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Alhazen's hyperbolic billiard problem

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Given two points inside a circle in the hyperbolic plane, we study the problem of finding an isosceles triangle inscribed in the circle so that the two points belong to distinct congruent sides. By means of a reduction to the corresponding result in Euclidean geometry, we prove that this problem cannot generally be solved with hyperbolic ruler and compass.

In his treatise on optics, written in Arabic, the scientist and mathematician Abu Ali al-Ḥasan ibn al-Haytham (965–1039) posed the problem of *finding the light path between a source and an observer by way of a fixed spherical mirror*, and gave a geometric solution for it. The problem may have been formulated much earlier, by the great Greek mathematicians of the Hellenistic era, but no surviving testimony confirms this. Thus it is fit that it carries al-Ḥasan ibn al-Haytham's name, which was rendered as Alhazen in the Latin translation of his book — a document that played an important role in the development of modern science.

Alhazen recognized that the problem is essentially two-dimensional — the path must lie in a plane determined by the center of the sphere, the source and the observer. His solution is long, in part because he is actually studying a more general problem; see [Sabra 1982] for details. It is not a ruler-and-compass construction, as it requires an auxiliary hyperbola; in fact, apart from special cases, the problem turns out not to be solvable with ruler and compass alone, though it seems this was only proved some 50 years ago ([Elkin 1965]; see also [Riede 1989; Neumann 1998]).

In this paper, we study the hyperbolic version of Alhazen's problem and relate it to its classical Euclidean counterpart. We use the following formulation of the problem: *Given a circle* (*in the Euclidean or the hyperbolic plane*) *and two points A and B inside it*, *construct an inscribed*, *isosceles triangle with A on one equal leg and B on the other.*

The isosceles condition is equivalent to the condition that the two legs meet at equal angles the diameter of the circle that goes through their common vertex, so this is Alhazen's problem all right. (One can also imagine a round billiard table

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with two points marked on the felt. A shot that goes through one of the points, hits the cushion and then goes through the other point is a solution to the problem.)

We show the following result:

Theorem. *For a given circle in the hyperbolic plane and a given circle in the Euclidean plane, there exists a bijection — indeed a homeomorphism — between Alhazen point configurations of one and those of the other, preserving in both directions the property of Alhazen constructibility with ruler and compass.*

That is, hyperbolic configurations whose Alhazen solution is constructible with (hyperbolic) ruler and compass correspond to Euclidean configurations whose Alhazen solution is constructible with (Euclidean) ruler and compass, and similarly for nonconstructible configurations.

This correspondence was unexpected to us, since hyperbolic triangles are so different from Euclidean ones — to begin with, their angles add up to arbitrary measures less than π. Generally, Euclidean ruler-and-compass constructions fail to carry over to the hyperbolic plane; even trisecting an arbitrary segment, something quite simple with Euclidean ruler and compass, cannot be done in the hyperbolic case! (See [Martin 1975, p. 483], for instance.)

Since, as already mentioned, Alhazen's problem is seldom solvable with ruler and compass in the Euclidean plane, we obtain (see Remark 2 on page 281):

Corollary. *The hyperbolic Alhazen problem is not generally solvable with ruler and compass. Indeed, for any fixed hyperbolic open disk D_H, the set of pairs of points $A, B \in D_H$ for which the Alhazen problem can be solved with ruler and compass has measure zero in $D_H \times D_H$.*

This paper is organized as follows. In Section 1 we describe the relevant models of the hyperbolic plane and spell out the hyperbolic Alhazen problem. In Section 2 we motivate the correspondence between Euclidean and hyperbolic constructions, observing it in action in a simple constructible case. The proof of the theorem is then given in Section 3.

1. Alhazen's billiard problem in hyperbolic geometry

Hyperbolic geometry has several *models*, that is, ways to name points and make calculations. We will need to use two: the Poincaré disk model and the Klein model.

