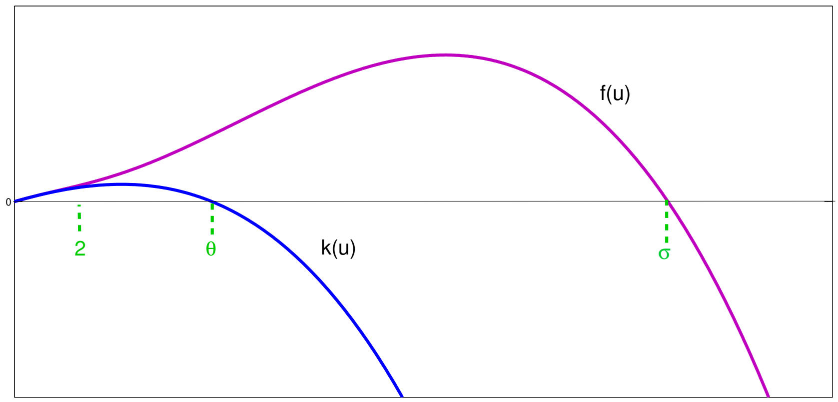

Figure 20

As shown in Figure 20, it is clear $f(u) \geq k(u)$. Any positive root of $k(u)$, θ, will occur before any positive root of $f(u)$, σ. Thus, if $r \in (0, \theta)$, then $r \in (0, \sigma)$.

Therefore, it suffices to show that $2 \in (0, \theta)$. By solving $k(u) = 0$, we obtain

$$\theta = \frac{(b-1-c)+\sqrt{4b+(b-1-c)^2}}{2}. \tag{5-6}$$

We want $\theta > 2$, so

$$\frac{(b-1-c)+\sqrt{4b+(b-1-c)^2}}{2} > 2 \tag{5-7}$$

Simplifying, we obtain

$$\sqrt{4b+(b-1-c)^2} > (5+c-b) \tag{5-8}$$

If $5+c-b \leq 0$, then it is clear the above inequality holds true. If $5+c-b > 0$, then $c > b-5$ and squaring and solving (5-8) gives

$$c < \tfrac{3}{2}b - 3. \tag{5-9}$$

Thus, if $b > 2$ and $c \in (\max\{0, b-5\}, \frac{3}{2}b-3)$, then $2 \in (0, \theta)$; hence, $2 \in (0, \sigma)$.

Step 2. Recall that

$$H(s) = F(s) - \frac{s}{2} f(s) = \frac{s^4}{4} - (b-1)\frac{s^3}{6} + c\left[\frac{s^3}{2(1+s^2)} - s + \arctan s\right].$$

Then,

$$H(2) = 4-(b-1)\tfrac{4}{3} + c[\tfrac{4}{5} - 2 + \arctan 2] \leq \frac{16-4b}{3} + c(-\tfrac{6}{5}+1) < 0$$

for all $b > 4$.

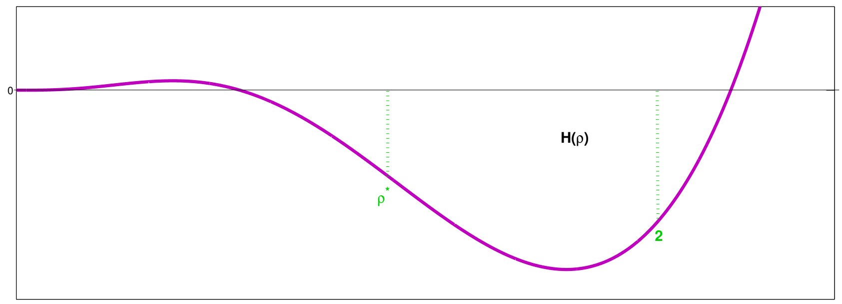

Figure 21

Step 3. Let $b > 4$ and $c \in (b-1, \frac{3}{2}b-3)$. Recall from Lemma 2.2 that

$$G'(\rho) = \int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{\frac{3}{2}}} dv. \tag{5-10}$$

