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The firefighter problem for regular infinite directed grids

Daniel P. Biebighauser, Lise E. Holte and Ryan M. Wagner

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We investigate the firefighter problem for regular infinite directed grids. We provide a complete classification of these grids by dividing them into two categories: grids where a single outbreak of fire can be contained with one firefighter per time step and grids that require a second firefighter at some time step. We then investigate infinite directed grids where the degrees of a single vertex are different from the degrees of all other vertices in the grid.

1. Introduction

The firefighter problem was introduced by Bert Hartnell at a conference talk [1995]. A fire breaks out at one or more vertices of a graph G at time zero. At each subsequent time step, one or more defenders are placed on nonburning and undefended vertices, and then the fire spreads from each burning vertex to all of its undefended neighbors. Once a vertex is burning or defended, it remains in that state for the duration of the problem. In particular, firefighters cannot move. The goal is to place firefighters in a way that achieves a desired optimal result, such as containing the fire in as few time steps as possible or minimizing the total number of burned vertices. For a comprehensive introduction to the problem, see [Finbow and MacGillivray 2009].

Question 26 in this last reference suggests investigating the firefighter problem for directed graphs. In this paper, we study infinite directed grids. An *infinite grid* is the graph with vertex set $\mathbb{Z} \times \mathbb{Z}$ where (x_1, y_1) is adjacent to (x_2, y_2) if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$. We consider the firefighter problem on *regular infinite directed grids*, which are infinite grids where a direction is assigned to each edge in such a way that every vertex has in-degree two and out-degree two. We will always consider our grids to be embedded in the plane such that each vertex (x, y) is on the lattice point (x, y).

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In this paper, we are concerned with the number of firefighters needed at each time step to eventually contain a fire that starts at a single vertex in a regular infinite directed grid. By "contain," we mean that there is some time step where no new vertices are burned. We are not necessarily interested in containing the fire as soon as possible, but in determining the minimum number of firefighters per time step needed for containment.

Fogarty [2003] proved that two firefighters per time step is necessary and sufficient to contain any finite outbreak of fire in an infinite grid. (Wang and Moeller [2002] had proved earlier that this number was necessary and sufficient for a single vertex initially on fire.) This number is sufficient for infinite directed grids, since directions on the arcs potentially restrict the movement of the fire. If there is an arc joining a burning vertex to an undefended vertex, the fire will spread to the undefended vertex on the next time step (as it would in an undirected graph), but if the arc points in the opposite direction, the fire will not spread along that arc. We will prove that, for regular infinite directed grids with a single vertex initially on fire, we can always contain the fire with fewer defenders.

Our main result is the following theorem, which we prove in Section 3. Without loss of generality, assume that the fire begins at the origin. We will say that an infinite directed grid is a *category A grid* if one firefighter per time step is sufficient to contain the fire. An infinite directed grid is a *category B grid* if one firefighter per time step is not enough to contain the fire, but one firefighter per time step and a second firefighter at any single time step is sufficient to contain the fire.

Theorem 1.1. *Let G be a regular infinite directed grid. Then G is either a category A or a category B grid.*

At the end of this paper, we consider infinite directed grids where at least one vertex has degrees other than in-degree two and out-degree two.

2. A lemma

Fogarty [2003] introduced a theorem with a "Hall-type condition" that is useful for proving that a certain number of defenders per time step is not enough to contain an outbreak of fire in an infinite graph. Her applications of this theorem were mostly to two-dimensional grids. Hartke [2004] extended Fogarty's result using a more general Hall-type condition that allowed him to make stronger statements about infinite grids in higher dimensions. We will use a modified version of Fogarty's theorem that applies to directed graphs. The proof is nearly identical to Fogarty's original proof and will not be included here.

Let G be a directed graph. Assume that one vertex catches on fire at time $t = 0$. Let D_k denote the set of vertices of distance k from the original burned vertex, where the distance from v to w is the length of a shortest directed path from v to w.

Let $B_k \subseteq D_k$ be the set of vertices in D_k that have been burned after time k. Let f_k denote the number of new firefighters available at time step k. Let r_k be the number of firefighters in D_{k+1}, D_{k+2}, ... after time k. We call these *reserve firefighters*. Let $N(S)$ be the neighborhood of a set of vertices S, that is, the set of vertices which are distance 1 from any vertex in S in the underlying undirected graph of G. For any subset $A \subseteq D_k$ let $N^+(A) = N(A) \cap D_{k+1}$.

Theorem 2.1. *Let G be a directed graph. For each k, if every $A \subseteq D_k$ satisfies $|N^+(A)| \geq |A| + f_k$, then $|B_n| \geq 1 + r_n$ for all n.*

We will now apply this theorem to prove the following lemma. An *infinite quarter-plane* is the subgraph of the infinite grid that includes all of the vertices and edges in the first quadrant, including the origin and the positive x- and y-axes.

