

involve

a journal of mathematics

Bochner (p, Y) -operator frames

Mohammad Hasan Faroughi, Reza Ahmadi and Morteza Rahmani



Bochner (p, Y) -operator frames

Mohammad Hasan Faroughi, Reza Ahmadi and Morteza Rahmani

(Communicated by David R. Larson)

Using the concepts of Bochner measurability and Bochner space, we introduce a continuous version of (p, Y) -operator frames for a Banach space. We also define independent Bochner (p, Y) -operator frames for a Banach space and discuss some properties of Bochner (p, Y) -operator frames.

1. Introduction and preliminaries

The concept of frames was first introduced in the context of nonharmonic Fourier series [Duffin and Schaeffer 1952], and after the publication of [Daubechies et al. 1986] it has found broad application in signal processing, image processing, data compression and sampling theory. In this paper we introduce *Bochner (p, Y) -operator frames*, which are the continuous version of (p, Y) -operator frames for a Banach space, introduced in [Cao et al. 2008]. The new frames also generalize the *continuous p -frames* introduced in [Faroughi and Osgooei 2011].

Throughout this paper H will be a Hilbert space and X will be a Banach space.

Definition 1.1. Let $\{f_i\}_{i \in I}$ be a sequence of elements of H . We say that $\{f_i\}_{i \in I}$ is a *frame* for H if there exist constants $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \sum_{i \in I} |\langle f_i, h \rangle|^2 \leq B\|h\|^2. \quad (1-1)$$

The constants A and B are called frame bounds. If A, B can be chosen so that $A = B$, we call this frame an A -tight frame and if $A = B = 1$ it is called a Parseval frame. If we only have the upper bound, we call $\{f_i\}_{i \in I}$ a Bessel sequence. If $\{f_i\}_{i \in I}$ is a Bessel sequence then the following operators are bounded:

$$T : l^2(I) \rightarrow H, \quad T(c_i) = \sum_{i \in I} c_i f_i, \quad (1-2)$$

$$T^* : H \rightarrow l^2(I), \quad T^*(f) = \{\langle f, f_i \rangle\}_{i \in I}, \quad (1-3)$$

MSC2010: 42C15, 46C05.

Keywords: Bochner (p, Y) -operator frame, Bochner measurability, Bochner space, Radon–Nikodym property.

called the *synthesis* and *analysis* operators, respectively. Hence the *frame operator* S , given by

$$Sf = TT^*f = \sum_{i \in I} \langle f, f_i \rangle f_i, \tag{1-4}$$

is also bounded.

The theory of frames has a continuous version, as follows.

Definition 1.2 [Rahimi et al. 2006]. Let (Ω, μ) be a measure space. Let $f : \Omega \rightarrow H$ be weakly measurable (i.e., for each $h \in H$, the mapping $\omega \rightarrow \langle f(\omega), h \rangle$ is measurable). Then f is called a *continuous frame* or *c-frame* for H if there exist constants $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \int_{\Omega} |\langle f(\omega), h \rangle|^2 d\mu \leq B\|h\|^2. \tag{1-5}$$

In this context the synthesis operator $T_f : L^2(X, \mu) \rightarrow H$ is defined by

$$\langle T_f\phi, h \rangle = \int_X \phi(x)\langle f(x), h \rangle d\mu(x); \tag{1-6}$$

the analysis operator $T_f^* : H \rightarrow L^2(X, \mu)$ by

$$(T_f^*h)(x) = \langle h, f(x) \rangle, \quad x \in X; \tag{1-7}$$

and the frame operator by

$$S_f = T_f T_f^*. \tag{1-8}$$

By Theorem 2.5 in [Rahimi et al. 2006], S_f is positive, self-adjoint and invertible.

Suppose (Ω, Σ, μ) is a measure space, where μ is a positive measure.

Definition 1.3. A function $f : \Omega \rightarrow X$ is called *simple* if there exist $x_1, \dots, x_n \in X$ and $E_1, \dots, E_n \in \Sigma$ such that $f = \sum_{i=1}^n x_i \chi_{E_i}$, where $\chi_{E_i}(\omega) = 1$ if $\omega \in E_i$ and $\chi_{E_i}(\omega) = 0$ if $\omega \in E_i^c$. If $\mu(E_i)$ is finite whenever $x_i \neq 0$ then the simple function f is *integrable*, and the integral is then defined by

$$\int_{\Omega} f(\omega) d\mu(\omega) = \sum_{i=1}^n \mu(E_i)x_i.$$

Definition 1.4. A function $f : \Omega \rightarrow X$ is called *Bochner-measurable* if there exists a sequence of simple functions $\{f_n\}_{n=1}^{\infty}$ such that

$$\lim_{n \rightarrow \infty} \|f_n(\omega) - f(\omega)\| = 0, \quad \mu\text{-a.e.}$$

