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The n-diameter of planar sets of constant width

Zair Ibragimov and Tuan Le

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We study the notion of n-diameter for sets of constant width. A convex set in the plane is said to be of *constant width* if the distance between two parallel support lines is constant, independent of the direction. The Reuleaux triangles are the well-known examples of sets of constant width that are not disks. The *n-diameter* of a compact set E in the plane is

$$d_n(E) = \max \left(\prod_{1 \le i < j \le n} |z_i - z_j| \right)^{\frac{2}{n(n-1)}},$$

where the maximum is taken over all $z_k \in E$, $k = 1, 2, \dots, n$. We prove that if $n = 5$, then the Reuleaux n-gons have the largest n-diameter among all sets of given constant width. The proof is based on the solution of an extremal problem for n-diameter.

1. Introduction

Sets of constant width have been an object of study by geometers for several centuries; some nontrivial examples of such sets were already known to Euler. A good summary of these studies is given in [Chakerian and Groemer 1983]; see also [Eggleston 1958]. A convex set in the plane is said to be of constant width if the distance between two parallel support lines is constant independent of the direction. Equivalently, a planar convex set W with nonempty interior is said to be of constant width if for each $\xi \in \partial W$ there exists $\eta \in \partial W$ with $|\xi - \eta| = \operatorname{diam} W$. While the disks are easily seen to be of constant width, the Reuleaux triangles are the well-known examples of sets of constant width that are not disks. In fact, sets of constant width can be thought of as generalizations of disks in that they share many properties with disks. For example, closed disks are *diametrically complete*, that is, addition of any point increases their diameter. This completeness notion characterizes the sets of constant width. Namely, the family of all complete sets is precisely the family of sets of constant width [Eggleston 1958, Theorem 52]. Another common property is

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that sets of constant width are precisely the sets of constant diameter. (A compact set E is said to be of constant diameter if $\max\{|x-y| : y \in E\} = \operatorname{diam} E$ for each $x \in \partial E$). Also, the definition of constant width sets using parallel support lines is also based on a property of disks, namely that the distance between any two parallel support lines of a disk is constant. Finally, one more property of sets of constant width that is common with disks is that the length of the boundary arc of a disk is equal to $\lambda\pi$, where λ is the diameter of the disk; the same is true of sets of constant width. Of course, not every property of the disks is shared by sets of constant width. For example, sets of constant width $\lambda > 0$ do not have to have the same area as disks of diameter λ or that sets of constant width do not have to have smooth boundaries. In fact, by the isoperimetric inequality disks of diameter λ have the largest area while by the Blaschke–Lebesgue theorem the Reuleaux triangles of diameter λ have the smallest area. The Reuleaux triangle (named after the nineteenth-century German engineer Franz Reuleaux) of diameter λ is constructed by connecting the vertices of an equilateral triangle of sidelength λ by arcs of circles of radius λ and centered at the vertices.

Sets of constant width arise in many areas of mathematics. For instance, every odd-term Fourier series gives rise to a planar set of constant width [Kelly 1957]. Constant width sets are used in cinematography and engineering. For example, they are used in the design of the Wankel engine [Berger 1994]. They are also aesthetically pleasing, frequently turning up in art and design contexts. For example, some Irish coins have constant width shapes because of their appealing character.

The 3-diameter of sets of constant width as well as the related notions of d_3-complete sets and sets of constant 3-diameter were first studied in [Hästö et al. 2012]. As mentioned above the disks have the largest area and the Reuleaux triangles have the smallest area among all sets of given constant width. Surprisingly, the roles of the isoperimetric inequality and the Blaschke–Lebesgue theorem are reversed when it comes to 3-diameter. More precisely, among the planar sets of constant width λ, Reuleaux triangles have the largest 3-diameter, namely λ, and disks have the smallest 3-diameter, $\sqrt{3}\lambda/2$ [Hästö et al. 2012, Theorem 3.1]. On the other hand, the Reuleaux triangles have the largest area among all sets with both the diameter and 3-diameter equal to λ [Hästö et al. 2012, Proposition 2.2]. As in the case of ordinary diameter, disks are both of constant 3-diameter and d_3-complete, and d_3-complete sets are of constant 3-diameter [Hästö et al. 2012, Theorem 5.2].