Then from Theorem 5.1, we conclude that $G'(\rho)$ begins positive. From Steps 1 and 2, we know $2 \in (0, \sigma)$ and $H(2) < 0$. Hence there exists $\rho^* \in (0, 2)$, (say, the first zero of $H(\rho)$), such that $G'(\rho^*) < 0$. (See Figure 16.) Also, $\lim_{\rho\to\sigma^-} G(\rho) = \infty$ by Lemma 5.2. Therefore, the graph of $G(\rho)$ must be at least S-shaped. □

Next we study the bifurcation diagram for a larger range of c values. First, we consider the case when the bifurcation curve is split, providing various numbers of positive solutions for different ranges of λ:

Theorem 5.3. *There exist $c_0, \hat{c}_1$ $(< c_1)$ such that for $c \in (c_0, \hat{c}_1)$, there exists a $\lambda_1 > (\pi/\sqrt{2b})^2$ such that* (1-1) *has at least one positive solution in $((\pi/\sqrt{2b})^2, \lambda_1)$, at least two positive solutions for $\lambda = \lambda_1$, and at least three positive solutions for $\lambda > \lambda_1$.*

Proof. By Lemma 5.1, $\lim_{\rho\to 0^+} G(\rho) = \pi/\sqrt{2b}$. Recall from Section 3 that for $c_0 < c < c_1$, $f(s)$ has the appearance shown in Figure 22.

Furthermore, recall from Section 3 that $G(\rho)$ is well-defined for $\rho \in (0, \sigma)$ and $\rho \in (\gamma, \sigma_1)$, and that $F(\gamma) = F(\sigma)$ (see the graph of $F(u)$ for $c \in (c_0, \hat{c}_1)$ in Figure 4, page 139). From Lemma 5.2, we know $\lim_{\rho\to\sigma^-} G(\rho) = \infty$. A similar argument can be applied to show that $\lim_{\rho\to\sigma_1^-} G(\rho) = \infty$. The proof of Theorem 5.3 is complete if $\lim_{\rho\to\gamma^+} G(\rho) = +\infty$. Such a result was proved in [Brown and Budin 1979], which we will recall now. First recall that $F(\gamma) = F(\sigma)$. We let $A = \max\{|f'(s)|; s \in [0, \sigma_1]\}$. Then we can note that $|f(s)| \le A|s - \sigma|$ for all $s \in [0, \sigma_1]$. Next we let $B = \max\{|f(s)|; 0 \le s \le \sigma_1\}$. Now, if $\sigma_1 > \rho > \gamma$ and $0 \le s < \rho$, then

$$F(\rho) - F(s) = F(\rho) - F(\gamma) + F(\sigma) - F(s).$$

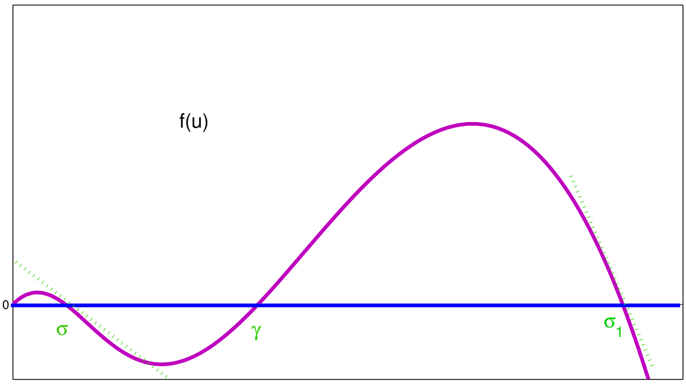

Figure 22

By the mean value theorem, we can then write this as

$$F(\rho)-F(s)=(\rho-\gamma)f(\xi)+(\sigma-s)f(\eta),$$

where $\xi\in(\gamma,\rho)$ and η lies between σ and s. Hence,

$$F(\rho)-F(s)\le B(\rho-\gamma)+A(\sigma-s)^2.$$

Recall that

$$G(\rho)=\sqrt{2}\int_0^\rho [F(\rho)-F(s)]^{-\frac{1}{2}}ds.$$

By substitution, we can write

$$G(\rho)\ge\int_0^\gamma \sqrt{2}\big[B(\rho-\gamma)+A(\sigma-s)^2\big]^{-\frac{1}{2}}ds.$$