Definition 1.5. A Bochner-measurable function $f : \Omega \rightarrow X$ is called *Bochner-integrable* if there exists a sequence of integrable simple functions $\{f_n\}_{n=1}^{\infty}$ such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \|f_n(\omega) - f(\omega)\| d\mu(\omega) = 0.$$

In this case, $\int_E f(\omega) d\mu(\omega)$ is defined by

$$\int_E f(\omega) d\mu(\omega) = \lim_{n \rightarrow \infty} \int_E f_n(\omega) d\mu(\omega), \quad E \in \Sigma.$$

Definition 1.6. A Banach space X has the *Radon–Nikodym property* if, for every finite measure space (Ω, Σ, μ) and every (finitely additive) X -valued measure γ on (Ω, Σ) that has bounded variation and is absolutely continuous with respect to μ , there is a Bochner-integrable function $g : \Omega \rightarrow X$ such that

$$\gamma(E) = \int_E g(\omega) d\mu(\omega)$$

for every measurable set $E \in \Sigma$.

Remark 1.7. Suppose that (Ω, Σ, μ) is a measure space and X^* has the Radon–Nikodym property. Let $1 \leq p \leq \infty$. The *Bochner space* $L^p(\mu, X)$ is defined to be the Banach space of (equivalence classes of) X -valued Bochner-measurable functions F on Ω whose L^p norm is finite; here the L^p norm is defined by

$$\|F\|_p = \left(\int_{\Omega} \|F(\omega)\|^p d\mu(\omega) \right)^{1/p}$$

if p is finite, and by the essential supremum of $\|F(\omega)\|$ if $p = \infty$. In [Diestel and Uhl 1977; Cengiz 1998; Fleming and Jamison 2008, p. 51] it is proved that if q is such that $\frac{1}{p} + \frac{1}{q} = 1$, then $L^q(\mu, X^*)$ is isometrically isomorphic to $(L^p(\mu, X))^*$ if and only if X^* has the Radon–Nikodym property. This isometric isomorphism

$$\psi : L^q(\mu, X^*) \rightarrow (L^p(\mu, X))^*$$

takes $g \in L^q(\mu, X^*)$ to ϕ_g , the linear map defined by

$$\phi_g(f) = \int_{\Omega} g(\omega)(f(\omega)) d\mu(\omega), \quad f \in L^p(\mu, X).$$

So for all $f \in L^p(\mu, X)$ and $g \in L^q(\mu, X^*)$ we have

$$\psi(g)(f) = \langle f, \psi(g) \rangle = \int_{\Omega} g(\omega)(f(\omega)) d\mu(\omega) = \int_{\Omega} \langle f(\omega), g(\omega) \rangle d\mu(\omega).$$

In the following, we use the notation $\langle f, g \rangle$ instead of $\langle f, \psi(g) \rangle$, so for all $f \in L^p(\mu, X)$ and $g \in L^q(\mu, X^*)$

$$\langle f, g \rangle = \int_{\Omega} \langle f(\omega), g(\omega) \rangle d\mu(\omega).$$

Hilbert spaces have the Radon–Nikodym property, so in particular, if H is a Hilbert space then $(L^p(\mu, H))^*$ is isometrically isomorphic to $L^q(\mu, H)$. So, for

all $f \in L^p(\mu, H)$ and $g \in L^q(\mu, H)$, we have

$$\langle f, g \rangle = \int_{\Omega} \langle f(\omega), g(\omega) \rangle d\mu(\omega),$$

in which $\langle f(\omega), g(\omega) \rangle$ does not mean the inner product of elements $f(\omega), g(\omega)$ in H , but

$$\langle f(\omega), g(\omega) \rangle = v(g(\omega))(f(\omega)),$$

where $v : H \rightarrow H^*$ is the isometric isomorphism between H and H^* .