Lemma 2.2. *Consider an infinite directed quarter-plane where all horizontal arcs point right and all vertical arcs point up. If the fire starts at the origin, one firefighter per time step is not enough to contain the fire. If we are given at least one firefighter per time step, and a second firefighter at any time step, then the fire can be contained.*

Proof. We first prove that one firefighter per time step is not enough to contain the fire. For each k, if $A \subseteq D_k$, we can see that

$$|N^+(A)| \geq |A| + 1,$$

since each vertex in D_k has exactly two neighbors in D_{k+1} and any two vertices in D_k can share at most one neighbor in D_{k+1}. So from Theorem 2.1, since the origin is initially on fire, for every k, we have $|B_k| \geq 1$. Thus one firefighter per time step is not enough to contain the fire.

If we are given at least one firefighter per time step, we can force the fire along an axis of the grid until we get a second firefighter, at which point the fire can be contained by placing this defender on the axis directly ahead of the fire. □

In terms of our categories, the grid in Lemma 2.2 is an example of a category B grid.

3. Regular infinite directed grids

We now prove Theorem 1.1 for regular infinite directed grids.

There are two cases that we must consider. First is the case in which the origin has two consecutive arcs (in cyclic order) facing out. The second case is where the two arcs facing out point in opposite directions. Without loss of generality, in the first case we can assume that the two arcs coming from the origin point along the positive x-axis and the positive y-axis and in the second case they point along the positive and negative y-axes.

The following theorem proves Theorem 1.1 for the first case.

Theorem 3.1. *Let G be a regular infinite directed grid where the two arcs coming from the origin point along the positive x-axis and the positive y-axis. If each arc in the first quadrant (including the axes) is facing either right or up, then G is a category B grid. If at least one arc in the first quadrant (including the axes) faces down or left, then G is a category A grid unless it is the exception shown in Figure 1, which is a category B grid.*

Proof. Suppose first that each arc in the first quadrant (including the axes) faces either right or up. Then the fire cannot leave this quadrant, and by Lemma 2.2, one firefighter per time step is not enough to contain the fire, but a second firefighter at any time step will allow us to contain the fire.

In most of the grids in the rest of this proof, we will show that we can contain the fire with one firefighter per time step. Our strategy will often be to "steer" the fire into a directed cycle. Then we can place defenders on outward neighbors not in the cycle until the fire returns to the first vertex in the cycle. In this way, we can always contain a fire once it reaches a directed cycle.

From now on, suppose that at least one arc in the first quadrant (including the axes) faces either left or down. Consider a closest arc in the first quadrant to the origin (where the distance is measured in the undirected grid from the origin to the head of the arc) that faces either left or down. Call such an arc e. The vertex, v, incident to the head of e must be on an axis, because, if it is not, then since v has in-degree two and out-degree two, at least one of the arcs coming from v must face down or left, and the vertex incident to the head of this arc must be closer to the origin than v was, contrary to our definitions of v and e.

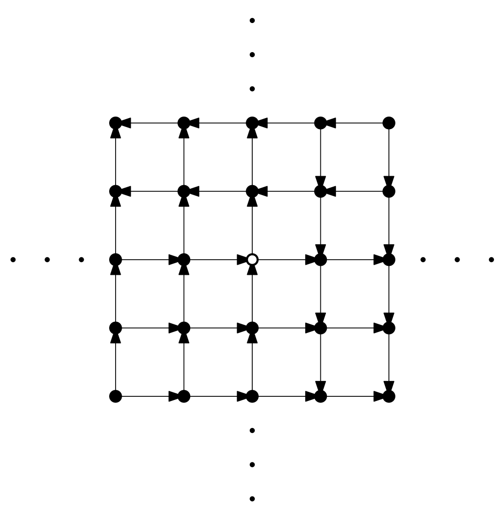


Figure 1. The exception to Theorem 3.1.

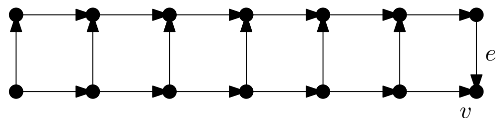


Figure 2. The grid when e is facing down.

Assume that the arc between (0, 1) and (1, 1) points right or the arc between (1, 0) and (1, 1) points up (or both). Thus at least one of these arcs could not be chosen as e. The special case when both of these arcs point toward the axes will be considered at the end of the proof.

Without loss of generality, assume v is on the positive x-axis. If there are multiple edges which could have v as their heads and could be considered to be e, we will break the tie by choosing the vertical arc that is pointing down. There are two cases that we must now consider. First, we will consider the case where e is facing down onto the axis. Since e is the closest arc facing down or to the left, all arcs closer to the origin than e must be facing either up or to the right. A picture of what this grid must look like is shown in Figure 2.

Let v_1 be the vertex directly above v, and label the vertices besides v and v_1 in the four faces containing v in the planar embedding of the grid in a clockwise cycle around v with v_2 through v_8 (so that v_8 is directly to the left of v_1). Let v_9 be the vertex directly above v_8, and let v_{10} be the vertex directly above v_1. See Figure 3.

We will defend along the line $y = 1$, forcing the fire to continue spreading to the right until the fire is at v_7, which is directly to the left of v. (It is possible for v_7 to be the origin.) Then we will defend on v, forcing the fire to spread up to v_8, and then defend on v_9 in order to force the fire to spread to v_1. Now we will defend on either v_{10} or v_2, whichever is incident with an arc coming from v_1. Then the fire could only spread to v, but since v is already defended, the fire is contained.