In this paper we study n-diameter of sets of constant width. Our study is based on the following extremal problem: *among all planar sets of cardinality n and of diameter less than or equal to 2, find one with the largest n-diameter*. We conjecture that the vertices of regular n-gons have the largest n-diameter if n is odd (Conjecture 2.8) and show the conjecture is equivalent to stating that the Reuleaux n-gons have the largest n-diameter among all sets of given constant width

(Theorem 4.3). Clearly, for $n = 3$ the vertices of equilateral triangles provide a solution to the extremal problem. In contrast, the vertices of the regular 4-gon do not have the largest 4-diameter (Lemma 2.9). We show that Conjecture 2.8 holds for $n = 5$ (Theorem 3.1), and also verify the conjecture for $n = 7$ under some additional assumptions (Proposition 3.3).

2. Extremal problem for n-diameter

Definition 2.1. The n-diameter of a compact set E in the complex plane $\mathbb{C}$ is defined by

$$d_n(E) = \max\Bigg(\prod_{1\le i<j\le n} |z_i - z_j|\Bigg)^{\frac{2}{n(n-1)}},$$

where the maximum is taken over all $z_k \in E$, $k = 1, 2, \ldots, n$.

Clearly, $d_2(E)$ is the ordinary diameter of E. That is,

$$d_2(E) = \operatorname{diam} E = \sup\{|z - w| : z, w \in E\}.$$

The n-diameter is weakly decreasing in n, that is, $d_n(E) \ge d_{n+1}(E)$ [Ahlfors 1973, p. 23]; see also [Hayman 1966, Theorem 1]. We give the proof for completeness. We have

$$d_{n+1}(E) = \prod_{1\le i<j\le n+1} |z_i - z_j|^{\frac{2}{n(n+1)}};$$

thus

$$\big(d_{n+1}(E)\big)^{n(n+1)/2} = \prod_{k=2}^{n+1} |z_1 - z_k| \prod_{2\le i<j\le n+1} |z_i - z_j| \le \prod_{k=2}^{n+1} |z_1 - z_k| \big(d_n(E)\big)^{n(n-1)/2}.$$

Similarly, for each $l = 2, 3, \ldots, n+1$ we have

$$\big(d_{n+1}(E)\big)^{n(n+1)/2} \le \prod_{\substack{k=1\\k\ne l}}^{n+1} |z_l - z_k| \big(d_n(E)\big)^{n(n-1)/2}.$$

Multiplying these expressions we obtain

$$\big(d_{n+1}(E)\big)^{n(n+1)^2/2} \le \big(d_{n+1}(E)\big)^{n(n+1)} \big(d_n(E)\big)^{n(n-1)(n+1)/2}$$

which yields $d_{n+1}(E) \le d_n(E)$, as required.

The *transfinite diameter* of E is defined by

$$d_\infty(E) = \lim_{n\to\infty} d_n(E).$$

The transfinite diameter of a line segment of length L is $L/4$ and the transfinite diameter of a disk of radius r is equal to r [Ahlfors 1973, p. 28; Goluzin 1969,

p. 298]. The notion of transfinite diameter is due to Fekete and plays an important role in complex analysis. It is related to the notions of logarithmic capacity and the Chebysheff constant [Ahlfors 1973; Hille 1962; Tsuji 1959]. Some extremal problems involving the transfinite diameter and n-diameter of planar sets were studied in [Burckel et al. 2008; Dubinin 1986; Duren and Schiffer 1991; Grandcolas 2000; Grandcolas 2002; Reich and Schiffer 1964].

The simplest examples of sets for computing the n-diameter are undoubtedly the *n-tuples*, that is, sets consisting of n distinct points. Let T_n denote the set of all n-tuples in $\mathbb{C}$ of diameter less than or equal to 2.