Lemma 1.8. *Let (Ω, Σ, μ) be a measure space and suppose there exists $k > 0$ such that $\mu(E) \geq k$ for every nonempty measurable set E of Ω . For every $\omega \in \Omega$, define $P_{\omega} : L^p(\mu, X) \rightarrow X, P_{\omega}(G) = G(\omega)$. Then $\|P_{\omega}\| \leq k^{-1/p}$.*

Proof. For a fix $\omega_0 \in \Omega$, put

$$\Delta = \{\omega \in \Omega \mid \|G(\omega)\| \geq \|G(\omega_0)\|\}.$$

Then

$$\|G\|_p^p = \int_{\Omega} \|G(\omega)\|^p d\mu(\omega) \geq \int_{\Delta} \|G(\omega)\|^p d\mu(\omega) \geq \mu(\Delta) \|G(\omega_0)\|^p \geq k \|G(\omega_0)\|^p.$$

Hence

$$\|P_{\omega_0}\| = \sup_{\|G\|_p \leq 1} \|P_{\omega_0}(G)\| = \sup_{\|G\|_p \leq 1} \|G(\omega_0)\| \leq \sup_{\|G\|_p \leq 1} k^{-1/p} \|G\|_p = k^{-1/p}. \quad \square$$

2. Bochner (p, Y) -Bessel mappings for X

Throughout this section and the next we will work with a second Banach space Y in addition to X . We denote by $B(X, Y)$ the space of bounded operators from X to Y .

Definition 2.1. Let $1 < p < \infty$, and let $F : \Omega \rightarrow B(X, Y)$ be a map; we write F_{ω} for $F(\omega)$. We say that F is a *Bochner (p, Y) -Bessel mapping for X* if the following conditions are met:

- (i) For each $x \in X$, the mapping $\omega \mapsto F_{\omega}(x)$ from Ω into Y is Bochner-measurable.
- (ii) There exists a positive constant B such that

$$\|F_{\cdot}(x)\|_p \leq B \|x\| \quad \text{for all } x \in X, \tag{2-1}$$

where

$$\|F_{\cdot}(x)\|_p = \left(\int_{\Omega} \|F_{\omega}(x)\|^p d\mu \right)^{1/p}. \tag{2-2}$$

We denote by $B_X^p(Y)$ the set of all Bochner (p, Y) -Bessel mappings for X . It

is easy to see that this set is closed under addition (defined in the obvious way: for $F, K \in B_X^p(Y)$, the sum $F + K$ satisfies $(F + K)_\omega(x) = F_\omega(x) + K_\omega(x)$ for all $x \in X$ and $\omega \in \Omega$) and under multiplication by scalars. Thus $B_X^p(Y)$ is a vector space. We give it a norm as follows. The *Bessel bound* of $F \in B_X^p(Y)$ is the number

$$B_F = \inf\{B > 0 : B \text{ satisfies (2-1)}\}.$$

For every $F \in B_X^p(Y)$, define $R_F : X \rightarrow L^p(\mu, Y)$ by $x \mapsto F_\cdot(x)$. This is clearly a linear map; we should that it is also bounded. For every $F \in B_X^p(Y)$,

$$\|R_F(x)\|_p = \|F_\cdot(x)\|_p \leq B\|x\|, \quad (2-3)$$

for any B satisfying (2-1). Together with the linearity of R_F this implies that

$$\|R_F\| \leq B_F; \quad (2-4)$$

that is, $R_F \in B(X, L^p(\mu, Y))$. Now set

$$\|F\|_p = \|R_F\|. \quad (2-5)$$

By (2-4), $\|F\|_p \leq B_F$. It is easy to show that this gives a norm on $B_X^p(Y)$.

Theorem 2.2. *Let (Ω, Σ, μ) be a measure space and suppose there exists $k > 0$ such that $\mu(E) \geq k$ for every nonempty measurable set E of Ω . For every $1 < p < \infty$, the mapping*

$$\Lambda : B_X^p(Y) \rightarrow B(X, L^p(\mu, Y))$$

given by $\Lambda(F) = R_F$ is a linear isometric isomorphism, and $B_X^p(Y)$ is a Banach space over \mathbb{C} .

Proof. Clearly, the mapping Λ is a linear isometry from $B_X^p(Y)$ into $B(X, L^p(\mu, Y))$. Next we prove that Λ is surjective.