The ***Poincaré model*** represents the hyperbolic plane by an open disk, that is, the set of points inside a fixed Euclidean circle.[1] This so-called *boundary circle* is not part of the hyperbolic plane. It is a "boundary" of the model only: the hyperbolic

[1] We haven't defined the hyperbolic plane. The reader new to hyperbolic geometry can imagine that it *is* the Poincaré model: the set of points (x, y) in $\mathbb{R}^2$ satisfying $x^2 + y^2 < 1$, with further features called (hyperbolic) distance, lines, angles, and so on, which we now describe.

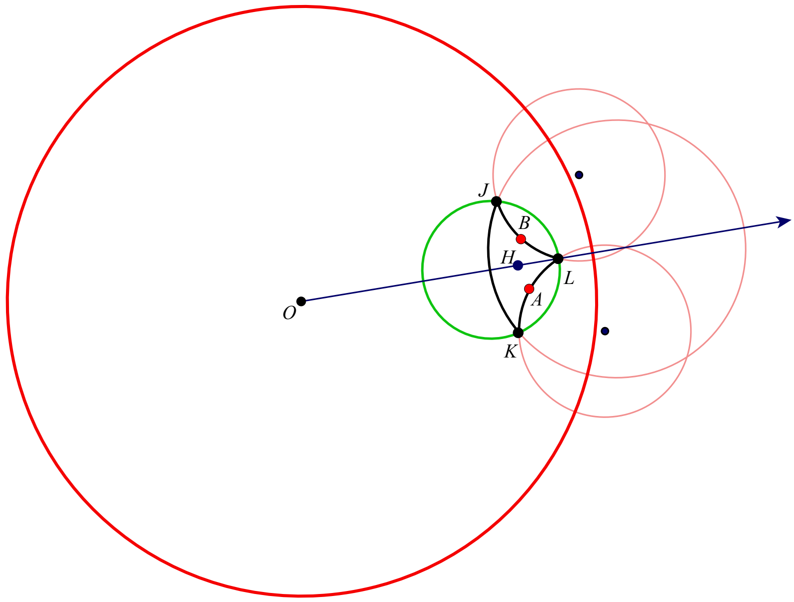

Figure 1. A hyperbolic Alhazen triangle, *JKL*. The hyperbolic plane is the interior of the disk with red boundary. The pink circles represent hyperbolic lines, to be determined in the solution of the problem, together with their intersections J, K, L. The givens of the problem consist of the small circle (on which we must place J, K, and L) and the points A and B inside it. Note the position of the center H of the given hyperbolic circle.

plane itself extends infinitely in all directions. The center of the boundary circle will be labeled O. It is not a special point in the hyperbolic plane — any point can be chosen for this honor — but it does enjoy special properties in the model.

Figure 1 illustrates the main features of the Poincaré model. The boundary circle is shown in red. Hyperbolic straight lines appear in the model either as Euclidean diameters (like the line OH) or as circles (in pink) orthogonal to the boundary circle — or rather, the portions of such circles inside the boundary circle. Hyperbolic circles (sets of points at a fixed hyperbolic distance from a center) appear as Euclidean circles contained in the open disk; the green circle is an example.

This is two-thirds of what we need in order to visualize the hyperbolic Alhazen problem. But how are we to recognize isosceles triangles? Hyperbolic distances cannot be discerned from appearances in the model: the formula to compute the hyperbolic distance between two points, given their coordinates in the Poincaré model, is very different from the formula giving the Euclidean distance. The ratio between the two is, roughly speaking, inversely proportional to the Euclidean distance to be boundary.

(Another manifestation of this is that the center of a hyperbolic circle does not match what appears to be the center in the model. The true center H is the point hyperbolically equidistant from the points on the circle; it can be found, for instance, as the intersection of two hyperbolic lines perpendicular to the circle. H always lies closer to the boundary than the apparent center, unless both coincide with O. In Figure 1, H is the center of the green circle.)