Definition 2.2. By the extremal problem for n-diameter we mean the problem of finding

$$\sup_{E\in T_n} d_n(E).$$

According to Jung's theorem each $E \in T_n$ is contained in a disk of radius r, where $1/2 \le r \le 2/\sqrt{3}$ [Berger 1994, Theorem 11.5.8]. Also, for any $E \subset \mathbb{C}$ and for any linear transformation $L(z) = az+b$ ($a, b \in \mathbb{C}$, $a \neq 0$) we have $d_n(L(E)) = |a| d_n(E)$. Consequently,

$$\sup_{E\in T_n} d_n(E) = \sup_{E\in T'_n} d_n(E), \qquad \text{where } T'_n = \{E \in T_n : E \subset \bar{B}(0, 2/\sqrt{3})\}.$$

Since the function $d_n : T'_n \to [0, 2]$ is continuous and since T'_n is a compact subset of the n-dimensional complex space $\mathbb{C}^n$, d_n achieves its maximum in T'_n.

Definition 2.3. An n-tuple $E' \in T_n$ is called extremal if

$$\sup_{E\in T_n} d_n(E) = d_n(E').$$

Let $E_n \subset T_n$ denote the set of all extremal n-tuples in T_n. Thus, the n-diameter problem is equivalent to finding a member of E_n and computing its n-diameter.

Lemma 2.4. *Given $E \in E_n$, for each $z \in E$ there exists $w \in E$ with $|z - w| = 2$. In particular, the n-gon with vertices in E is convex.*

Proof. Let $E = \{z_1, z_2, \dots, z_n\} \in E_n$ and suppose that there exists k such that $|z_k - z_l| < 2$ for all $l = 1, 2, \dots, n$. Then there exists a disk D centered at z_k such that $|z - z_l| < 2$ for all $z \in D$ and for all $l = 1, 2, \dots, n$. Since the function

$$P(z) = \prod_{\substack{l=1\\ l\neq k}}^{n} (z - z_l)$$

is analytic in D, its modulus $|P(z)|$ cannot achieve its maximum at z_k. Hence there exists a point $z'_k \in D$ such that the n-tuple $E' = \{z_1, z_2, \dots, z'_k, \dots, z_n\}$ belongs to T_n and that $d_n(E') > d_n(E)$. Hence $E \notin E_n$, which is the required contradiction.

Let $\mathscr{C}(E)$ be the *convex hull* of E, that is, the smallest convex set containing E. Then

$$\mathscr{C}(E) = \left\{ \sum_{k=1}^{3} \lambda_k \alpha_k \mid \alpha_k \in E,\ \lambda_k \geq 0,\ \sum_{k=1}^{3} \lambda_k = 1 \right\}$$

by Carathéodory's theorem [Berger 1994, 11.1.8.6]. The first part of the lemma implies that the points $z_1, z_2, \dots, z_n$ can only lie on the corners of $\mathscr{C}(E)$. Hence $\mathscr{C}(E)$ is the n-gon with vertices at the points $z_1, z_2, \dots, z_n$. □

The following corollary is an immediate consequence of Lemma 2.4.

Corollary 2.5. *Let $n \geq 3$ be an odd integer. Then for each $E \in E_n$ there exist $z, w_1, w_2 \in E$ such that $|z - w_1| = |z - w_2| = 2$.*

Let $\omega = e^{2\pi i/n}$ be the nth root of unity and put

$$\mathscr{E}_n = \{1, \omega, \omega^2, \dots, \omega^{n-1}\}.$$

Let $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disk in $\mathbb{C}$. The following observation is credited to Pólya [Overholt and Schober 1989, p. 279]:

Theorem 2.6 (Pólya extremal problem).

$$\max d_n(\{z_1, z_2, \dots, z_n\}) = d_n(\mathscr{E}_n) = n^{1/(n-1)}$$

where the maximum is taken over all points $z_1, z_2, \dots, z_n$ in $\overline{\mathbb{D}}$.