Choose $\omega \in \Omega$. For every $A \in B(X, L^p(\mu, Y))$, define $F_\omega^A : X \rightarrow Y$ by

$$F_\omega^A(x) = P_\omega(A(x)) = A(x)(\omega), \quad x \in X.$$

By Lemma 1.8, we have $\|P_\omega\| \leq k^{-1/p}$; hence $F_\omega^A \in B(X, Y)$ for all $\omega \in \Omega$. Now, consider the mapping

$$F^A : \Omega \rightarrow B(X, Y)$$

given by $\omega \mapsto F_\omega^A$. Since $F^A(x) = A(x)(\cdot) : \Omega \rightarrow Y$ for each $x \in X$, the mapping $\omega \mapsto F_\omega^A(x)$ from Ω into Y is Bochner-measurable and

$$\|A(x)\|_p = \int_\Omega \|A(x)(\omega)\|^p d\mu(\omega) = \int_\Omega \|F_\omega^A(x)\|^p d\mu(\omega) = \|F_\cdot^A(x)\|_p.$$

Therefore

$$\|F_\cdot^A(x)\|_p = \|A(x)\|_p \leq \|A\|\|x\|.$$

Hence $F^A \in B_X^p(Y)$. Also, for all $\omega \in \Omega$ we have $R_{F^A}(x)(\omega) = F_\omega^A(x) = A(x)(\omega)$. Thus $R_{F^A}(x) = A(x)$ for all $x \in X$. This shows that $\Lambda(F^A) = R_{F^A} = A$; thus Λ is surjective and so bijective. Consequently, $B_X^p(Y)$ is isometrically isomorphic to the Banach space $B(X, L^p(\mu, Y))$. Therefore, $B_X^p(Y)$ is a Banach space over \mathbb{C} . \square

Theorem 2.3. *Let $1 < p < \infty$ and $F \in B_X^p(Y)$. Then, for every $y^* \in Y^*$, the mapping $F_\cdot^*(y^*) : \Omega \rightarrow X^*$, $F_\cdot^*(y^*)(\omega) = F_\omega^*(y^*)$ is a Bochner pg-Bessel mapping for X with respect to \mathbb{C} .*

Proof. Let $y^* \in Y^*$ and $x \in X$. Clearly for each $x \in X$ the map $\omega \mapsto \langle x, F_\omega^*(y^*) \rangle$ from Ω into \mathbb{C} is measurable and

$$\begin{aligned} \int_\Omega |\langle x, F_\omega^*(y^*) \rangle|^p d\mu(\omega) &= \int_\Omega |\langle F_\omega(x), y^* \rangle|^p d\mu(\omega) \\ &\leq (\|y^*\|^p) \left(\int_\Omega \|F_\omega(x)\|^p d\mu(\omega) \right) \\ &\leq \|y^*\|^p B_F^p \|x\|^p. \end{aligned} \quad \square$$

Theorem 2.4. *Let (Ω, μ) be a σ -finite measure space with positive measure μ and let $\Omega = \bigcup_{n \in \mathbb{N}} K_n$ with $K_n \subseteq K_{n+1}$. Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $F : \Omega \rightarrow B(X, Y)$. The following assertions are equivalent:*

- (i) $F \in B_X^p(Y)$.
- (ii) For each $x \in X$, $\int_\Omega \|F_\omega(x)\|^p d\mu(\omega) < \infty$.
- (iii) For each $G \in L^q(Y^*)$, $\sup_{\|x\| \leq 1} |\int_\Omega \langle x, F_\omega^*(G(\omega)) \rangle d\mu(\omega)| < \infty$.
- (iv) The operator $S_F : L^q(Y^*) \rightarrow X^*$ defined by

$$\langle x, S_F(G) \rangle = \int_\Omega \langle x, F_\omega^*(G(\omega)) \rangle d\mu(\omega) \quad \text{for } x \in X$$

is well defined and bounded.

Proof. (i) \Rightarrow (ii) This is obvious.

(ii) \Rightarrow (i) Define $A_n : X \rightarrow L^p(Y)$ by $A_n(x)(\omega) = \chi_{K_n}(\omega)F_\omega(x)$. For every $n \in \mathbb{N}$, we have

$$\|A_n\| = \sup_{\|x\| \leq 1} \|A_n(x)\|_p \leq \|F_\omega\|.$$

Hence, for all $n \in \mathbb{N}$, $A_n \in B(X, L^p(Y))$. By the definition of R_F , for every $n \in \mathbb{N}$,

$$\begin{aligned} \|(R_F - A_n)(x)\|_p^p &= \int_\Omega \|R_F(x)(\omega) - A_n(x)(\omega)\|^p d\mu(\omega) \\ &= \int_\Omega \|F_\omega(x) - \chi_{K_n}(\omega)F_\omega(x)\|^p d\mu(\omega) \\ &= \int_{\Omega - K_n} \|F_\omega(x)\|^p d\mu(\omega). \end{aligned}$$

This converges to 0 as $n \rightarrow \infty$, proving that $\lim_{n \rightarrow \infty} A_n(x) = R_F(x)$ for all $x \in X$. By the Banach–Steinhaus theorem, $R_F \in B(X, L^p(Y))$ and $\|R_F\| = \sup \|A_n\| < \infty$. Hence $F \in B_X^p(Y)$.