Next we will consider the case where e is horizontal on the x-axis, facing left toward v. See Figure 4. By our choice of e, the arc between v and v_1 must come from v (if not, we would have chosen this arc as e by our tie-breaking procedure). Also, the arc between v_1 and v_2 must come from v_1, because otherwise there would

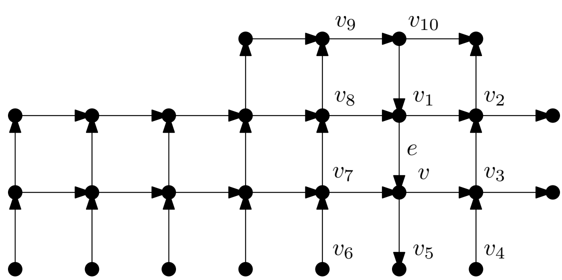


Figure 3. The vertices near v when e is facing down.

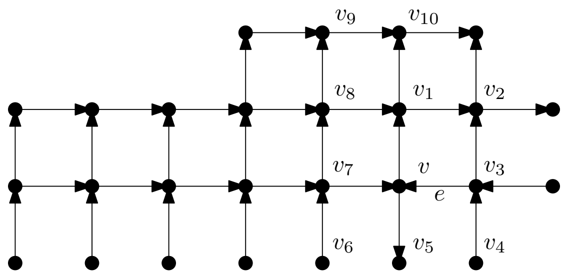


Figure 4. The vertices near v when e is facing left.

be either a downward arc with its head on the positive x-axis to the left of v, or there would be a path of leftward arcs from v_1 that would necessarily follow the line $y = 1$ to the positive y-axis at the point $(1, 0)$. In either case, this contradicts our choices of v and e. (We are still assuming that at least one of the two arcs between (1, 1) and the axes points away from the axis the arc touches.) There are now two subcases that must be considered. The subcases are the arc between v_2 and v_3 facing up or facing down.

If the arc faces down, then there is a directed cycle from v to v_1 to v_2 to v_3 and back to v. We will defend along the line $y = 1$ until the fire spreads to v. Next we defend the outward neighbors of v, v_1, v_2, and v_3 that are not in the cycle until the fire returns to v, at which point we have contained the fire.

If the arc between v_2 and v_3 faces up, then since every vertex has in-degree two and out-degree two, the arc between v_3 and v_4 must face up, the arc between v and v_5 must face down, and the arc between v_6 and v_7 must face up because of our choice of v. The arc between v_4 and v_5 can face either left or right. If it faces right, then there is a directed cycle from v to v_5 to v_4 to v_3 and back to v. If it faces left, then the arc between v_5 and v_6 must also face left because v_5 must have out-degree two. This gives a directed cycle from v_7 to v to v_5 to v_6 and back to v_7. In either case, we can defend along $y = 1$ until the fire reaches v or v_7, respectively, and then contain the fire once it enters the directed cycle.

We are now left with the case where the arc between (0, 1) and (1, 1) points left and the arc between $(1, 0)$ and $(1, 1)$ points down. We will show that any grid in this case can be defended with one firefighter per time step except for the exception in Figure 1. We now have three subcases to consider. The first subcase is when at least one arc on the positive x-axis is pointing left or at least one arc on the positive y-axis is pointing down. The second subcase is when all arcs on the positive x-axis face right, all arcs on the positive y-axis face up, and at least one arc in the first quadrant (not including the axes) faces up or right. The third subcase is when all arcs on the positive x-axis face right, all arcs on the positive y-axis face up, and no arcs in the first quadrant (not including the axes) face up or right.

For the first subcase, consider a nearest arc on a positive axis facing the origin. Without loss of generality, assume it is on the x-axis. Let the vertex at the head of this arc be called u. All arcs to the left of u on the positive x-axis must point right. The arc directly above u must point up. The arc between $(2, 1)$ and $(1, 1)$ must point left. If the arc between $(2, 0)$ and $(2, 1)$ points up, then we have a directed cycle and we can contain the fire. If not, then the arc between $(3, 1)$ and $(2, 1)$ must face left. Again, if the arc between $(3, 0)$ and $(3, 1)$ points up, we have a directed cycle. The only way we might not be able to contain the fire is if all arcs face down from the line $y = 1$ to the positive x-axis. However, as we said, the arc directly above u must face up. Therefore, we will have a directed cycle at or before u, and we can defend the fire so that it spreads along the positive x-axis until it reaches this directed cycle. Therefore, we can contain the fire with one defender per time step for any grid in this subcase.

For the second subcase, when all arcs on the positive x-axis face right, all arcs on the positive y-axis face up, and at least one arc in the first quadrant (not including the axes) faces up or right, choose a closest arc (in terms of the underlying undirected grid) to the origin in the first quadrant (not including the axes) facing up or right and call it e'. We claim that e' has its tail on an axis. Suppose not. Then the arcs directly below the tail and directly to the left of the tail must be facing down and left, respectively, because e' was the closest arc facing up or right. However, then the tail of e' has out-degree at least three, which is not possible. Therefore, e' must have its tail on an axis.