Observe that

$$\operatorname{diam} \mathscr{E}_n = 2 \quad \text{if } n \text{ is even.}$$

On the other hand, if n is odd, then

$$\operatorname{diam} \mathscr{E}_n = |1 - \omega^{(n-1)/2}| = 2\sin((n-1)\pi/2n) = 2\sin(\pi/2 - \pi/2n) = 2\cos(\pi/2n).$$

Put

$$r_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \sec(\pi/2n) & \text{if } n \text{ is odd,} \end{cases}$$

and let

$$r_n\mathscr{E}_n = \{r_n, r_n\omega, r_n\omega^2, \dots, r_n\omega^{n-1}\}.$$

Note that

$$d_n(r_n\mathscr{E}_n) = r_n d_n(\mathscr{E}_n) = \begin{cases} n^{1/(n-1)} & \text{if } n \text{ is even,} \\ \sec(\pi/2n)n^{1/(n-1)} & \text{if } n \text{ is odd.} \end{cases} \tag{2-7}$$

Since $r_n\mathscr{E}_n \in T_n'$, we have

$$\sup_{E \in T_n} d_n(E) = \sup_{E \in T_n'} d_n(E) \geq d_n(r_n\mathscr{E}_n) = \begin{cases} n^{1/(n-1)} & \text{if } n \text{ is even,} \\ \sec(\pi/2n)n^{1/(n-1)} & \text{if } n \text{ is odd.} \end{cases}$$

Conjecture 2.8. *If n is odd, then $d_n(E) \le d_n(r_n \mathscr{E}_n)$ for each $E \in T_n$.*

Conjecture 2.8 predicts that if n is odd, then the vertices of the regular n-gons are extremal. In contrast, the vertices of the regular 4-gon are not extremal.

Lemma 2.9. *Let $E = \{r_3, r_3\omega, r_3 x, r_3\omega^2\} = r_3(\mathscr{E}_3 \cup \{x\})$, where $x = 1 - \sqrt{3}$. Then $d_4(E) > d_4(\mathscr{E}_4)$.*

Proof. Recall that $r_3 = 2/\sqrt{3}$ and $\mathscr{E}_3 = \{1, \omega, \omega^2\}$, where $w = e^{2\pi i/3}$ is the third root of unity. Then $|x-1| = \sqrt{3}$ and $|x-\omega| = |x-\omega^2| = \sqrt{6-3\sqrt{3}}$. Clearly, $E \in T_4$ and hence

$$\begin{aligned} d_4(E) &= r_3(|x-1|\,|x-\omega|\,|x-\omega^2|\,|1-\omega|\,|1-\omega^2|\,|\omega-\omega^2|)^{1/6} \\ &= \frac{2}{\sqrt{3}}(|x-1|\,|x-\omega|\,|x-\omega^2|)^{1/6}\big[(|1-\omega|\,|1-\omega^2|\,|\omega-\omega^2|)^{1/3}\big]^{1/2} \\ &= \frac{2}{\sqrt{3}}(6\sqrt{3}-9)^{1/6}(d_3(\mathscr{E}_3))^{1/2} = \frac{2}{\sqrt{3}}(6\sqrt{3}-9)^{1/6}3^{1/4} \\ &= 2(2-\sqrt{3})^{1/6} > 4^{1/3} = d_4(\mathscr{E}_4), \end{aligned}$$

as required. □

Unfortunately, this idea does not seem to extend to even integers greater than 4. More precisely, some tedious computations show that if

$$E = r_{n-1}(\mathscr{E}_{n-1} \cup \{x_n\}), \qquad \text{where } x_n = 1 - 2\cos\frac{\pi}{2(n-1)},$$

then $d_n(E) < d_n(\mathscr{E}_n)$ for $n = 6, 8, 10$.

3. Cases $n = 5$ and $n = 7$

In this section we discuss Conjecture 2.8 for $n=5$ and $n=7$. We will make a frequent use of the well-known Ptolemy's inequality and the AM-GM inequality as well as Reinhardt's theorem. Recall that Ptolemy's inequality says that $|a-b|\,|c-d| \le |a-c|\,|b-d| + |a-d|\,|b-c|$ for all $a, b, c, d \in \mathbb{C}$ and that the equality occurs if and only if the points a, b, c, d lie on a circle in this order. The AM-GM inequality says that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \ge (x_1 \cdot x_2 \cdots x_n)^{1/n}$$

for all nonnegative real numbers $x_1, x_2, \ldots, x_n$, and that the equality occurs when $x_1 = x_2 = \cdots = x_n$. Reinhardt's theorem says that if n is odd, then the regular n-gon has the largest perimeter among all convex n-gons of fixed diameter [Mossinghoff 2006].