(i) \Rightarrow (iii) Let $G \in L^q(\mu, Y^*)$ be arbitrary. By the Hölder inequality, we have

$$\begin{aligned} & \sup_{\|x\| \leq 1} \left| \int_{\Omega} \langle x, F_{\omega}^*(G(\omega)) \rangle d\mu(\omega) \right| \\ &= \sup_{\|x\| \leq 1} \left| \int_{\Omega} \langle F_{\omega}(x), G(\omega) \rangle d\mu(\omega) \right| \\ &\leq \sup_{\|x\| \leq 1} \left(\int_{\Omega} \|F_{\omega}(x)\|^p d\mu(\omega) \right)^{1/p} \left(\int_{\Omega} \|G\omega\|^q d\mu(\omega) \right)^{1/q} \leq B_F \|G\|_q < \infty. \end{aligned}$$

(iii) \Rightarrow (iv) Clearly S_F is well defined and by the proof of (i) \Rightarrow (iii) we have

$$\|S_F\| = \sup_{\|G\|_q \leq 1} \|S_F(G)\| = \sup_{\|G\|_q \leq 1} \sup_{\|x\| \leq 1} \langle S_F(G), x \rangle \leq B_F < \infty.$$

(iv) \Rightarrow (i) Take $G \in L^q(\mu, Y^*)$ such that $\|G(\omega)\| = 1$ for every $\omega \in \Omega$ and

$$\|F_{\omega}(x)\| = \langle F_{\omega}(x), G(\omega) \rangle = \langle x, F_{\omega}^*(G(\omega)) \rangle \quad \text{for all } x \in X.$$

Define $\alpha_n : \Omega \rightarrow Y^*$ by $\alpha_n(\omega) = \chi_{K_n}(\omega) \|F_{\omega}(x)\|^{p-1} G(\omega)$. Then

$$\begin{aligned} \|\alpha_n\|_q &= \left(\int_{\Omega} \|\chi_{K_n}(\omega) \|F_{\omega}(x)\|^{p-1} G(\omega)\|^q d\mu(\omega) \right)^{1/q} \\ &= \left(\int_{K_n} \|F_{\omega}(x)\|^{q(p-1)} d\mu(\omega) \right)^{1/q} = \left(\int_{K_n} \|F_{\omega}(x)\|^p d\mu(\omega) \right)^{1/q}. \end{aligned}$$

Now, we have

$$\begin{aligned} \int_{K_n} \|F_{\omega}(x)\|^p d\mu(\omega) &= \int_{K_n} \langle x, \|F_{\omega}(x)\|^{p-1} F_{\omega}^*(G(\omega)) \rangle d\mu(\omega) \\ &= \int_{\Omega} \langle x, \chi_{K_n}(\omega) \|F_{\omega}(x)\|^{p-1} F_{\omega}^*(G(\omega)) \rangle d\mu(\omega) = \langle x, S_F(\alpha_n) \rangle \\ &\leq \|x\| \|S_F\| \|\alpha_n\|_q = \|x\| \|S_F\| \left(\int_{K_n} \|F_{\omega}(x)\|^p d\mu(\omega) \right)^{1/q}. \end{aligned}$$

Thus

$$\left(\int_{K_n} \|F_{\omega}(x)\|^p d\mu(\omega) \right)^{1/p} \leq \|x\| \|S_F\|. \quad (2-6)$$

By letting $n \rightarrow \infty$ in (2-6), we get $F \in B_X^p(Y)$. \square

3. Bochner (p, Y) -operator frames

Definition 3.1. Let $1 < p < \infty$. A mapping $F : \Omega \rightarrow B(X, Y)$ is called a *Bochner (p, Y) -operator frame* for X if the following conditions hold:

- (i) For each $x \in X$, the mapping $\omega \mapsto F_\omega(x)$ from Ω into Y is Bochner-measurable.
- (ii) There exist positive constants A and B such that

$$A\|x\| \leq \|F_\cdot(x)\|_p \leq B\|x\| \quad \text{for all } x \in X, \tag{3-1}$$

where $\|F_\cdot(x)\|_p$ is as in (2-2). The *lower* and *upper bounds* of F are then given by

$$A_F = \sup\{A > 0 : A \text{ satisfies (3-1)}\}, \quad B_F = \inf\{B > 0 : B \text{ satisfies (3-1)}\},$$

We denote by $F_X^p(Y)$ the set of all Bochner (p, Y) -operator frames for X .

