

involve

a journal of mathematics

Properties of generalized derangement graphs

Hannah Jackson, Kathryn Nyman and Les Reid



Properties of generalized derangement graphs

Hannah Jackson, Kathryn Nyman and Les Reid

(Communicated by Ann Trenk)

A permutation on n elements is called a k -derangement ($k \leq n$) if no k -element subset is mapped to itself. One can form the k -derangement graph on the set of all permutations on n elements by connecting two permutations σ and τ if $\sigma\tau^{-1}$ is a k -derangement. We characterize when such a graph is connected or Eulerian. For n an odd prime power, we determine the independence, clique and chromatic numbers of the 2-derangement graph.

1. Introduction

Permutations which leave no element fixed, known as derangements, were first considered in [de Montmort 1708] and have been extensively studied since. A derangement graph is a graph whose vertices are the elements of the symmetric group S_n and whose edges connect two permutations that differ by a derangement. Derangement graphs have been shown to be connected (for $n > 3$) and Hamiltonian, and their independence number, clique number, and chromatic number have been calculated [Renteln 2007].

Here we consider the generalization of derangements known as k -derangements, which are those permutations in S_n that do not fix any k -element subset of the set being permuted. A k -derangement graph is defined in an analogous manner to a derangement graph. We examine some of the graph-theoretical properties of k -derangement graphs.

2. Preliminaries

Let S_n be the group of permutations on the set $\{1, 2, \dots, n\}$. A permutation $\sigma \in S_n$ maps any k -element subset of $\{1, \dots, n\}$ to a k -element subset of $\{1, \dots, n\}$; in the usual notation,

$$\sigma(\{a_1, \dots, a_k\}) = \{\sigma(a_1), \dots, \sigma(a_k)\}.$$

If $\{a_1, \dots, a_k\} = \{\sigma(a_1), \dots, \sigma(a_k)\}$ (as sets, that is, without regard to order), we

MSC2010: primary 05C69, 05A05; secondary 05C45.

Keywords: derangements, Eulerian, chromatic number, maximal clique, Cayley graph, independent set.

say that σ fixes the unordered k -tuple $\{a_1, \dots, a_k\}$. (“Unordered k -tuple” is another name for a k -element set.)

If σ does not map *any* of the $\binom{n}{k}$ possible unordered k -tuples to itself, we say that σ is a k -derangement. For example, with $n = 4$, the cyclic permutation $\sigma = (1234)$ is a 2-derangement, because (taking $k = 2$) we have

$$\begin{aligned} (1234)(\{1, 2\}) &= \{(1234)(1), (1234)(2)\} = \{2, 3\}, \\ (1234)(\{1, 3\}) &= \{(1234)(1), (1234)(3)\} = \{2, 4\}, \\ (1234)(\{1, 4\}) &= \{(1234)(1), (1234)(4)\} = \{2, 1\} = \{1, 2\}, \\ (1234)(\{2, 3\}) &= \{(1234)(2), (1234)(3)\} = \{3, 4\}, \\ (1234)(\{2, 4\}) &= \{(1234)(2), (1234)(4)\} = \{3, 1\} = \{1, 3\}, \\ (1234)(\{3, 4\}) &= \{(1234)(3), (1234)(4)\} = \{4, 1\} = \{1, 4\}. \end{aligned}$$

This extends the ordinary notion of a derangement, defined as a permutation $\sigma \in S_n$ such that $\sigma(x) \neq x$ for all $x \in \{1, \dots, n\}$.

The set of k -derangements in S_n is denoted by $\mathcal{D}_{k,n}$, and its cardinality $|\mathcal{D}_{k,n}|$ — the number of k -derangements in S_n — is denoted by $D_k(n)$. As we have seen, (1234) is in $\mathcal{D}_{2,4}$. Specifically,

$$\begin{aligned} \mathcal{D}_{2,4} = \{ &(1234), (1243), (1324), (1342), (1423), (1432), (123)(4), (124)(3), \\ &(132)(4), (134)(2), (142)(3), (143)(2), (234)(1), (243)(1)\}, \end{aligned}$$

and thus $D_2(4) = 14$. The sequence $D_2(n)$ appears as A137482 in the *On-Line Encyclopedia of Integer Sequences*; see [Henshaw 2008]. The number $D_1(n)$ is also simply called the derangement number.