Theorem 3.1. *Conjecture 2.8 is true for $n = 5$.*

Proof. Let $E = \{z_1, z_2, z_3, z_4, z_5\}$ be any 5-tuple in T_5. Without loss of generality we can assume that E is extremal, that is, $E \in E_5$. Let $\mathcal{P}_5(E)$ denote the 5-gon with vertices in E. Note that $\mathcal{P}_5(E)$ is convex by Lemma 2.4. Let P denote the perimeter of $\mathcal{P}_5(E)$. That is,

$$P = |z_1 - z_2| + |z_2 - z_3| + |z_3 - z_4| + |z_4 - z_5| + |z_5 - z_1|.$$

Clearly,

$$|z_1 - z_3|\,|z_1 - z_4|\,|z_2 - z_4|\,|z_2 - z_5|\,|z_3 - z_5| \le 2^5$$

and the equality holds if $\mathcal{P}_5(E)$ is regular. Using the AM-GM inequality we obtain

$$|z_1 - z_2|\,|z_2 - z_3|\,|z_3 - z_4|\,|z_4 - z_5|\,|z_5 - z_1| \le (P/5)^5.$$

Since $\mathcal{P}_5(E)$ is convex and $\operatorname{diam} \mathcal{P}_5(E) = 2$, by Reinhardt's theorem P is less than or equal to the perimeter of a regular 5-gon of diameter 2. Computations show that such a 5-gon has a side-length $l = \sec(\pi/5)$ and is inscribed in a circle of radius

$$r = \csc(2\pi/5) = \csc(\pi/2 - \pi/10) = \sec(\pi/10).$$

Hence

$$d_5(E) = \prod_{1 \le i < j \le 5} |z_i - z_j|^{1/10} \le \sqrt{2\sec(\pi/5)}.$$

Observe that

$$\sqrt{2\sec(\pi/5)} = \sec(\pi/10)5^{1/4}.$$

Indeed, by Theorem 2.6 we have $d_5(\mathscr{E}_5) = 5^{1/4}$ and a direct computation yields

$$d_5(\mathscr{E}_5) = \sqrt{|1-\omega|\,|1-\omega^2|} = 2\sqrt{\sin(\pi/5)\sin(2\pi/5)}.$$

Hence $2\sqrt{\sin(\pi/5)\sin(2\pi/5)} = 5^{1/4}$ and it remains to show that

$$2\sec(\pi/5) = 4\sec^2(\pi/10)\sin(\pi/5)\sin(2\pi/5).$$

Equivalently,

$$\cos^2(\pi/10) = 2\cos(\pi/5)\sin(\pi/5)\sin(2\pi/5).$$

We have

$$2\cos(\pi/5)\sin(\pi/5)\sin(2\pi/5) = \sin^2(2\pi/5) = \sin^2(\pi/2 - \pi/10) = \cos^2(\pi/10),$$

as required.

Finally, the equality holds if $\mathcal{P}_5(E)$ is regular, that is, $E = L(r_5\mathscr{E}_5)$ for some $L(z) = az + b$ with $|a| = 1$. Thus,

$$d_5(E) \le \sqrt{2\sec(\pi/5)} = \sec(\pi/10)5^{1/4} = d_n(r_5\mathscr{E}_5),$$

completing the proof. □

Next, we discuss Conjecture 2.8 for $n = 7$. While we cannot verify if the conjecture is true for $n = 7$, we provide its validity under the following additional condition on 7-tuples. Given a 7-tuple $E = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7\}$, suppose that the 7-gon $\mathcal{P}_7(E)$ with vertices in E is convex and that

$$\sum_{k=1}^{7} |z_k - z_{k+1}|\,|z_{k+2} - z_{k+3}| \leq \frac{1}{2}\sum_{k=1}^{7} |z_k - z_{k+1}|\big(|z_{k+1} - z_{k+2}| + |z_{k+3} - z_{k+4}|\big), \tag{3-2}$$

where $z_8 = z_1$, $z_9 = z_2$, $z_{10} = z_3$, $z_{11} = z_4$. Observe that the regular 7-gons satisfy condition (3-2).