Definition 3.2. A Bochner (p, Y) -operator frame F is called *tight* if $A_F = B_F$. If $A_F = B_F = 1$, we call F *normalized*. We denote by $TF_X^p(Y)$ and $NF_X^p(Y)$, respectively, the sets of all tight and normalized Bochner (p, Y) -operator frames for X .

Corollary 3.3. Let $F \in B_X^p(Y)$.

- (i) $F \in F_X^p(Y)$ if and only if R_F is bounded below if and only if R_F^* is surjective.
- (ii) $F \in TF_X^p(Y)$ if and only if R_F is a scaled isometry.

Lemma 3.4. (i) If $F \in B_X^p(Y)$ then $R_F^*\psi = S_F$.

- (ii) If Y is reflexive then $L^p(\mu, Y)$ is reflexive.

Proof. (i) For all $g \in L^q(\mu, Y^*)$ and $x \in X$, we have

$$\begin{aligned} \langle x, R_F^*\psi(g) \rangle &= \langle R_F x, \psi(g) \rangle = \int_\Omega \langle F_\omega(x), g(\omega) \rangle d\mu(\omega) \\ &= \int_\Omega \langle x, F_\omega^*(g(\omega)) \rangle d\mu(\omega) = \langle x, S_F g \rangle. \end{aligned}$$

(ii) Let $J_Y : Y \rightarrow Y^{**}$ be the canonical mapping. Suppose that Y is reflexive, that is $J_Y(Y) = Y^{**}$. For every $f \in L^p(\mu, Y)$, define $L^p(J_Y)(f(\omega)) = J_Y f(\omega)$, $\omega \in \Omega$. This gives a bijection $L^p(J_Y) : L^p(\mu, Y) \rightarrow L^p(\mu, Y^{**})$. By using Remark 1.7, we know that the mapping $\psi : L^q(\mu, Y^*) \rightarrow (L^p(\mu, Y))^*$ is a bijective bounded operator and so the adjoint $\psi^* : (L^p(\mu, Y))^{**} \rightarrow (L^q(\mu, Y^*))^*$ is bijective.

By using Remark 1.7 again, we obtain a bijective bounded operator

$$\psi' : L^p(\mu, Y^{**}) \rightarrow (L^q(\mu, Y^*))^*$$

such that for all $f \in L^p(\mu, Y^{**})$ and $g \in L^q(\mu, Y^*)$

$$\langle f, \psi' g \rangle = \int_{\Omega} \langle f(\omega), g(\omega) \rangle d\mu(\omega).$$

For all $f \in L^p(\mu, Y)$, $g \in L^q(\mu, Y^*)$ we have

$$\langle g, (\psi^* \circ J_{L^p(\mu, Y)}) f \rangle = \langle \psi(g), J_{L^p(\mu, Y)} f \rangle = \langle f, \psi(g) \rangle = \int_{\Omega} \langle f(\omega), g(\omega) \rangle d\mu(\omega)$$

and

$$\begin{aligned} \langle g, (\psi' \circ L^p(J_Y)) f \rangle &= \langle g, (\psi'(J_Y f(\cdot))) \rangle \\ &= \int_{\Omega} \langle g(\omega), J_Y f(\omega) \rangle d\mu(\omega) \\ &= \int_{\Omega} \langle f(\omega), g(\omega) \rangle d\mu(\omega). \end{aligned}$$

Therefore, $\psi^* \circ J_{L^p(\mu, Y)} = \psi' \circ L^p(J_Y)$ and hence $J_{L^p(\mu, Y)} = (\psi^*)^{-1} \circ \psi' \circ L^p(J_Y)$, which is a bijection. Hence $L^p(\mu, Y)$ is reflexive. \square

Theorem 3.5. *Let $F \in B_X^p(Y)$, $G \in F_X^p(Y)$ and $\|F\|_p \leq A_G$. Then*

$$F \pm G \in F_X^p(Y).$$

Proof. For each $x \in X$, we have

$$\|(F \pm G).(x)\|_p = \|F.(x) \pm G.(x)\|_p \geq \|G.(x)\|_p - \|F.(x)\|_p \geq (A_G - \|F\|_p)\|x\|$$

and

$$\|(F \pm G).(x)\|_p \leq (\|F\|_p + \|G\|_p)\|x\|.$$

So $F \pm G \in F_X^p(Y)$. \square

Theorem 3.6. *Let $F \in F_X^p(Y)$. Then for each $x^* \in X^*$, there exists an element $G \in L^p(\mu, Y^*)$ such that*

$$\langle y, x^* \rangle = \int_{\Omega} \langle y, F_{\omega}^*(G(\omega)) \rangle d\mu(\omega), \quad y \in X.$$