The cycle structure of a permutation σ , denoted by C_σ , is the multiset of the lengths of the cycles in its cycle decomposition (e.g., $C_{(12)(3)(45)} = \{2, 2, 1\}$). Note that the cycle structure of $\sigma \in S_n$ is a partition of n . Given a partition r of n , let P_r be the set of all permutations in S_n whose cycle structure is r . For example (as usual, excluding singletons in our notation) $P_{\{2,1,1\}} = \{(12), (13), (14), (23), (24), (34)\}$.

We first note that if the cycle structure of a permutation σ contains a multiset which partitions k , then σ is not a k -derangement. For example, $(12)(34)$ is a 3-derangement in S_4 , but $(12)(3)(4)$ is not, because it fixes the set $\{1, 2, 3\}$, for example. And we see that $\{2, 1\} \subseteq C_{(12)(3)(4)} = \{2, 1, 1\}$ is a partition of 3. Thus we observe that the cycle structure of a permutation determines whether or not it is a k -derangement, and we have the following.

Proposition 1. *A permutation $\sigma \in S_n$ is a k -derangement if and only if the cycle decomposition of σ does not contain a set of cycles whose lengths partition k .*

Proof. If $\{q, r, \dots, s\}$ is a partition of k , and $(a_1 \cdots a_q)(b_1 \cdots b_r) \cdots (c_1 \cdots c_s)$ are cycles of σ , then, for $x = \{a_1, \dots, a_q, b_1, \dots, b_r, c_1, \dots, c_s\}$, $\sigma(x) = x$. Conversely,

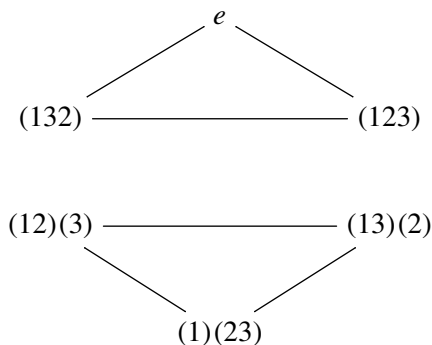


Figure 1. The 2-derangement graph on 6 vertices, $\Gamma_{2,3}$.

if σ has no set of cycles whose lengths partition k , then, given any k -element subset x of $\{1, \dots, n\}$, there is a cycle in σ which contains at least one element in x and contains some element not in x . Hence σ sends an element in x to an element not in x and so $\sigma(x) \neq x$. \square

Let $CD_{k,n}$ be the set of cycle structures corresponding to k -derangements in S_n ; for example, $CD_{2,4} = \{\{4\}, \{3, 1\}\}$. Since a cycle structure C_σ is in $CD_{k,n}$ if and only if it is in $CD_{n-k,n}$, we have $\mathcal{D}_{k,n} = \mathcal{D}_{n-k,n}$.

Let G be a group, and let S be a subset of G that is closed under taking inverses. The *Cayley graph* $\Gamma(G, S)$ is the graph whose vertices are the elements of G such that an edge connects two vertices $u, v \in G$ if $su = v$ for some $s \in S$. A *k -derangement graph* is a Cayley graph defined by $\Gamma_{k,n} := \Gamma(S_n, \mathcal{D}_{k,n})$. (Note that $\mathcal{D}_{k,n}$ is symmetric, as the inverse of a k -derangement is a k -derangement, and thus satisfies the requirements for a Cayley graph.) It is worth noting that $\Gamma_{k,n}$ is, by construction, $D_k(n)$ -regular, and that, since $\mathcal{D}_{k,n} = \mathcal{D}_{(n-k),n}$, $\Gamma_{k,n} = \Gamma_{(n-k),n}$. Figure 1 illustrates the 2-derangement graph on 6 vertices, $\Gamma_{2,3}$.