Proposition 3.3. *Suppose that the 7-tuple $E = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7\}$ is in T_7 and satisfies condition* (3-2) *and that $\mathcal{P}_7(E)$ is convex. Then*

$$d_7(E) \leq 2(2\sin(\pi/14))^{1/2}(1 + 2\sin(\pi/14))^{1/6}.$$

Equality holds $\mathcal{P}_7(E)$ is a regular 7-gon of side-length

$$l = 2\sec(\pi/14)\sin(\pi/7) = 4\sin(\pi/14).$$

In particular,

$$2(2\sin(\pi/14))^{1/2}(1 + 2\sin(\pi/14))^{1/6} = \sec(\pi/14)7^{1/6}.$$

Proof. The product $\prod_{1\leq k<l\leq 7} |z_k - z_l|$ can be split into three parts:

$$|z_1 - z_2|\,|z_2 - z_3|\,|z_3 - z_4|\,|z_4 - z_5|\,|z_5 - z_6|\,|z_6 - z_7|\,|z_1 - z_7|,$$

$$|z_1 - z_4|\,|z_1 - z_5|\,|z_2 - z_5|\,|z_2 - z_6|\,|z_3 - z_6|\,|z_3 - z_7|\,|z_4 - z_7|,$$

$$|z_1 - z_3|\,|z_2 - z_4|\,|z_3 - z_5|\,|z_4 - z_6|\,|z_5 - z_7|\,|z_1 - z_6|\,|z_2 - z_7|.$$

It follows from the AM-GM inequality and Reinhardt's theorem that

$$|z_1 - z_2|\,|z_2 - z_3|\,|z_3 - z_4|\,|z_4 - z_5|\,|z_5 - z_6|\,|z_6 - z_7|\,|z_1 - z_7| \leq (P/7)^7,$$

where P is the perimeter of a regular 7-gon with side-length $4\sin\frac{\pi}{14}$. Therefore,

$$|z_1 - z_2|\,|z_2 - z_3|\,|z_3 - z_4|\,|z_4 - z_5|\,|z_5 - z_6|\,|z_6 - z_7|\,|z_1 - z_7| \leq [4\sin(\pi/14)]^7$$

and since $E \in T_7$, we also obtain

$$|z_1 - z_4|\,|z_1 - z_5|\,|z_2 - z_5|\,|z_2 - z_6|\,|z_3 - z_6|\,|z_3 - z_7|\,|z_4 - z_7| \leq 2^7.$$

Moreover, the equality holds for a regular 7-gon in both of these inequalities.

It remains to find the maximum value of

$$|z_1 - z_3|\,|z_2 - z_4|\,|z_3 - z_5|\,|z_4 - z_6|\,|z_5 - z_7|\,|z_1 - z_6|\,|z_2 - z_7|.$$

To achieve this goal we will use Ptolemy's inequality. We have

$$|z_1-z_3|\,|z_2-z_4| \le |z_1-z_4|\,|z_2-z_3|+|z_1-z_2|\,|z_3-z_4| \le 2|z_2-z_3|+|z_1-z_2|\,|z_3-z_4|.$$

Hence

$$|z_1-z_3|\,|z_2-z_4| \le 2|z_2-z_3|+|z_1-z_2|\,|z_3-z_4|.$$

In a similar fashion we obtain

$$\begin{aligned}
|z_2-z_4|\,|z_3-z_5| &\le 2|z_3-z_4|+|z_2-z_3|\,|z_4-z_5|,\\
|z_3-z_5|\,|z_4-z_6| &\le 2|z_4-z_5|+|z_3-z_4|\,|z_5-z_6|,\\
|z_4-z_6|\,|z_5-z_7| &\le 2|z_5-z_6|+|z_4-z_5|\,|z_6-z_7|,\\
|z_5-z_7|\,|z_1-z_6| &\le 2|z_6-z_7|+|z_5-z_6|\,|z_1-z_7|,\\
|z_1-z_6|\,|z_2-z_7| &\le 2|z_1-z_7|+|z_6-z_7|\,|z_1-z_2|,\\
|z_2-z_7|\,|z_1-z_3| &\le 2|z_1-z_2|+|z_1-z_7|\,|z_2-z_3|.
\end{aligned}$$

Notice that the equalities hold if $\mathcal{P}_7(E)$ is a regular 7-gon in T_7, since the vertices of such a 7-gon lie on a circle and that $|z_k-z_{k+3}|=2$ for each $k=1,2,\dots,7$.