Let $H_\rho(s)=\sqrt{2}\big[B(\rho-\gamma)+A(\sigma-s)^2\big]^{-\frac{1}{2}}$. Since H_ρ is nondecreasing as $\rho\to\gamma^+$, we can apply the monotonic convergence theorem, which gives us

$$\lim_{\rho\to\gamma^+}G(\rho)\ge\lim_{\rho\to\gamma^+}\int_0^\gamma H_\rho(s)\,ds=\int_0^\gamma\sqrt{2}A^{-\frac{1}{2}}|\sigma-s|^{-1}ds$$
$$=\sqrt{2}A^{-\frac{1}{2}}\int_0^\sigma(\sigma-s)^{-1}ds+\int_\sigma^\gamma(s-\sigma)^{-1}ds.$$

Both these integrals diverge to $+\infty$, so $\lim_{\rho\to\gamma^+}G(\rho)=+\infty$. □

Thus, we obtain the bifurcation diagram represented in Figure 23 (see next page). Next we establish a result for large values of c:

Theorem 5.4. *There exists a $\tilde{c}$ such that if $c>\tilde{c}$, then* (1-1) *has a unique positive solution for all $\lambda>(\pi/\sqrt{2b})^2$.*

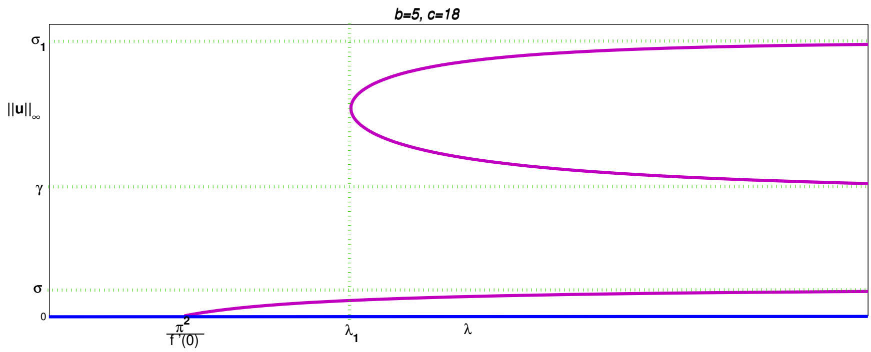

Figure 23

Proof. From Section 3 we know that for $c > \hat{c}_1$, $G(\rho)$ is only defined for $\rho \in (0, \sigma)$. With $f_1(u) = u(u+1)(b-u)$ and $f_2(u) = cu^2/(1+u^2)$, the graph of $f_1 - f_2$ is illustrated in Figure 24. Recall the equality

$$G'(\rho) = \int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{\frac{3}{2}}} dv$$

with

$$H(s) = F(s) - \tfrac{1}{2} s f(s), \quad H'(s) = \tfrac{1}{2}[f(s) - s f'(s)], \quad H''(s) = -\tfrac{1}{2} s f''(s).$$

We wish to show that $f''(s) < 0$ for $0 < s < \sigma$. This will alternatively imply that $H''(s) > 0$ for $0 < s < \sigma$, noting once again that $H(0) = H'(0) = 0$. Therefore, showing $H'(s) > 0$ for $0 < s < \sigma$ implies $G'(\rho) > 0$ for $0 < \rho < \sigma$, as shown in the bifurcation diagram in Figure 25.

We begin with the analysis of $f''(s)$.

$$f''(s) = -6s + 2(b-1) - c\frac{2 - 6s^2}{(1+s^2)^3} \tag{5-11}$$

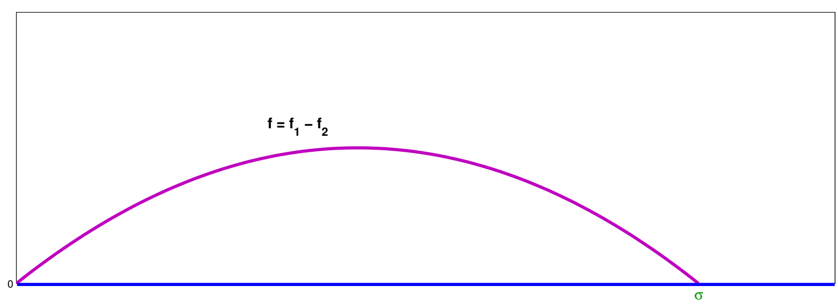

Figure 24. $f(u) = f_1 - f_2$.