Proof. By Lemma 3.4, we have $R_F^* \psi = S_F$. Since $F \in F_X^p(Y)$, it follows from Corollary 3.3 that R_F^* is surjective. Thus the operator $S_F : L^q(\mu, Y^*) \rightarrow X^*$ is a surjection. Let $x^* \in X^*$; then there exists a $G \in L^p(\mu, Y^*)$ such that $x^* = S_F(G)$, so

$$\langle y, x^* \rangle = \int_{\Omega} \langle y, F_{\omega}^*(G(\omega)) \rangle d\mu(\omega), \quad y \in X. \quad \square$$

Definition 3.7. A Bochner (p, Y) -operator frame for X is called *independent* if the operator S_F is injective, i.e., if for every $f \neq 0$ there exists $x \in X$ such that

$$\int_{\Omega} \langle x, F_{\omega}^*(f(\omega)) \rangle d\mu(\omega) \neq 0.$$

We denote by $IF_X^p(Y)$ the set of all independent Bochner (p, Y) -operator frames for X .

Theorem 3.8. *Let F be an independent Bochner (p, Y) -operator frame for X . Then R_F is invertible.*

Proof. We already know that S_F is injective. By Lemma 3.4 and Corollary 3.3, we know that R_F^* is bijective. Hence R_F is invertible. \square

Theorem 3.9. *Let (Ω, Σ, μ) be a measure space and suppose there exists $k > 0$ such that $\mu(E) \geq k$ for every nonempty measurable set E of Ω . For each $F \in IF_X^p(Y)$, there exists a unique Bochner (q, Y^*) -operator frame Q for X^* such that for all $y \in X$*

$$\langle y, x^* \rangle = \int_{\Omega} \langle y, F_{\omega}^* R_Q x^*(\omega) \rangle d\mu(\omega).$$

Proof. Let F be an independent Bochner (p, Y) -operator frame for X . Then Theorem 3.8 yields that the operator R_F is invertible, so by Lemma 3.4, S_F is invertible. We can define $Q_{\omega} = P_{\omega} S_F^{-1}$, $\omega \in \Omega$, where $P_{\omega} : L^q(\mu, Y^*) \rightarrow Y^*$ is defined by $P_{\omega}(G) = G(\omega)$. By Lemma 1.8, P_{ω} is bounded. Therefore $Q_{\omega} \in B(X^*, Y^*)$, $\omega \in \Omega$. For each $x^* \in X^*$, we have $Q_{\omega}(x^*) = S_F^{-1}(x^*)$, so for each $x^* \in X^*$, the mapping $\omega \mapsto Q_{\omega}(x^*)$ is Bochner-measurable and

$$\frac{1}{\|S_F\|} \|x^*\| \leq \left(\int_{\Omega} \|Q_{\omega}(x^*)\|^q d\mu \right)^{1/q} = \|S_F^{-1}(x^*)\| \leq \|S_F^{-1}\| \|x^*\|.$$

Hence, Q is a Bochner (q, Y^*) -operator frame for X^* with bounds $\|S_F\|^{-1}$ and $\|S_F^{-1}\|$. By the definition of Q , we obtain that $R_Q = S_F^{-1}$ and so $x^* = S_F R_Q x^*$, $x^* \in X^*$. Thus

$$\langle y, x^* \rangle = \int_{\Omega} \langle y, F_{\omega}^* R_Q x^*(\omega) \rangle d\mu(\omega), \quad y \in X.$$

Next, we will show the uniqueness of Q . Let W be a Bochner (q, Y^*) -operator frame for X^* such that for all $y \in X$

$$\langle y, x^* \rangle = \int_{\Omega} \langle y, F_{\omega}^* R_W x^*(\omega) \rangle d\mu(\omega), \quad x^* \in X^*.$$