Multiplying these inequalities and applying the AM-GM inequality we obtain

$$\begin{aligned}
\prod_{1\le k\le 7}|z_k-z_{k+2}|^2 &= \big(|z_1-z_3|\,|z_2-z_4|\,|z_3-z_5|\,|z_4-z_6|\,|z_5-z_7|\,|z_1-z_6|\,|z_2-z_7|\big)^2\\
&\le \prod_{1\le k\le 7}\big(2|z_{k+1}-z_{k+2}|+|z_k-z_{k+1}|\,|z_{k+2}-z_{k+3}|\big)\\
&\le \frac{1}{7^7}\left(2\sum_{k=1}^{7}|z_k-z_{k+1}|+\sum_{k=1}^{7}|z_k-z_{k+1}|\,|z_{k+2}-z_{k+3}|\right)^7.
\end{aligned}$$

Reinhardt's theorem implies

$$2\sum_{k=1}^{7}|z_k-z_{k+1}| \le 2P = 56\sin\frac{\pi}{14}$$

Once again we have the equality if $\mathcal{P}_7(E)$ is a regular 7-gon in T_7.

By our assumption we have

$$\sum_{k=1}^{7}|z_k-z_{k+1}|\,|z_{k+2}-z_{k+3}| \le \frac{1}{2}\sum_{k=1}^{7}|z_k-z_{k+1}|\big(|z_{k+1}-z_{k+2}|+|z_{k+3}-z_{k+4}|\big)$$

with equality for a regular 7-gon. Applying the AM-GM inequality for each pair of $|z_k-z_{k+1}|^2+|z_l-z_{l+1}|^2$ $(1\le k<l\le 7)$ and Reinhardt's theorem we have

$$\begin{aligned}
28^2 \sin^2 \frac{\pi}{14} &\geq \left(\sum_{k=1}^{7} |z_k - z_{k+1}| \right)^2 \\
&\geq \frac{7}{3} \sum_{k=1}^{7} |z_k - z_{k+1}| \big(|z_{k+2} - z_{k+3}| + |z_{k+1} - z_{k+2}| + |z_{k+3} - z_{k+4}| \big) \\
&\geq 7 \sum_{k=1}^{7} |z_k - z_{k+1}| \, |z_{k+2} - z_{k+3}|
\end{aligned}$$

Thus,

$$\sum_{k=1}^{7} |z_k - z_{k+1}| \, |z_{k+2} - z_{k+3}| \leq 112 \sin^2 \frac{\pi}{14},$$

which means that

$$\prod_{k=1}^{7} |z_k - z_{k+2}| \leq \left(8 \sin \frac{\pi}{14} + 16 \sin^2 \frac{\pi}{14} \right)^{7/2}.$$

This completes our proof in this case. The equality indeed occurs when z_k's are vertices of the regular 7-gon whose side-length $l = 4 \sin \pi/14$. □

4. The *n*-diameter of sets of constant width

Let n be odd and consider the n-tuple $r_n \mathscr{E}_n$. Recall that $r_n = \sec(\pi/2n)$ and $\mathscr{E}_n = \{1, \omega, \omega^2, \dots, \omega^{n-1}\}$. Put $z_k = r_n \omega^k$, $k = 0, 1, 2, \dots, n-1$. Connect the consecutive points z_k and z_{k+1} by an arc of a circle whose center is the unique point z_l in the set $\{z_1, z_2, \dots, z_n\}$ which is equidistant from the points z_k and z_{k+1}. The radii of all such circles are the same and is equal to 2. For example, the circle centered at z_n and radius λ joins the points $z_{(n-1)/2}$ and $z_{(n+1)/2}$. The resulting set, denoted by $\mathscr{R}_n$, is of constant width 2. It is called a *Reuleaux n-gon* and the points $\{z_1, z_2, \dots, z_n\}$ are called *the vertices* of $\mathscr{R}_n$ [Chakerian and Groemer 1983, p. 59]. It follows from the construction of $\mathscr{R}_n$ that if W is any set of constant width 2 containing $r_n \mathscr{E}_n$, then $W \subset \mathscr{R}_n$. A Reuleaux n-gon of width $\lambda > 0$ is constructed in a similar fashion.