It is possible to consider k -derangements in S_n for any positive k and n . However, if $k = n$, there will be no k -derangements in S_n , since every partition in S_n will have a cycle structure such that the cycle lengths partition k . As such, $\Gamma_{k,n}$ will be the empty (edgeless) graph on n vertices. If $k > n$, then every permutation in S_n is a k -derangement vacuously, and thus $\Gamma_{k,n}$ will be the complete graph on $|S_n|$ vertices. As neither of these cases is particularly interesting, henceforth we will only consider k -derangements where $k < n$.

3. Properties of derangement graphs

Figure 1 shows that $\Gamma_{2,3}$ is not a connected graph, and, since $\Gamma_{2,3} = \Gamma_{1,3}$, we see that $\Gamma_{k,3}$ is disconnected for all $k < n$. But this is an exception rather than the rule, as the following theorem demonstrates.

Theorem 2. *For $n > 3$ and $k < n$, $\Gamma_{k,n}$ is connected.*

Proof. Every permutation in S_n can be written as the product of adjacent transpositions $(h \ h+1)$. These, in turn, can be expressed as products of two k -derangements, so long as $n > 3$, as we will demonstrate. As a result, for $n > 3$, the elements of $\mathcal{D}_{k,n}$ generate S_n , which means that every vertex of $\Gamma_{k,n}$ can be reached by a path from the identity.

We show that the permutation $(1 \ 2)$ can be written as the product of two k -derangements and then note that, since it is the form and not the individual labels that are important, any adjacent transposition can be written as the product of two k -derangements. We consider two cases: $k = 1$ and $k \geq 2$.

Case 1: If $k = 1$, then $(1 \ 2) = (1 \ 2 \ \dots \ n)^2 \cdot (n \ (n-1) \ \dots \ 1)^2(1 \ 2)$. We claim that $(1 \ 2 \ \dots \ n)^2$ and $(n \ (n-1) \ \dots \ 1)^2(1 \ 2)$ are each 1-derangements in S_n for all $n > 3$. If n is even, then $(1 \ 2 \ \dots \ n)^2 = (1 \ 3 \ \dots \ (n-3) \ (n-1))(2 \ 4 \ \dots \ (n-2) \ n)$, which is a 1-derangement in S_n for all n . Additionally,

$$(n \ (n-1) \ \dots \ 1)^2(1 \ 2) = (1 \ n \ (n-2) \ (n-4) \ \dots \ 2 \ (n-1) \ (n-3) \ \dots \ 3),$$

which is also a 1-derangement in S_n for any n .

On the other hand, if n is odd, then

$$(1 \ 2 \ \dots \ n)^2 = (1 \ 3 \ \dots \ (n-2) \ n \ 2 \ 4 \ \dots \ (n-3) \ (n-1)),$$

which is a 1-derangement in S_n for all n . And

$$\begin{aligned} (n \ (n-1) \ \dots \ 1)^2(1 \ 2) &= (n \ (n-2) \ (n-4) \ \dots \ 3 \ 1 \ (n-1) \ (n-3) \ \dots \ 4 \ 2)(1 \ 2) \\ &= (1 \ n \ (n-2) \ (n-4) \ \dots \ 3)(2 \ (n-1) \ (n-3) \ \dots \ 4), \end{aligned}$$

which is a 1-derangement in S_n so long as $n > 3$. (If $n = 3$, $(312)(12) = (13)(2)$, which is not a 1-derangement.)

Thus, for $n > 3$, we have shown that $(1 \ 2)$ can be written as the product of two 1-derangements, and, by extension, every adjacent transposition can be written as the product of two 1-derangements.