Without loss of generality, assume the tail of e' is on the x-axis and therefore e' is pointing up. All other vertical arcs directly to the left of e' between the positive x-axis and $y = 1$ and to the right of the positive y-axis must point down by our choice of e'. All arcs on $y = 1$ to the left of e' and to the right of the positive y-axis must point left, or there would be an up or right arc closer to the origin than e'. Thus there is a directed cycle along the positive x-axis, starting at $(1, 0)$, through e', then back along $y = 1$ to the downward arc from $(1, 1)$ to $(1, 0)$. By first defending $(0, 1)$, we force the fire into this directed cycle, and therefore can contain the fire.

For the third subcase, when all arcs on the positive x-axis face right, all arcs on the positive y-axis face up, and no arcs in the first quadrant (not including the axes) face up or right, Figure 5 shows all of the arcs that have predetermined directions.

If the arc between $(1, -1)$ and $(0, -1)$ points left, it completes a directed cycle including these vertices and $(0, 0)$ and $(1, 0)$. In this case, we could contain the fire with one firefighter per time step. If this arc points right, then it forces all of the arcs on $y = -1$ to the right of this arc to point right as well. It also forces the arc between $(0, -2)$ and $(0, -1)$ to point up, while all other vertical arcs directly to the right of this arc point down. This is shown in Figure 6.

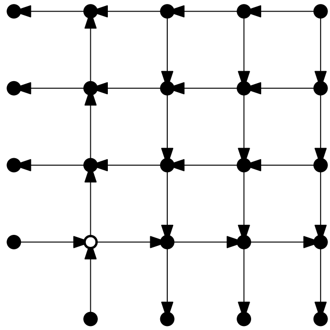


Figure 5. The directions of the arcs in the third subcase.

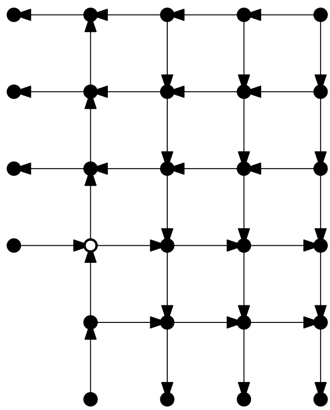


Figure 6. Another level of arcs in the third subcase.

We can continue this process, inductively going level by level down in the fourth quadrant. We will always be able to contain the fire, unless all vertical arcs on the negative y-axis point up, all vertical arcs to the right of the negative y-axis point down and all horizontal arcs below the positive x-axis point right.

By a similar argument to that of the fourth quadrant, in the second quadrant we can always contain the fire unless all horizontal arcs on the negative x-axis point right, all horizontal arcs above the negative x-axis point left, and all vertical arcs in the second quadrant point up.

Notice that, at this point, the arcs in the first, second, and fourth quadrants are the same as in the exception in Figure 1. Since every vertex has in-degree two and out-degree two, the arcs in the third quadrant are forced to be the same as the arcs in the third quadrant of the exception. We can see this by arguing inductively out from the second and fourth quadrants.

Finally, we prove that this exception is in category B. Assume we have one defender per time step. No matter where we put the first defender, the fire will

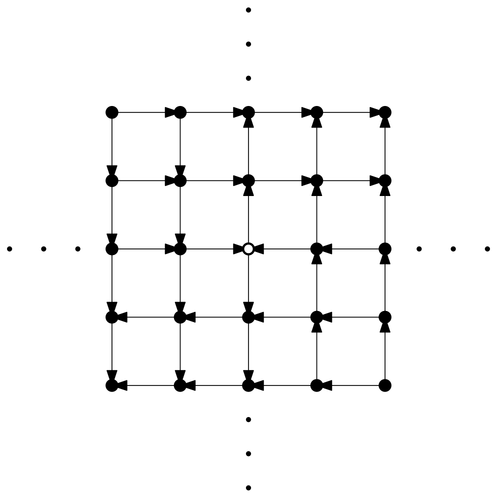


Figure 7. The exception to Theorem 3.2.

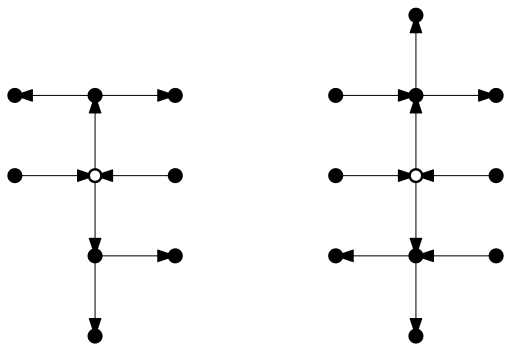
spread to at least one of $(1, 0)$ and $(0, 1)$. Without loss of generality, assume the fire moves to $(1, 0)$. By Lemma 2.2, considering $(1, 0)$ to be the origin of an infinite directed quarter-plane contained in the fourth quadrant, one firefighter per time step cannot contain the fire, but a second firefighter at some time step will allow us to contain the fire. If we get the second firefighter at time step $t = 1$, then we can immediately contain the fire. □

The second case is when the two arcs facing out point in opposite directions. Without loss of generality, assume they point along the positive and negative y-axis. The following theorem classifies which grids are in category A and which grids are in category B in this case.