Thus $S_F R_W = I_{X^*}$, or $R_W = S_F^{-1} = R_Q$. Therefore, $W = Q$. \square

References

- [Cao et al. 2008] H.-X. Cao, L. Li, Q.-J. Chen, and G.-X. Ji, “ (p, Y) -operator frames for a Banach space”, *J. Math. Anal. Appl.* **347**:2 (2008), 583–591. MR 2009h:46024 Zbl 05344335
- [Cengiz 1998] B. Cengiz, “The dual of the Bochner space $L^p(\mu, E)$ for arbitrary μ ”, *Turkish J. Math.* **22**:3 (1998), 343–348. MR 99k:46061 Zbl 0930.46034
- [Daubechies et al. 1986] I. Daubechies, A. Grossmann, and Y. Meyer, “Painless nonorthogonal expansions”, *J. Math. Phys.* **27**:5 (1986), 1271–1283. MR 87e:81089 Zbl 0608.46014
- [Diestel and Uhl 1977] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Mathematical Surveys **15**, American Mathematical Society, Providence, RI, 1977. MR 56 #12216 Zbl 0369.46039
- [Duffin and Schaeffer 1952] R. J. Duffin and A. C. Schaeffer, “A class of nonharmonic Fourier series”, *Trans. Amer. Math. Soc.* **72** (1952), 341–366. MR 13,839a Zbl 0049.32401
- [Faroughi and Osgooei 2011] M. H. Faroughi and E. Osgooei, “Continuous p -Bessel mappings and continuous p -frames in Banach spaces”, *Involve* **4**:2 (2011), 167–186. MR 2012m:42045 Zbl 1235.42026
- [Fleming and Jamison 2008] R. J. Fleming and J. E. Jamison, *Isometries on Banach spaces: vector-valued function spaces*, vol. 2, Monographs and Surveys in Pure and Applied Mathematics **138**, CRC, Boca Raton, FL, 2008. MR 2009i:46001 Zbl 1139.46001
- [Rahimi et al. 2006] A. Rahimi, A. Najati, and Y. N. Dehghan, “Continuous frames in Hilbert spaces”, *Methods Funct. Anal. Topology* **12**:2 (2006), 170–182. MR 2007d:42061 Zbl 1120.42019

Received: 2011-04-16 Accepted: 2012-01-21

mhfaroughi@yahoo.com	<i>Department of Pure Mathematics, Faculty of Mathematical Science, University of Tabriz, 29 Bahman Street, Tabriz 5166614766, Iran</i>
reza.ahmadi84@yahoo.com	<i>Department of Pure Mathematics, Faculty of Mathematical Science, University of Tabriz, 29 Bahman Street, Tabriz 5166614766, Iran</i>
m_rahmani26@yahoo.com	<i>Department of Pure Mathematics, Faculty of Mathematical Science, University of Tabriz, 29 Bahman Street, Tabriz 5166614766, Iran Department of Mathematics, Ilam University, P.O. Box 69315516, Ilam, Iran</i>

involve

msp.org/involve

EDITORS

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	Victoria University, Australia pietro.cerone@vu.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moselehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tobrie1@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsteam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA jgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

PRODUCTION


Silvio Levy, Scientific Editor

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2012 is US \$105/year for the electronic version, and \$145/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2012 Mathematical Sciences Publishers

involve

2012

vol. 5

no. 3

Analysis of the steady states of a mathematical model for Chagas disease MARY CLAUSON, ALBERT HARRISON, LAURA SHUMAN, MEIR SHILLOR AND ANNA MARIA SPAGNUOLO	237
Bounds on the artificial phase transition for perfect simulation of hard core Gibbs processes MARK L. HUBER, ELISE VILLELLA, DANIEL ROZENFELD AND JASON XU	247
A nonextendable Diophantine quadruple arising from a triple of Lucas numbers A. M. S. RAMASAMY AND D. SARASWATHY	257
Alhazen's hyperbolic billiard problem NATHAN POIRIER AND MICHAEL MCDANIEL	273
Bochner (p, Y) -operator frames MOHAMMAD HASAN FAROUGH, REZA AHMADI AND MORTEZA RAHMANI	283
k -furgus semigroups NICHOLAS R. BAETH AND KAITLYN CASSITY	295
Studying the impacts of changing climate on the Finger Lakes wine industry BRIAN MCGAUVRAN AND THOMAS J. PFAFF	303
A graph-theoretical approach to solving Scramble Squares puzzles SARAH MASON AND MALI ZHANG	313
The n -diameter of planar sets of constant width ZAIR IBRAGIMOV AND TUAN LE	327
Boolean elements in the Bruhat order on twisted involutions DELONG MENG	339
Statistical analysis of diagnostic accuracy with applications to cricket LAUREN MONDIN, COURTNEY WEBER, SCOTT CLARK, JESSICA WINBORN, MELINDA M. HOLT AND ANANDA B. W. MANAGE	349
Vertex polygons CANDICE NIELSEN	361
Optimal trees for functions of internal distance ALEX COLLINS, FEDELIS MUTISO AND HUA WANG	371



1944-4176(2012)5:3;1-A