Case 2: For $k \geq 2$, $(1 \ 2) = (1 \ 2 \ \dots \ n)^{-1}(1 \ 3 \ 4 \ \dots \ n)$. We know $(1 \ 2 \ \dots \ n)^{-1}$ is a k -derangement for all k since the inverse of a k -derangement is a k -derangement. And, by the cycle structure, we see that $(1 \ 3 \ 4 \ \dots \ n) = (1 \ 3 \ 4 \ \dots \ n)(2)$ is a k -derangement for all k , except $k = 1$ and $k = (n-1)$ (however, since $\Gamma_{1,n} = \Gamma_{(n-1),n}$, Case 1 addresses $(n-1)$ -derangements as well as 1-derangements).

So we have shown that, for $k \geq 2$, $(1 \ 2)$ can be written as the product of two k -derangements, and again, by extension, we can write any adjacent transposition as the product of two k -derangements. Thus every vertex is connected by a path to the identity, and $\Gamma_{k,n}$ is connected. \square

It is worth noting that Theorem 2 holds for $n = 2$ as well. Since we are only interested in k -derangements in S_n such that $k < n$, when $n = 2$, k must equal 1, and so $\Gamma_{1,2}$ is the connected graph on two vertices.

Next, we give a characterization in terms of n and k for when a derangement graph is Eulerian. We will require the following result.

Lemma 3. *If a cycle structure includes a cycle of length greater than 2, then there are an even number of permutations with that cycle structure.*

Proof. Consider P_r , the set of permutations with a given cycle structure, r . We can pair each $\sigma \in P_r$ with its inverse $\sigma^{-1} \in P_r$, and, so long as $\sigma \neq \sigma^{-1}$ for any $\sigma \in P_r$, $|P_r|$ will be even. Suppose there exists a $\sigma \in P_r$ such that $\sigma = \sigma^{-1}$. Then $\sigma^2 = e$, and so the order of σ is at most 2. The order of a permutation is the least common multiple of the orders of the elements of its cycle structure, so σ must not include a cycle of length greater than 2. This is a contradiction; thus $|P_r|$ is even. \square

Theorem 4. *For $n > 3$ and $k < n$, $\Gamma_{k,n}$ is Eulerian if and only if k is even or k and n are both odd.*

Proof. A graph is Eulerian if and only if it is connected and each vertex has an even degree. In light of Theorem 2 and the previously noted fact that $\Gamma_{k,n}$ is $D_k(n)$ -regular, in order to ascertain if $\Gamma_{k,n}$ is Eulerian, we must determine whether $D_k(n)$ is even or odd.

If k is even, we claim that $D_k(n)$ is the sum of even numbers. Any cycle structure composed entirely of 2- or 1-cycles will partition an even k , and thus any permutation which is in $\mathcal{D}_{k,n}$ for an even k will contain a cycle of length 3 or greater in its cycle decomposition. Now, $\mathcal{D}_{k,n} = P_{r_1} \dot{\cup} P_{r_2} \dot{\cup} \dots \dot{\cup} P_{r_m}$ (disjoint union) such that no r_i partitions k , and, by Lemma 3, $|P_{r_i}|$ is even for all $i \in \{1, \dots, m\}$. Thus, when k is even, $D_k(n)$ is even.

If k and n are both odd, again we see that every permutation in $\mathcal{D}_{k,n}$ will contain a cycle of length 3 or greater in its cycle decomposition, since an odd k can be partitioned by a set of cycles of lengths 1 or 2 if there is at least one 1-cycle. Furthermore, since n is odd, there are no permutations whose cycle structure is composed only of length-2 cycles. Thus, $D_k(n)$ is even.