Theorem 3.2. *Let G be a regular infinite directed grid where the vertical arc directly above the origin faces up and the vertical arc directly below the origin faces down. Then G is a category A grid unless the grid is the exception shown in Figure 7 or a reflection of this figure across the y-axis. These exceptions are in category B.*

Proof. Two cases must be considered. The first case is when both of the horizontal arcs incident on at least one of $(0, 1)$ and $(0, -1)$ point away from the vertex. The second case is when both $(0, 1)$ and $(0, -1)$ have one of their horizontal arcs facing them and one pointing away. These two cases are shown in Figure 8.

In the first case, assume without loss of generality that both horizontal arcs at $(0, 1)$ point away from $(0, 1)$. At least one of the horizontal arcs at $(0, -1)$ points away from $(0, -1)$, and we can assume, without loss of generality, that this arc is the arc directly to its right. If the arc between $(1, 0)$ and $(1, 1)$ faces down, there is



Case 1 Case 2

Figure 8. Two cases for Theorem 3.2.

a directed cycle from $(0, 0)$ to $(0, 1)$, to $(1, 1)$, to $(1, 0)$, and back to $(0, 0)$. If this arc faces up, though, it forces the arc between $(1, 0)$ and $(1, -1)$ to face up as well. We then have a directed cycle from $(0, 0)$ to $(0, -1)$, to $(1, -1)$, to $(1, 0)$, and back to $(0, 0)$. Therefore, for any arrangement of the remaining arcs in this first case, the grid can be defended by one firefighter per time step.

For the second case, if the horizontal arcs that point away from $(0, 1)$ and $(0, -1)$ point in the same direction, then by the same argument as that of the first case, the grid can be defended by one firefighter per time step. If the horizontal arcs pointing away from $(0, 1)$ and $(0, -1)$ point in opposite directions, without loss of generality, we can assume the arc between $(0, 1)$ and $(1, 1)$ points right and the arc between $(0, -1)$ and $(-1, -1)$ points left.

Suppose one or more arcs lying in the quarter-plane determined by $x \geq 0$ and $y \geq 1$ points left or down, or one or more arcs lying in the quarter-plane determined by $x \leq 0$ and $y \leq -1$ points right or up. Without loss of generality assume one or more arcs lying in the quarter-plane determined by $x \geq 0$ and $y \geq 1$ points left or down. We place our first defender at $(0, -1)$, forcing the fire to spread to $(0, 1)$. Unless all arcs lying in the quarter-plane determined by $x \geq 0$ and $y \geq 1$ look like the first quadrant in the exception of Theorem 3.1 (i.e., all arcs lying in the quarter-plane determined by $x > 0$ and $y > 1$ point left or down, all arcs directly to the right of $(0, 1)$ point right, and all arcs directly above $(0, 1)$ point up), we now know by Theorem 3.1 that we can contain the fire, treating the arcs lying in the quarter-plane determined by $x \geq 0$ and $y \geq 1$ as the first quadrant in Theorem 3.1.

If all arcs lying in the quarter-plane determined by $x > 0$ and $y > 1$ point left or down, all arcs directly to the right of $(0, 1)$ point right, and all arcs directly above $(0, 1)$ point up, then the arc between $(1, 0)$ and $(1, 1)$ must point down, completing a directed cycle from $(0, 0)$ to $(0, 1)$, to $(1, 1)$, to $(1, 0)$, and back to $(0, 0)$. This case can therefore be defended by one firefighter per time step.

Finally, if all arcs lying in the quarter-plane determined by $x \geq 0$ and $y \geq 1$ point up or right, and all arcs lying in the quarter-plane determined by $x \leq 0$ and $y \leq -1$ point down or left, then all of the directions of these arcs are the same as the directions of these arcs in the exception in Figure 7. Since every vertex has in-degree two and out-degree two and we know the direction of the arcs at the origin, we can see that the remaining vertical arcs in the first quadrant must point up and the remaining vertical arcs in the third quadrant must point down. Then, level by level, the remaining arcs in the second and fourth quadrants (including the axes) are forced to match the directions of the arcs in the exception.

We now prove that this exception is a category B grid. Assume we have one firefighter per time step. No matter where we put the first defender, the fire will spread to at least one of (0, 1) and (0, −1). Without loss of generality, assume the fire moves to (0, 1). If we treat (0, 1) as the origin, then by Lemma 2.2, one firefighter per time step is not enough to contain the fire. By this same lemma, a second firefighter at any time step allows us to contain the fire. If we get the second firefighter at time step $t = 1$, we can contain the fire immediately. Thus this grid is a category B grid. Notice that if we had assumed the arc between (0, 1) and (1, 1) points left and the arc between (0, −1) and (−1, −1) points right, then we would have the reflection of this exception over the y-axis. □

4. Other infinite directed grids

As a variation of the work done in the previous section, we will now consider an infinite directed grid where all vertices have in-degree two and out-degree two except for a single vertex. We will only investigate the cases when this vertex has in-degree three and out-degree one or in-degree four and out-degree zero. We will think of the construction of one of these grids as a process, starting with a grid where each vertex has in-degree two and out-degree two. We will then change the directions of one or more arcs at a single vertex so that it has the desired degrees and then change arcs at other vertices in such a way that all other vertices still have in-degree two and out-degree two. Note that we may not always make a minimum number of changes in order for this to be the case.