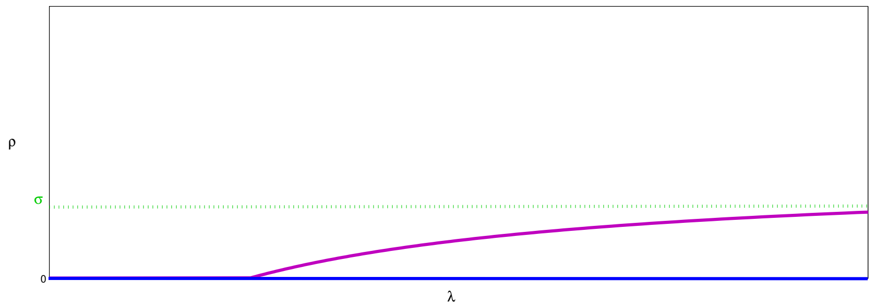

Figure 25

We can then bound this function by a larger one, so we choose

$$f''(s) \le 2(b-1) - c\frac{2-6s^2}{(1+s^2)^3}. \tag{5-12}$$

Let

$$B(s) = \frac{2-6s^2}{(1+s^2)^3}.$$

Note that $B(s) > 0$ on $0 \le s < \frac{1}{\sqrt{3}}$. For $c \gg 1$, we can assume that $\sigma < \frac{1}{2\sqrt{3}}$. Hence, for $c \gg 1$, there exists $\delta > 0$ such that $B(s) \ge \delta$ for all $0 \le s \le \sigma$. Thus,

$$f''(s) \le 2(b-1) - c\delta \quad \text{for all } \ 0 \le s \le \sigma. \tag{5-13}$$

Therefore, for $c \gg 1$, $f''(s) < 0$ for $0 < s < \sigma$. Hence, we know then that $H''(s) > 0$ for $0 < s < \sigma$ and $G'(\rho) > 0$ for $0 < \rho < \sigma$. □

The corresponding bifurcation diagram is illustrated in Figure 25.

6. Computational results for logistic growth

In [Lee et al. 2011], the effect of grazing on a logistic growth rate was studied. Several bifurcation diagrams were provided, but a complete bifurcation evolution for the one-dimensional case as c varies was not provided. It is useful to compare these computational results to those of the weak and strong Allee effect. The combination of grazing with a logistic growth rate can be illustrated by the following equation:

$$\hat{f}(u) = u(1-bu) - c\frac{u^2}{1+u^2}; \ \ b > 0, c \ge 0. \tag{6-1}$$

We obtain our evolution results via the quadrature method and Mathematica computations. The following figures illustrate this evolution for a fixed b as c increases.

If $b \in (0, b_0)$, then there exist $\hat{c}_0, \hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{c}_4 > 0$ such that:

1. If $c \in [0, \hat{c}_0)$, there exists a $\Lambda > 0$ such that (6-1) has

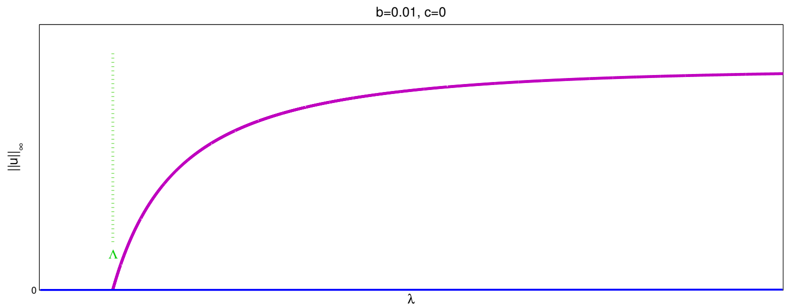

Figure 26. Illustration of Case 1.

- no positive solution for $\lambda \in (0, \Lambda)$, and
- exactly 1 positive solution for $\lambda \in [\Lambda, \infty)$.

(See illustration in Figure 26.)