Finally, we show that, if k is odd and n is even, then $\Gamma_{k,n}$ is not Eulerian. In this case, $P_{\{2,2,\dots,2\}}$ is in $CD_{k,n}$. By choosing pairs of elements for the cycles and dividing by the number of ways to order the cycles, we see that the number of permutations in $P_{\{2,2,\dots,2\}}$ is given by

$$\begin{aligned} \frac{\binom{n}{2} \binom{n-2}{2} \cdots \binom{2}{2}}{\left(\frac{n}{2}\right)!} &= \frac{n(n-1)(n-2) \cdots (3)(2)(1)}{\left(2 \cdot \frac{n}{2}\right) \left(2 \cdot \left(\frac{n}{2} - 1\right)\right) \cdots (6)(4)(2)} \\ &= \frac{n(n-1)(n-2) \cdots (3)(2)(1)}{n(n-2) \cdots (6)(4)(2)} = (n-1)(n-3) \cdots (5)(3)(1). \end{aligned}$$

Since n is even, the product $(n-1)(n-3)\cdots(5)(3)(1)$ is odd. Every other k -derangement in S_n will contain a cycle with length greater than 2, since any combination of 1-cycles or 1- and 2-cycles will partition k . So $D_k(n)$ is the sum of one odd number and even numbers, and so is odd. \square

4. Chromatic, independence and clique numbers for $k = 2$ and n an odd prime power

For the majority of this section, we will think of permutations in terms of the result of their application to the ordering $\{1, 2, 3, \dots, n\}$. Thus, $\{2, 3, 1, 4, 5\}$ represents the permutation which has moved 2 to the first position, 3 to the second, 1 to the third, and left 4 and 5 fixed; that is, the permutation $(132)(4)(5)$ in cycle notation, or the inverse of the permutation $\begin{pmatrix} 12345 \\ 23145 \end{pmatrix}$ in two line notation.

We note that in order for vu^{-1} (or, equivalently, $v^{-1}u$) to be a k -derangement, it is necessary and sufficient that no unordered k -tuple of elements be sent to the same unordered k -tuple of positions by both u and v . For example, the permutations $u = \{2, 3, 1, 4, 5\}$ and $v = \{4, 1, 3, 5, 2\}$ both send the pair $\{1, 3\}$ to the second and third positions. Thus $(vu^{-1})(\{2, 3\}) = \{2, 3\}$, and so vu^{-1} is not a 2-derangement and there is no edge between u and v in the 2-derangement graph. More formally, suppose u and v both send the k -tuple $M' = \{a'_1, a'_2, \dots, a'_k\}$ to positions $M = \{a_1, a_2, \dots, a_k\}$. Then, $(vu^{-1})(M) = v(M') = M$. Thus, vu^{-1} is not a k -derangement.

On the other hand, if u and v send no k -tuple to the same positions we claim vu^{-1} is a k -derangement. Consider an arbitrary k -tuple, $M = \{a_1, a_2, \dots, a_k\}$, and suppose u maps the k -tuple $M' = \{a'_1, a'_2, \dots, a'_k\}$ to the positions given in M . Then $(vu^{-1})(M) = v(M') \neq M$ since v cannot send the k -tuple M' to the same positions as u does. Thus, vu^{-1} is a k -derangement.

In Theorem 6, we find the clique number of the 2-derangement graph, $\omega(\Gamma_{2,n})$, for n an odd prime power, by constructing a clique of maximal size. Before establishing this clique number, we note an upper bound on the clique number of a general k -derangement graph.

Lemma 5. For $k < n$, $\omega(\Gamma_{k,n}) \leq \binom{n}{k}$.

Proof. The clique number of the k -derangement graph, $\omega(\Gamma_{k,n})$, cannot be greater than $\binom{n}{k}$, since there are only $\binom{n}{k}$ subsets of size k and hence at most $\binom{n}{k}$ different unordered k -tuples of positions for an arbitrary k -tuple of elements to be sent under a permutation. \square

Theorem 6. If n is an odd prime power, then $\omega(\Gamma_{2,n}) = \binom{n}{2}$.