What saves the day is an important feature of the Poincaré model: it is *conformal*, meaning that it renders angles faithfully. The true angle between two hyperbolic lines equals the (Euclidean) angle between the circles representing the same lines in the model. Thus a hyperbolic Alhazen triangle has two equal angles in the Poincaré model, as exemplified by the triangle JKL in Figure 1. (This example is special in that the triangle's axis of symmetry, the line OH, is a diameter of the model, so the triangle also appears "isosceles", that is, symmetric, to Euclidean eyes. This would not generally be the case.)

The ***Klein model*** of the hyperbolic plane also uses a Euclidean disk to represent its points, but in this model hyperbolic lines correspond to Euclidean chords. Euclidean appearances are even more deceiving here, because hyperbolic angle measures are not the Euclidean ones visible in the model. However, the property that hyperbolic and Euclidean notions of straightness coincide in this model will be helpful.

There exists an isomorphism between the Poincaré and Klein models, based on stereographic projection, which will be the key in Section 3 to our correspondence between the hyperbolic and Euclidean Alhazen problems. To describe it, we work in (Euclidean) three-dimensional space, with both models lying on the horizontal coordinate plane. We rest a sphere on this plane, as shown in Figure 2: the radius of the sphere is half the radius of the Poincaré model, and its south pole is the center O of both models. Given a point R in the Poincaré model, we find its counterpart in the Klein model by first mapping the point onto the sphere via *stereographic projection* (central projection from the north pole); this gives a point P, somewhere on the south hemisphere. We then project P directly down onto the horizontal plane, obtaining Q; this is the counterpart of R in the Klein model.

Stereographic projection maps circles to circles and preserves orthogonality. An arc of circle orthogonal to the red boundary of the Poincaré model projects onto the sphere as a semicircle orthogonal to the equator. When projected again straight down, this gives a line segment. This confirms that in the Klein model hyperbolic lines are represented by Euclidean chords.

Finally, we observe that if a hyperbolic circle happens to be centered at O in the Poincaré model (in which case its hyperbolic and Euclidean centers coincide), it will map to a horizontal circle on the sphere, and from there down to a circle in the Klein model, again centered at O. These are the only hyperbolic circles that look like Euclidean circles in Klein model: other circles look like Euclidean ellipses.

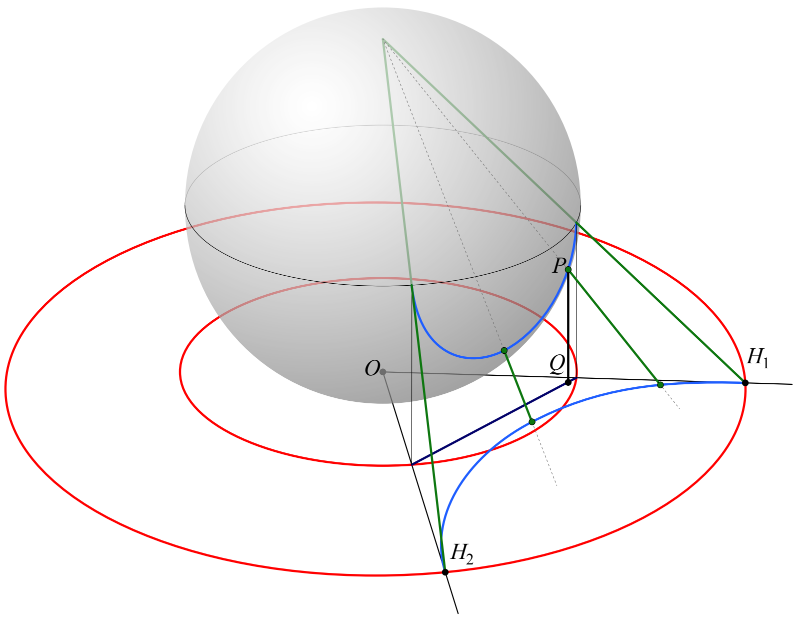

Figure 2. Correspondence between the Poincaré and Klein models.