Any time a single arc between u and v is changed in a grid, if all vertices except v are required to maintain their original in-degree and out-degree, then a trail of vertices from v must be changed. If the arc had been facing from v to u, then, when it is changed to point toward v, one of the arcs that had previously been facing away from u must be changed to point toward it. Call the vertex that this arc had previously been facing w. Now, in order for w to continue to have the same in-degree and out-degree, another arc that had previously been facing away from w must now face towards it. This continues, forming a trail of changed arcs. Moreover, this trail is directed in such a way that, from any point on the trail, we could follow the arcs on the trail back to v. In the other case, when the arc between u and v was facing toward v, then the trail would face the other way, and following the arcs would take us away from v.

Let us now consider the case where v changes to have in-degree three and out-degree one. We will consider this case for most of the rest of this section. Since all vertices except v still have in-degree two and out-degree two, the grid is very similar to the type of grids investigated in Section 3. For this reason, we will refer often to the defense strategies provided for those grids.

Even though it is possible to form more than one trail of changed arcs as we change the degrees of v, we will now suppose that our grid where v has in-degree three and out-degree one contains only one trail of changed arcs. The situation where more than one trail is formed is considered later in this section — in particular, in Figures 10–13. We will show that the single changed trail can only either help move a grid from category B to category A or keep a grid in its original category. It can never bring a grid from category A to category B. We will first prove that a grid cannot go from category A to category B, which implies that all of the grids in category A must remain in category A. We will then determine which category B grids move to category A, and which category B grids stay in category B.

Theorem 4.1. *Suppose we have a category A infinite directed grid where each vertex has in-degree two and out-degree two. If one vertex*, v, *changes to have in-degree three and out-degree one in such a way that it creates only one trail of changed arcs*, *then this grid must remain in category A.*

Proof. As discussed above, if there is only one trail of changed arcs, then it must be an infinite directed trail where the arcs point toward v. Call this trail T.

We need to make an observation about how we defend the fire in category A grids where every vertex has in-degree two and out-degree two. In our proofs of Theorems 3.1 and 3.2, when we are able to contain the fire with one firefighter per time step, at each time step the fire could possibly move from a burning vertex to two neighbors since that vertex has out-degree two. We always place our firefighter at one of these two neighbors, and the fire moves to the other neighbor unless that other neighbor has already been burned or defended, in which case we finish containing the fire. Since this is true at every time step, there is at most one new burning vertex at each time step. Thus the burned vertices in all of our containment strategies follow a single directed path from the origin, which we will call P.

If T and P have no vertices in common, then we can use the same defense strategy as would have been used in Theorems 3.1 or 3.2, following P until it has been contained. Otherwise, consider the first vertex on P that is also on T. This situation is shown in Figure 9. This vertex could be v itself, or any other vertex

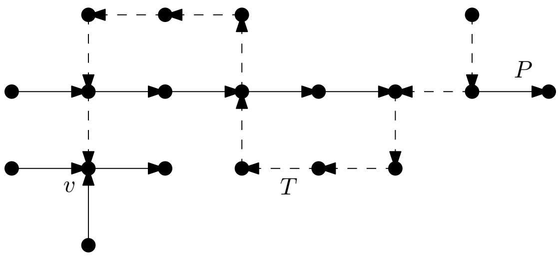


Figure 9. T intersects P (the dashed arcs are T).

on T. We begin by defending as we would have before, forcing the fire along P, until the fire reaches this first shared vertex. At this point we will force the fire to follow T until it reaches v. This is possible because every vertex of T other than v has in-degree two and out-degree two. Once the fire has spread to v, there is at most one vertex to which it can spread since v has out-degree one. We then defend this vertex, if necessary, and therefore contain the fire. In either case, we are still able to contain the fire with one firefighter per time step. The changed grid is therefore still in category A. □

We will now determine which category B grids are able to become category A grids after the change to v results in one changed trail, T. From Theorem 3.1, one type of category B grid is the grid where all arcs in a single quadrant (including its axes) face away from the origin (without loss of generality, assume this is the first quadrant); the directions of the arcs in the remaining quadrants are irrelevant. If v is in the first quadrant (including its axes), then the fire can be forced to v, at which point we are able to contain the fire. If v is not in the first quadrant, but T contains any arcs that are in the first quadrant, then the vertex w of T that is both in the first quadrant and is closest to v on T must be on an axis. We can force the fire along this axis until it reaches w and then force the fire to follow T to v, where we can contain the fire. If, however, v is not in the first quadrant and T does not affect any arcs in the first quadrant (as an example, consider when v is any vertex in the third quadrant and T consists of precisely the edges to the left of v), then one firefighter per time step will still not be enough to contain the fire. This is the only situation where a category B grid of this type remains in category B.