2. If $c \in (\hat{c}_0, \hat{c}_1)$, there exist $\Lambda, \lambda_0, \lambda_1 > 0$ such that (6-1) has
   - no positive solution for $\lambda \in (0, \Lambda)$,
   - exactly 1 positive solution for $\lambda \in [\Lambda, \lambda_0)$,
   - exactly 2 positive solutions for $\lambda = \lambda_0$,
   - exactly 3 positive solutions for $\lambda \in (\lambda_0, \lambda_1]$, and
   - exactly 1 positive solution for $\lambda \in (\lambda_1, \infty)$.

   (See illustration in Figure 27.)

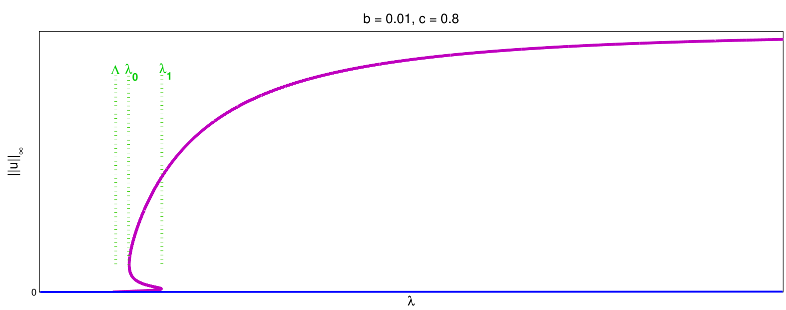

Figure 27. Illustration of Case 2.

3. If $c \in (\hat{c}_2, \hat{c}_3)$, there exist $\Lambda, \lambda_0 > 0$ such that (6-1) has
   - no positive solution for $\lambda \in (0, \Lambda)$,
   - exactly 1 positive solution for $\lambda \in (\Lambda, \lambda_0)$,
   - exactly 2 positive solutions for $\lambda = \lambda_0$, and
   - exactly 3 positive solutions for $\lambda \in (\lambda_0, \infty]$.

   (See illustrations in Figure 28.)
4. If $c \in (\hat{c}_3, \hat{c}_4)$, there exists a $\Lambda > 0$, such that (6-1) has
   - no positive solution for $\lambda \in (0, \Lambda)$, and

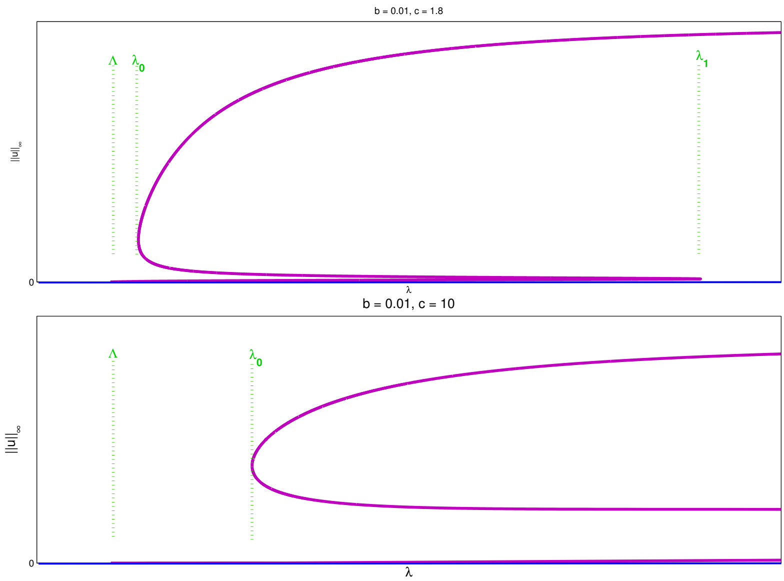

Figure 28. Illustrations of Case 3.