2. A constructible example

The Euclidean Alhazen problem has an obvious solution when the given points A and B lie on a diameter of the given circle and are equidistant from the center (Figure 3, left). We simply construct the perpendicular bisector of $\overline{AB}$ — the horizontal diameter in Figure 3, right. Each of the two points where this line

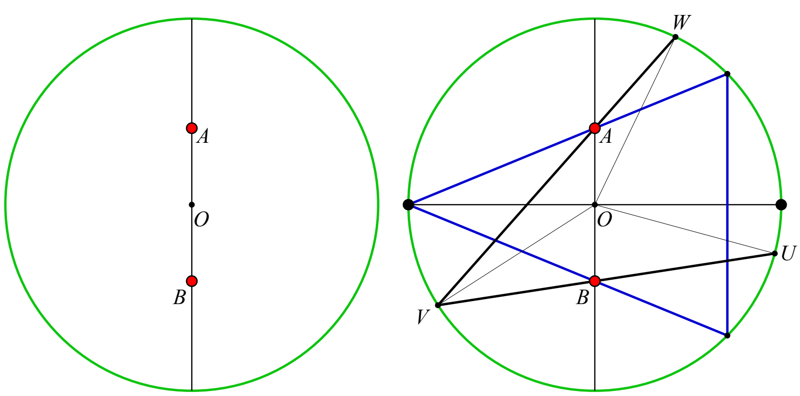

Figure 3. Solving the Alhazen problem in a special case: A and B are diametrically opposed and equidistant from the center of the circle. The triangle UVW is a solution if and only if $VO \perp AB$.

intersects the given circle provides a solution to the problem, since the angles subtended by OA and OB from these points are the same.

Moreover, these are the only solutions. To see this, take a point V on the given circle, and form the inscribed triangle whose sides lie on the lines VA and VB. If this triangle is isosceles, its median VO is also an altitude, so V lies on the perpendicular bisector of AB. (If we allow degenerate triangles then of course the line AB also provides a solution.)

For this simple situation, the reasoning in the hyperbolic case is identical. Given a hyperbolic circle of center H and two points A and B, diametrically opposed and equidistant from H, we draw the perpendicular bisector of A and B using our (hyperbolic!) compass, just as we would in the Euclidean case, and mark off its intersections with the circle, each of which provides a solution to the Alhazen problem. Seen in the Klein model, the picture would look exactly the same as Figure 3, provided we took the precaution of starting with a circle whose center coincides with the center O of the model! In the Poincaré model, with the same precaution, we'd have Figure 4.

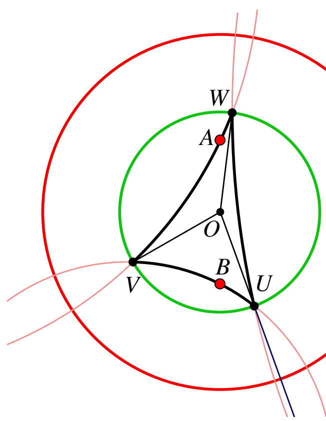

Figure 4. Special case of Alhazen problem in the Poincaré model; compare Figure 3, right. The triangle UVW is a solution if and only if $VO \perp AB$.

This leads to an important digression. With Euclidean constructions, we have physical tools (paper, ruler and compass) at our disposal, which are sufficiently accurate to help build intuition. Alas, we don't have a physical hyperbolic compass at our disposal, nor hyperbolic paper. What tools can we use to explore?