The other category B grid from Theorem 3.1 is the exception in that theorem (see Figure 1). We will show that, wherever v is on the grid, it will become a category A grid. If v is in the second or fourth quadrants (not including their axes), then we are able to force the fire to v, at which point we are able to contain the fire. If v is in the first quadrant (including the axes), then the construction will create a trail, T, of changed arcs that must intersect an axis at some vertex. We are able to force the fire along that axis to the first vertex on the axis that is also on T. Now we force the fire along T until we reach v, where the fire can be contained. If v is in the third quadrant (including the axes), since T is changing arcs in such a way that the trail points toward v, it will at some point either enter the second or fourth quadrant (not including their axes) or it will pass through the origin. If T reaches the second or fourth quadrant, we can force the fire to T and then follow T to v, where the fire is contained. If T passes through the origin, then from the very first time step we should force the fire to follow T until it reaches v.

The exception in Figure 7 from Theorem 3.2 is the only category B grid in that theorem (along with its reflection across the y-axis). This grid also becomes a category A grid, regardless of the position of v. For clarity in this proof, we will identify four regions in this grid. Region 1 is where $x \geq 0$ and $y \geq 1$; Region 2 is where $x \leq 0$ and $y \geq -1$; Region 3 is where $x \leq 0$ and $y \leq -1$; Region 4 is where $x \geq 0$ and $y \leq 1$. If v is in Regions 1 or 3 (including the boundaries), then we can force the fire to v and it can be contained. If v is in Regions 2 or 4 (not including the boundaries), then T must either reach Region 1 or 3 or it must pass through the origin. If T reaches Region 1 or 3, then it must reach a boundary in that region; we can force the fire along that boundary to T, and then force the fire to follow T to v. If T passes through the origin, then from the first time step we can force the fire to follow T to v. The only remaining case is when v is at the origin, in which case we are able to contain the fire on the first time step with one firefighter.

We can now see that the only type of category B grid that stays in category B is the grid where all arcs in a quadrant face away from the origin and v does not lie in that quadrant nor does T affect any arcs in that quadrant.

When changing v so that it has in-degree three, out-degree one, and only one trail, T, of changed arcs, we have seen that all category A grids remain in category A, some category B grids remain in category B, and some category B grids become category A grids. It might appear that changing v so that it has in-degree three and out-degree one could only help us contain the fire with one firefighter per time step since it has out-degree one, never permitting category A grids to become category B. However, if v creates more than one trail of changed arcs, it is possible for grids in either category to stay in that category or to switch to the other category. We now provide examples of each situation below. In each example, the white vertex is the origin, and the dashed arcs are the arcs that changed directions.

If we change vertex v so that it has in-degree four and out-degree zero, it creates an even number of two or more trails of changed arcs throughout the grid. If only two trails are created, then they both face toward v, so, similar to Theorem 4.1, they can never change a category A grid to category B. If, however, there are four or more trails created by changing v, it is possible for grids in either category to stay in that category or switch to the other category. Examples of this are similar to those in Figures 10–13.

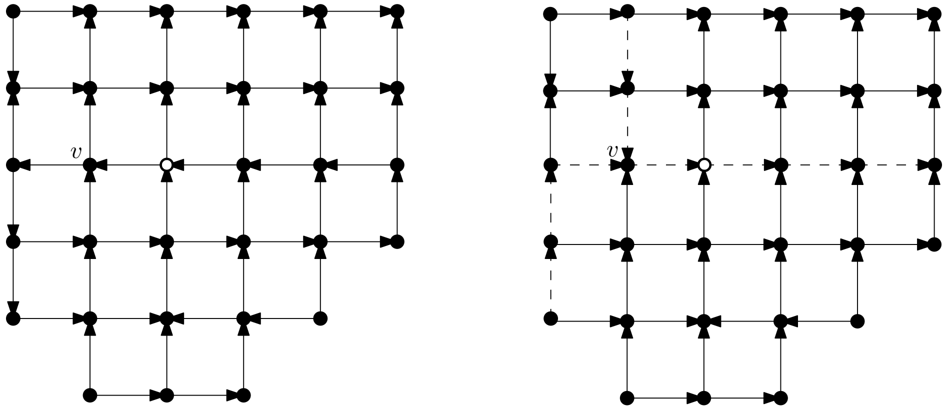


Figure 10. A category A grid that becomes a category B grid.

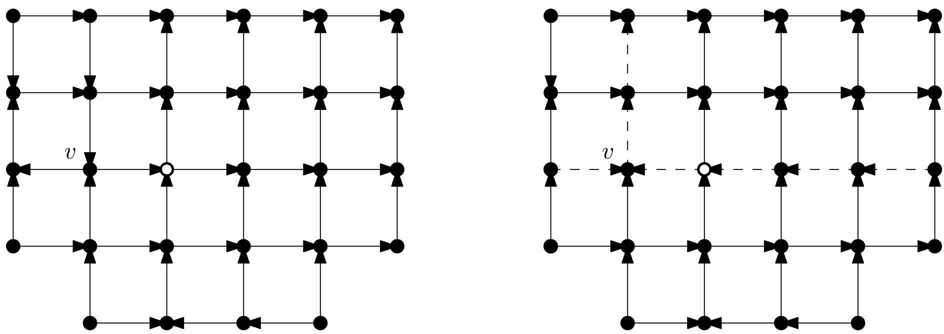


Figure 11. A category B grid that becomes a category A grid.