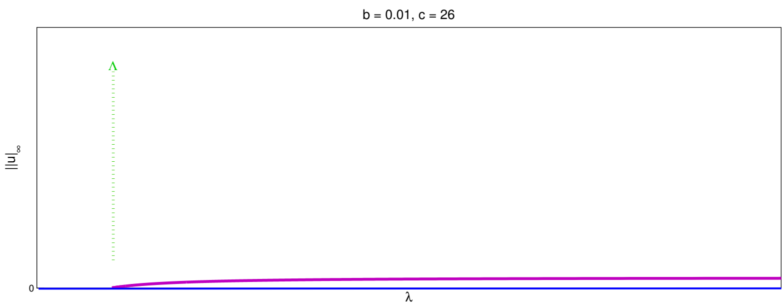

Figure 29. Illustration of Case 4.

- exactly 1 positive solution for $\lambda \in [\Lambda, \infty)$.

(See illustration in Figure 29.)

7. Computational results for strong Allee effect

This section describes the case in which a grazing term is combined with a strong Allee growth rate. Thus, our reaction term is

$$\bar{f}(u) = u(u-1)(b-u) - \frac{cu^2}{1+u^2}, \quad b > 1, \ c \geq 0.$$

As in Section 4, we obtain our results via the quadrature method detailed in Section 2 and apply Mathematica to complete our computations. In the top left part of Figure 30, we present the bifurcation curve with no grazing term ($c = 0$). The

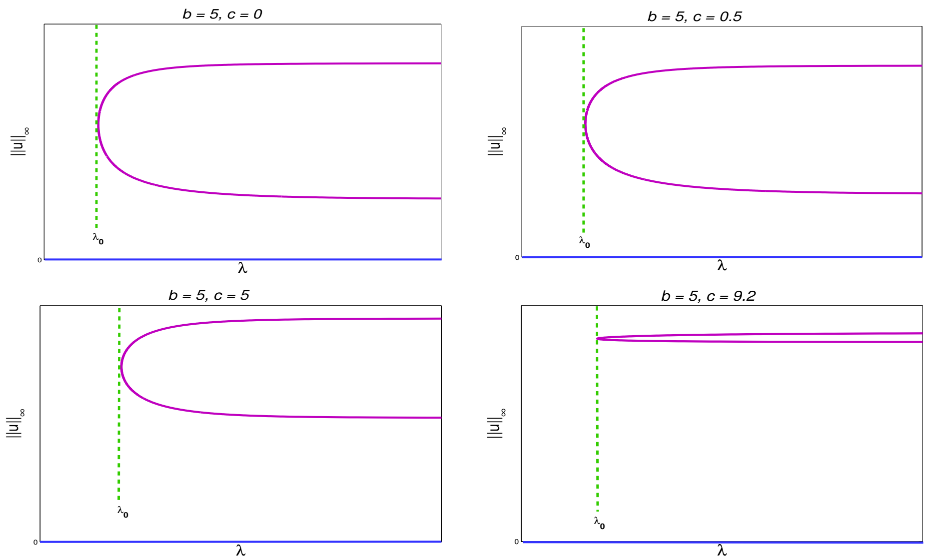

Figure 30. Evolution of the bifurcation curve as c increases.

resulting bifurcation curve evolution for increasing c is briefly exemplified in the remaining three parts of the figure.

After careful computational analysis and application of values for ranges of $b > 1$ and $c \geq 0$, we observe that the grazing term ultimately has little effect on the overall structure of the resulting bifurcation curve. However, we must note that for large values of c the grazing term will overcome the strong Allee effect and there will no longer exist any steady states. Thus, in this case the population will die out.

8. Biological implications

Analysis of the steady states of (1-1) provides valuable information on the long-term survival of a population. Given an initial population size and grazing rate, the bifurcation diagrams we have included provide ranges of λ for which the population persists. Depending on the range of λ, the persistence is either conditional or unconditional.

When λ is small, the diffusion coefficient is large enough to cause a population to die out despite its initial size. This is clearly illustrated through the bifurcation diagrams for the range $\lambda \in (0, \lambda_0)$. Using Figure 31 as an example, it is clear that whether a population has an initial size of k or of l, it will still die out.