Proof. We will explicitly construct a clique with $\binom{n}{2}$ elements. Let $n = p^r$, with p a prime greater than 2, and let \mathbb{F}_{p^r} denote the field with p^r elements. Rather than

letting S_n act on $\{1, \dots, n\}$, we will let it act on \mathbb{F}_{p^r} and construct $\Gamma_{2,n}$ accordingly. Let $v = (x_1, \dots, x_n)$ be an ordered n -tuple whose entries are the elements of \mathbb{F}_{p^r} in some order. Given any function $\phi : \mathbb{F}_{p^r} \rightarrow \mathbb{F}_{p^r}$, we define $\phi(v) = (\phi(x_1), \dots, \phi(x_n))$. Partition the nonzero elements of \mathbb{F}_{p^r} by pairing each element with its (additive) inverse, and let T be a set obtained by choosing exactly one element from each pair, giving $|T| = (p^r - 1)/2$.

Define $f_{s,\alpha}(x) = sx + \alpha$, and consider the set $X = \{f_{s,\alpha}(v) \mid s \in T \text{ and } \alpha \in \mathbb{F}_{p^r}\}$. Since $s \neq 0$, $f_{s,\alpha}$ is a bijection and $f_{s,\alpha}(v)$ is a permutation of the elements of \mathbb{F}_{p^r} . We claim that X is a clique in $\Gamma_{2,n}$. Suppose not; that is, suppose there are $s, t \in T$ and $\alpha, \beta \in \mathbb{F}_{p^r}$, $(s, \alpha) \neq (t, \beta)$, such that $f_{s,\alpha}(v)$ is not a 2-derangement of $f_{s,\beta}(v)$. In that case there exist $x, y \in \mathbb{F}_{p^r}$, $x \neq y$, such that either $f_{s,\alpha}(x) = f_{t,\beta}(x)$ and $f_{s,\alpha}(y) = f_{t,\beta}(y)$ or $f_{s,\alpha}(x) = f_{t,\beta}(y)$ and $f_{s,\alpha}(y) = f_{t,\beta}(x)$. In the first case, subtracting the two equations and rewriting yields $(s - t)(x - y) = 0$. If $s = t$, then $\alpha = \beta$, giving a contradiction. If $s \neq t$, then $x = y$ and again we have a contradiction. In the second case, subtracting and rewriting yields $(s + t)(x - y) = 0$ and, since $s + t \neq 0$ for $s, t \in T$, $x = y$ and this also give a contradiction. Thus, X is a clique of size $p^r(p^r - 1)/2 = \binom{n}{2}$. \square

The next example illustrates the construction when $n = 7$.

Example 7. We build a clique of size $\binom{7}{2}$ in the derangement graph $\Gamma_{2,7}$ consisting of $\frac{7-1}{2}$ blocks, each of which contains 7 permutations. We let $v = (1, 2, 3, 4, 5, 6, 7)$ (writing 7 instead of 0) and take $T = \{1, 4, 5\}$. Then

$$\begin{aligned} f_{1,0}(v) &= (1, 2, 3, 4, 5, 6, 7), & f_{4,0}(v) &= (4, 1, 5, 2, 6, 3, 7), \\ f_{5,0}(v) &= (5, 3, 1, 6, 4, 2, 7). \end{aligned}$$

Increasing α from 0 cyclically permutes the 7-tuples. Block 1 consists of the arrangements $\{f_{1,\alpha}(v) \mid \alpha \in \mathbb{F}_7\}$, that is, the arrangement $(1, 2, 3, 4, 5, 6, 7)$ and the remaining 6 rotations of this arrangement (e.g., $(2, 3, 4, 5, 6, 7, 1)$, $(3, 4, 5, 6, 7, 1, 2)$, etc.). Block 2 consists of the arrangement $f_{4,0}(v)$ along with all of its rotations. Finally, block 3 consists of $f_{5,0}(v)$ and its rotations. To see that these permutations form a clique, consider, for example, the pair $\{1, 2\}$. These elements are one position apart in block 1, two positions apart in block 2 and three positions apart in block 3 (counting the shortest distance between them either forwards or backwards). So the pair $\{1, 2\}$ cannot occupy the same positions in two permutations which appear in different blocks. Furthermore, within a block, the rotations insure that the pair never occupies the same positions.