That's where the two models come in. In the Klein model, a Euclidean ruler is a proxy for a hyperbolic one, but the same cannot be said about the compass: hyperbolic circles look like ellipses in the model. What about the Poincaré model? Hyperbolic lines look like lines or circles in the model, and hyperbolic circles look like circles, so it's at least conceivable that what can be done with hyperbolic ruler and compass can also be done in the Poincaré model with Euclidean ruler and compass. And that turns out to be so:

Fact 1. *Any point in the hyperbolic plane obtained from initial data by using only* (*hyperbolic*) *ruler and compass can be obtained using ruler and compass in the Euclidean plane that supports the Poincaré model.*

This is proved by considering each building block of ruler-and-compass constructions. For instance, the problem of drawing a line through two points translates into finding a circle perpendicular to the boundary of the model and going through the given points; this *can* be done with Euclidean ruler and compass in a few steps. Drawing a (hyperbolic) circle centered at a point and going through another point translates into finding the Euclidean center of the desired circle in the model, and so on. See [Goodman-Strauss 2001] for a pleasant exposition of these constructions.

Fact 2. *Conversely*, *any point in the Poincaré model obtained from initial data by using Euclidean ruler and compass can also be obtained using intrinsic* (*hyperbolic*) *ruler and compass.*

This is perhaps more surprising than Fact 1, since Euclidean manipulations in the Poincaré model can involve objects that are not actually in the hyperbolic plane, but rather on the boundary and the exterior of the disk. Fact 2 was apparently first proved in [Curtis 1990] — specifically in §6, but the whole article is recommended for its lucid discussion, references to earlier work, and a proof of the 90-year old result of D. Mordukhai-Boltovskoi that states exactly which lengths are constructible in hyperbolic geometry. (Warning: The "Klein conformal model" to which Curtis refers is a slight variation of the Poincaré model, rather than the projective Klein model explained on page 276.)

With this we can close our digression, confident that in talking about ruler-and-compass constructibility, it makes no difference whether we use intrinsic hyperbolic tools or Euclidean tools in the Poincaré model!

3. Correspondence between the hyperbolic and Euclidean Alhazen problems

We now prove the Theorem stated on page 274. We are given a disk D_H in the hyperbolic plane and a disk D_E in the Euclidean plane.

Lemma 1. *We can assume without loss of generality that D_H and D_E are centered at the origin O of the Cartesian plane.* (In the hyperbolic case, we understand the Cartesian plane as underlying the Poincaré model.)

This may seem obvious, but it merits discussion: the theorem does talk of arbitrary circles, after all. The proof of the lemma follows from three observations:

Any isometry T of either the Euclidean or the hyperbolic plane can be implemented with ruler and compass, in the following sense: Let T be defined by some known data, such as the images $T(a), T(b), T(c)$ of three noncollinear points a, b, c. Then, given any point x, one can construct $T(x)$ with ruler and compass, starting from x and the data defining T.

This is usually taken for granted for Euclidean constructions, and indeed it is easy to show — we leave it as an exercise. Your proof for the Euclidean case will quite likely carry over to the hyperbolic case (with intrinsic ruler and compass).

The second observation is that *any given disk is mapped by some isometry T to some disk centered at O*. This is because the hyperbolic and Euclidean planes are homogeneous: given two points in the Euclidean plane, there is an isometry taking one to the other — and of course such an isometry maps a disk to a disk. Similarly for the hyperbolic plane — having in mind hyperbolic isometries, of course.

The third observation ties it all together: The theorem asserts a constructibility-preserving bijection between D_H- and D_E-Alhazen configurations. If such a bijection is known for D_H and D_E of the special form in the lemma, it can be defined for *any* D_H and D_E, by using isometries to bridge between configurations in the old and the new D_H, and between configurations in the old and the new D_E. Because isometries are constructible, these bridges take constructible configurations to constructible configurations; and because they do so in both directions, they also take nonconstructible configurations to nonconstructible configurations.

This formally justifies the usual cavalier attitude about isometries when dealing with constructibility questions.

Let ϕ denote the map taking the Poincaré model to the Klein model, described in Figure 2 as the composition of stereographic projection and vertical projection. We need one more normalization.