Between λ_1 and λ_2, we have unconditional persistence. That is, a population with an initial size of m will decrease until achieving stability at the bottom branch. However, between λ_2 and λ_3, the stability of the steady states results in conditional persistence. In this range, the top and bottom branches are stable solutions while the

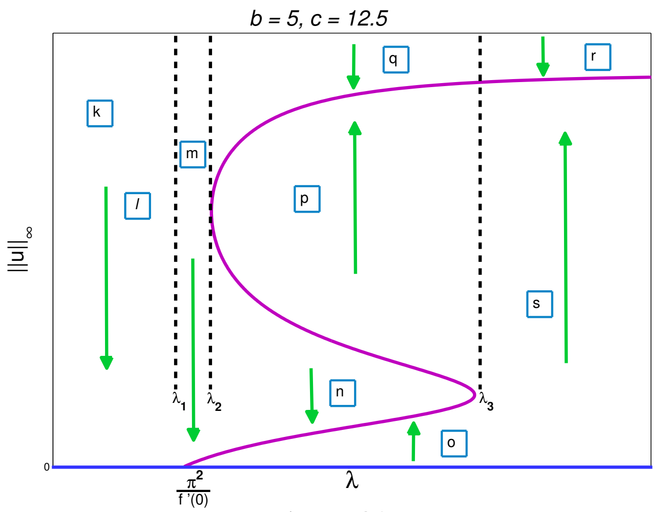

Figure 31

middle branch is unstable. Thus, a population's persistence is dictated by its initial population size. For example, a population beginning with a size of n will decline until reaching stability at the bottom branch; whereas, a population beginning with a size of o will increase until reaching stability at the bottom branch. Furthermore, for an initial size of p, the population will grow until obtaining stability at the top branch while an initial size of q will diminish until obtaining stability at the top branch.

For $\lambda > \lambda_3$, the population will unconditionally persist. With an initial size of r, a population will decrease until stabilizing at the top branch while an initial size of s will increase until stabilizing at the top branch.

9. Appendix: Proofs of Lemma 5.1 and Lemma 5.2

Proof of Lemma 5.1. (See also [Laetsch 1970].)

$$\lim_{\rho\to 0^+} G(\rho) = \lim_{\rho\to 0^+} \int_0^\rho \frac{dz}{\sqrt{F(\rho)-F(z)}} = \lim_{\rho\to 0^+} \int_0^1 \frac{\rho\, dv}{\sqrt{F(\rho)-F(\rho v)}} = \lim_{\rho\to 0^+} \int_0^1 \frac{dv}{\sqrt{\dfrac{F(\rho)-F(\rho v)}{\rho^2}}} \tag{9-1}$$

By Lebesgue's dominated convergence theorem, the limit can be moved inside the integral. Thus, we evaluate

$$\lim_{\rho\to 0^+} \frac{F(\rho)-F(\rho v)}{\rho^2}.$$

After applying L'Hospital's rule twice, we obtain

$$\lim_{\rho\to 0^+} \frac{f'(\rho)-v^2 f'(\rho v)}{2} = \frac{f'(0)}{2}(1-v^2). \tag{9-2}$$

Combining (9-1) and (9-2), we have

$$\lim_{\rho\to 0^+} G(\rho) = \sqrt{\frac{2}{f'(0)}} \int_0^1 \frac{dv}{\sqrt{1-v^2}} = \frac{\pi}{\sqrt{2f'(0)}} = \frac{\pi}{\sqrt{2b}}. \qquad \square$$

Proof of Lemma 5.2. (See also [Laetsch 1970].) Let $N > 0$ be large enough such that $f(u) \le N(\sigma - u)$ for all $0 \le u \le \sigma$. By the mean value theorem,

$$F(\rho) - F(s) = F'(\theta)(\rho - s) = f(\theta)(\rho - s) \le N(\sigma - \theta)(\rho - s) \le N(\sigma - s)^2.$$

Then $\sqrt{F(\rho) - F(s)} \le \sqrt{N}(\sigma - s)$, or, with $n = 1/\sqrt{N}$,

$$\frac{1}{\sqrt{F(\rho)-F(s)}} \ge \frac{n}{\sigma - s}.$$

Integrating both sides gives

$$\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}} \ge n \int_0^\rho \frac{ds}{\sigma - s} G(\rho) \ge -n\ln(\sigma-\rho) + n\ln(\sigma).$$

As $\rho \to \sigma^-$, the right side of the inequality approaches ∞. Thus, $G(\rho) \to \infty$ as $\rho \to \sigma^-$. $\square$

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