Remark 8. Cliques achieving the upper bound of Lemma 5 are known as *sharply k -homogeneous sets* of permutations. A corollary in [Nomura 1985] shows that, for $2k \leq n$, the existence of such a k -homogeneous set implies $n + 1 \equiv 0 \pmod k$. Thus Theorem 6 cannot be extended to even n , and we have the following.

Corollary 9. For n even and $n \geq 4$, $\omega(\Gamma_{2,n}) < \binom{n}{2}$.

A computer search confirms that $\omega(\Gamma_{2,4}) = 5 < \binom{4}{2}$.

Next we turn to the independence number $\alpha(\Gamma_{k,n})$ and the chromatic number $\chi(\Gamma_{k,n})$ of the k -derangement graph. We will require the following lemma which has been adapted from Frankl and Deza's lemma [1977] and applied to k -tuples of elements.

Lemma 10. For $k < n$, $\alpha(\Gamma_{k,n})\omega(\Gamma_{k,n}) \leq n!$.

Proof. Let \mathcal{P} be a set of permutations in S_n , every pair of which has at least one unordered k -tuple of elements in the same unordered k -tuple of positions. That is, for any $u, v \in \mathcal{P}$, there exists a set $M = \{a_1, \dots, a_k\} \subseteq \{1, \dots, n\}$ such that $(v^{-1}u)(M) = M$. Note that \mathcal{P} is an independent set in the k -derangement graph. Let \mathcal{Q} be a set of permutations in S_n such that each pair of permutations has no k -tuple of elements in the same positions; that is, \mathcal{Q} is a clique in the k -derangement graph. We claim that products of the form PQ with $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ give distinct permutations of n . Suppose, for the sake of contradiction, that $P_1Q_1 = P_2Q_2$ for $P_1, P_2 \in \mathcal{P}$ and $Q_1, Q_2 \in \mathcal{Q}$ with $P_1 \neq P_2$ and $Q_1 \neq Q_2$. This implies that $P_1^{-1}P_2 = Q_1Q_2^{-1}$. Now, since P_1 and P_2 are in \mathcal{P} , there is a k -tuple of elements $M = \{a_1, \dots, a_k\}$ such that $(P_1^{-1}P_2)(M) = M$. However, this implies $(Q_1Q_2^{-1})(M) = M$. But we know that the permutations in \mathcal{Q} agree on no k -tuples, and so we must have $Q_1 = Q_2$ and, hence, $P_1 = P_2$. Finally, since each product gives a unique permutation of n , there can be no more than $n!$ such products. \square

Theorem 11. For $k < n$, $\alpha(\Gamma_{k,n}) \geq k!(n-k)!$ and $\chi(\Gamma_{k,n}) \leq \binom{n}{k}$.

Proof. Consider H , the set of all permutations in S_n that send $\{1, 2, \dots, k\}$ to itself (and hence $\{k+1, \dots, n\}$ to itself). It is clear that H is a subgroup of S_n isomorphic to $S_k \times S_{n-k}$ and that $|H| = k!(n-k)!$. Since the unordered k -tuple $\{1, 2, \dots, k\}$ is fixed, none of these are k -derangements of each other, so H is an independent set and $\alpha(\Gamma_{k,n}) \geq k!(n-k)!$.

The cosets of H partition S_n , and each forms an independent set, since $\tau_1, \tau_2 \in \sigma H$ implies that $\tau_1^{-1}\tau_2 \in H$ is not a k -derangement and hence the vertices associated to τ_1 and τ_2 are not connected by an edge. Giving each of the $\frac{n!}{k!(n-k)!} = \binom{n}{k}$ cosets a different color results in a valid coloring of $\Gamma_{k,n}$, so $\chi(\Gamma_{k,n}) \leq \binom{n}{k}$. \square

Corollary 12. For n an odd prime power, $\alpha(\Gamma_{2,n}) = 2(n-2)!$ and $\chi(\Gamma_{2,n}) = \binom{n}{2}$.