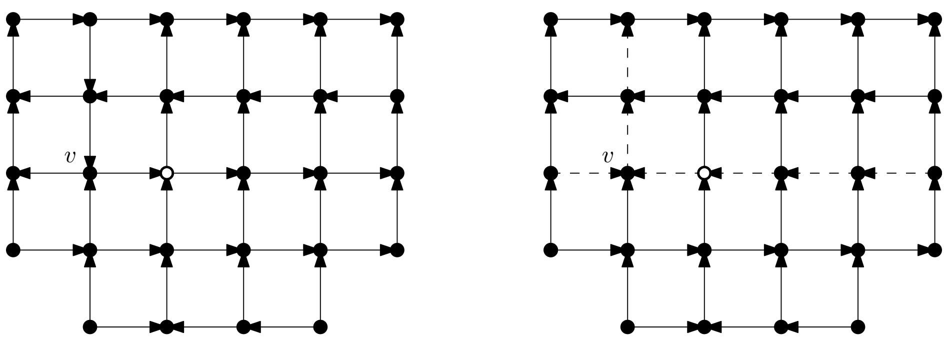


Figure 12. A category A grid that stays category A.

We close with a conjecture for general infinite directed grids. As mentioned in the introduction, we know that two firefighters per time step is sufficient to contain the fire if it begins at the origin. If the grid is regular, by Theorem 1.1, we know that either one firefighter per time step or one firefighter per time step with an additional

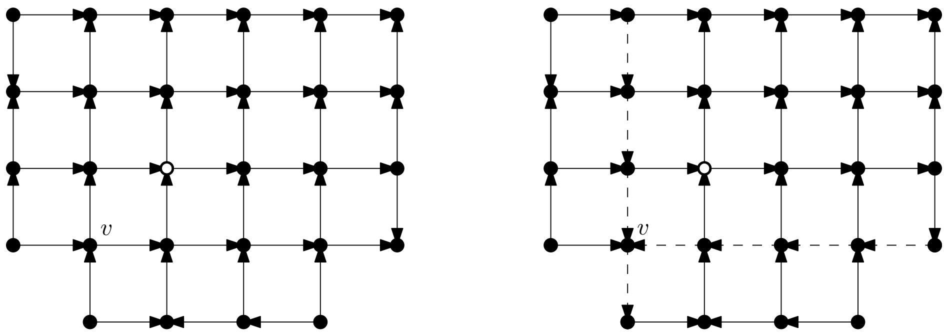


Figure 13. A category B grid that stays category B.

firefighter at some time step is sufficient to contain the fire. In general, we believe the following conjecture holds.

Conjecture 4.2. Let G be an infinite directed grid, and assume that the fire begins at the origin. If we are given one firefighter per time step and an occasional second firefighter is given on some finite number of time steps (the number may depend on the grid), the fire can be contained.

In some related work, Messinger [2008] and Ng and Raff [2008] provide containment strategies for undirected grids utilizing one firefighter on some time steps and two firefighters on other time steps. Their strategies, however, make assumptions as to which time steps a second firefighter will be available. Thus their strategies can be used for some instances of our conjecture, but they do not settle the general case.

The worst possible scenario for an infinite directed grid appears to be the grid where all horizontal arcs in the half-plane $x > 0$ point right, all horizontal arcs in the half-plane $x < 0$ point left, all vertical arcs in the half-plane $y > 0$ point up, and all vertical arcs in the half-plane $y < 0$ point down, seemingly allowing the fire to spread as much as possible. All four of the origin's incident arcs are directed away from the origin, and all other vertices on the x- and y-axes are in-degree one and out-degree three. The remaining vertices have in-degree two and out-degree two. Here is a defense strategy for this grid. In the first time step, we place a firefighter directly to the left of the origin, and we continue to place firefighters vertically above this vertex until a second firefighter is available, which allows us to push the fire to the right instead of simply maintaining it with this continuing vertical line of firefighters. Single firefighters are then again used to maintain the fire in a horizontal fashion until extra firefighters allow us to begin to push the line of defense downwards. In this general pattern, we can corral the fire quadrant by quadrant in a clockwise direction, maintaining the direction of the fire when given only one firefighter, and steering it in a clockwise direction when given an extra firefighter. Using this strategy, we will contain the fire after finitely many time steps.

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biebigha@cord.edu *Department of Mathematics, Concordia College, 901 8th Street South, Moorhead, MN 56562, United States*

leholte@ncsu.edu *Department of Mathematics, North Carolina State University, 2200 Hillsborough Street, Raleigh, NC 27695, United States*

rmwagner@cord.edu *Department of Mathematics, Concordia College, 901 8th Street South, Moorhead, MN 56562, United States*

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