A *Reuleaux polygon* of width λ is a set of constant width λ whose boundary consists of a finitely many (necessarily odd) circular arcs of radius λ [Eggleston 1958, p. 128]. Let $\mathscr{R}$ be a Reuleaux polygon and let D be a unique disk of smallest radius containing $\mathscr{R}$. If all the corners of $\mathscr{R}$ (i.e., the intersection points of the boundary arcs of $\mathscr{R}$) are contained on ∂D, then $\mathscr{R}$ is a Reuleaux n-gon, where n is the number of corners.

Lemma 4.1. *If n is odd, then $d_n(\mathscr{R}_n) = d_n(r_n \mathscr{E}_n)$.*

Proof. Let $D = \bar{B}(0, r_n)$. Since $\mathcal{R}_n \subset D$, we have $d_n(\mathcal{R}_n) \le d_n(D)$. Since the corners $\{z_1, z_2, \dots, z_n\}$ of $\mathcal{R}_n$ are equally spaced on ∂D, we have

$$d_n(\mathcal{R}_n) \ge \prod_{1\le k<l\le n} |z_k - z_l|^{\frac{2}{n(n-1)}} = d_n(D).$$

Thus, $d_n(\mathcal{R}_n) = d_n(D) = r_n d_n(\overline{\mathbb{D}}) = \sec(\pi/2n) n^{1/(n-1)} = d_n(r_n \mathcal{E}_n)$. □

Conjecture 4.2. *If n is odd, the Reuleaux n-gons have the largest n-diameter among all sets of the same constant width.*

Theorem 4.3. *Conjecture 4.2 is equivalent to Conjecture 2.8.*

Proof. Suppose that Conjecture 2.8 is true and let W be a set of constant width λ. Let $\mathcal{R}_n(\lambda)$ be a Reuleaux n-gon of width λ. Note that if $L(z) = az + b$ with $a \ne 0$, then the set $L(W)$ is of constant width $|a|\lambda$. Let E be an n-tuple of points in ∂W with $d_n(W) = d_n(E)$. Since $(2/\lambda)E \in T_n$, using Conjecture 2.8 and Lemma 4.1 we obtain

$$d_n(W) = d_n(E) = \frac{\lambda}{2} d_n((2/\lambda)E) \le \frac{\lambda}{2} d_n(r_n \mathcal{E}_n) = \frac{\lambda}{2} d_n(\mathcal{R}_n) = d_n(\mathcal{R}_n(\lambda)).$$

Conversely, suppose that Conjecture 4.2 is true and let E be any n-tuple in T_n. Then E is contained in a set W of constant width 2 [Eggleston 1958, Theorem 54]. Using Conjecture 4.2 and Lemma 4.1 we obtain

$$d_n(E) \le d_n(W) \le d_n(\mathcal{R}_n) = d_n(r_n \mathcal{E}_n).$$ □

Conjecture 4.2, if true, would imply the following corollary (see also [Hille 1962]).

Corollary 4.4. *Let $A \subset \mathbb{C}$ be any set with* diam $A = \lambda > 0$ *and let D be a disk with* diam $A = \lambda$. *Then*

$$d_\infty(A) \le d_\infty(D) = \frac{\lambda}{2}.$$

Proof. Clearly, $d_\infty(D) = \lambda/2$. The set A is contained in a set W of constant width λ. If n is odd, then using (2-7), Lemma 4.1 and Conjecture 4.2 we obtain

$$d_n(A) \le d_n(W) \le \frac{\lambda}{2} d_n(\mathcal{R}_n(\lambda) = \frac{\lambda}{2} \sec(\pi/2n) n^{1/(n-1)}.$$

By letting n tend to infinity we obtain $d_\infty(A) \le \lambda/2$. □

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