Lemma 2. *We can assume, moreover, that D_H is taken to D_E under the Poincaré-to-Klein map ϕ.*

The radius of D_H cannot be tampered with,[2] but fortunately the radius of D_E can, by applying a homothety (scaling transformation). Homotheties preserve constructibility, being themselves constructible (same logic as in the third observation above). So we just scale D_E to make it coincide with $\phi(D_H)$.

Lemma 3. *The Poincaré-to-Klein correspondence ϕ, applied to Alhazen configurations in our normalized D_H and D_E, preserves constructibility.*

[2]The hyperbolic plane has no similarities other than isometries, so we cannot hope to reduce the hyperbolic Alhazen problem to a single circle size, as we're accustomed to doing with Euclidean problems. Another way to say this is that hyperbolic disks of different sizes are not scaled images of one another — in fact, the larger the circle, the greater the ratio between circumference and diameter!

Indeed, the action of ϕ on single points can be implemented with (Euclidean) ruler and compass. This is obvious for points on the red boundary circle,[3] since such points are just pulled halfway toward O — a homothety of ratio $\frac{1}{2}$. Now let P be a point inside the Poincaré disk. Use Poincaré (i.e., Euclidean) ruler and compass to draw any two hyperbolic lines crossing at P; this is possible by Fact 1. Mark the intersections of these lines with the Poincaré red circle (like points H_1 and H_2 in Figure 2). Apply ϕ to the four points thus determined on the Poincaré red circle, to find the corresponding points on the Klein red circle. Obtain $\phi(P)$ as the intersection of the two line segments connecting pairs of opposite points.

The key observation now is that *the hyperbolic Alhazen problem in D_H with initial data A, B has S as a solution if and only if the* ***Euclidean*** *Alhazen problem in D_E with initial data $\phi(A)$, $\phi(B)$ has $\phi(S)$ as a solution.* Here we're thinking of the solution as a single point — the reflection point on the circle between A and B.

The statement in italics applies to solutions in general, whether or not they are constructible. To wrap up the proof, we resort again to the bridge idea used for the first two lemmas. We spell it out here, since we didn't before. Because ϕ and its inverse are constructible, they preserve the constructibility status of solutions. That is, if we can get from A and B to the solution S, then we can get from $\phi(A)$ and $\phi(B)$ to $\phi(S)$, via A $(= \phi^{-1}(\phi(A)))$ and B and then S. Conversely, if we can get from $\phi(A)$ and $\phi(B)$ to $\phi(S)$, we can get from A and B to S. This finishes the proof of the Theorem.

Remark 1. What makes the Alhazen problem special is that we can write that boldfaced "***Euclidean***" above. For any problem, ϕ transforms hyperbolic solutions in the Poincaré model into hyperbolic solutions in the Klein model; but only here is the hyperbolic solution also a Euclidean solution, and only because we chose D_H and D_E judiciously in Lemmas 1 and 2, rendering irrelevant the difference between the Euclidean metric and the hyperbolic metric in the Klein model. That's why the proof fails for a problem such as finding a point T a third of the way from A to B (as already mentioned, the hyperbolic version of this problem is not constructible).

Remark 2. The bijection we have constructed between D_H- and D_E-Alhazen configurations is just a componentwise application of the Poincaré-to-Klein map ϕ, and is therefore very well behaved (homeomorphic, locally bi-Lipschitz). It follows that, for each circle radius, not only are there D_H-Alhazen configurations that are not solvable with ruler and compass, but in fact they are the rule. Solvable configurations are the exception — they form a set of measure zero in $D_H \times D_H$, corresponding to a set of measure zero of solvable configurations in the Euclidean case [Neumann 1998, p. 527]. This proves the Corollary.

[3]That these points are not in the hyperbolic plane itself doesn't matter: we're using them as stepping stones, and ϕ is obviously defined on them.

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