Proof. By Lemma 10 and Theorem 6, we have $\binom{n}{2} \cdot \alpha(\Gamma_{2,n}) \leq n!$. Thus

$$\alpha(\Gamma_{2,n}) \leq n! \cdot \frac{2(n-2)!}{n!} = 2(n-2)!$$

and Theorem 11 gives the reverse inequality. For any graph G , $\chi(G) \geq \omega(G)$, so, by Theorem 6, $\chi(\Gamma_{2,n}) \geq \binom{n}{2}$ and again Theorem 11 gives the reverse inequality. \square

5. Further questions

In the last section, we showed that the clique number of the 2-derangement graph is equal to $\binom{n}{2}$ when n is an odd prime power and strictly less than that if n is even (and at least 4). The clique construction of Theorem 6 fails to work when n is odd and not a prime power since there is no field of that cardinality. We believe that in this case the clique number is strictly smaller than $\binom{n}{2}$. For arbitrary k , we have some faint hope that the bounds given in Theorem 11 for $\alpha(\Gamma_{k,n})$ and $\chi(\Gamma_{k,n})$ are actually equalities, but the situation for $\omega(\Gamma_{k,n})$ remains unclear.

In another direction, the numerical evidence is overwhelming that the derangement graphs are Hamiltonian. We hope to explore these and other questions in future work.

Acknowledgements

We would like to acknowledge Sam Tencer's contribution to this investigation. We are also grateful to an anonymous referee for spotting the connection between cliques attaining the bound of Lemma 5 and sharply k -homogeneous sets of permutations.

The first author worked on this topic with the third author during an REU at Missouri State University in the summer of 2009 (award #0552573).

References

- [Frankl and Deza 1977] P. Frankl and M. Deza, "On the maximum number of permutations with given maximal or minimal distance", *J. Combinatorial Theory Ser. A* **22**:3 (1977), 352–360. MR 55 #12534 Zbl 0352.05003
- [Henshaw 2008] J. Henshaw, "A137482: Number of permutations of n objects such that no two-element subset is preserved", entry A137482 in *The On-Line Encyclopedia of Integer Sequences* (<http://oeis.org>), 2008.
- [de Montmort 1708] P. R. de Montmort, *Essay d'analyse sur les jeux de hazard*, 1st ed., Jacques Quillau, Paris, 1708.
- [Nomura 1985] K. Nomura, "On t -homogeneous permutation sets", *Arch. Math. (Basel)* **44**:6 (1985), 485–487. MR 87e:20003 Zbl 0547.20003
- [Renteln 2007] P. Renteln, "On the spectrum of the derangement graph", *Electron. J. Combin.* **14**:1 (2007), Research Paper 82, 17. MR 2008j:05217 Zbl 1183.05047

Received: 2011-09-14 Revised: 2012-05-22 Accepted: 2012-07-13

hljackso@syr.edu

Mathematics Department, Syracuse University, 215 Carnegie, Syracuse, NY 13244, United States

knyman@willamette.edu

Mathematics Department, Willamette University, 900 State Street, Salem, OR 97301, United States

lesreid@missouristate.edu

Mathematics Department, Missouri State University, 901 South National Avenue, Springfield, MO 65897, United States

involve

msp.org/involve

EDITORS

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	Victoria University, Australia pietro.cerone@vu.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moselehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tobrie1@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisyjs@potsgdam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA jgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sngupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

PRODUCTION


Silvio Levy, Scientific Editor

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2013 is US \$105/year for the electronic version, and \$145/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2013 Mathematical Sciences Publishers

involve

2013

vol. 6

no. 1

Refined inertias of tree sign patterns of orders 2 and 3 D. D. OLESKY, MICHAEL F. REMPEL AND P. VAN DEN DRIESSCHE	1
The group of primitive almost pythagorean triples NIKOLAI A. KRYLOV AND LINDSAY M. KULZER	13
Properties of generalized derangement graphs HANNAH JACKSON, KATHRYN NYMAN AND LES REID	25
Rook polynomials in three and higher dimensions FERYAL ALAYONT AND NICHOLAS KRZYWONOS	35
New confidence intervals for the AR(1) parameter FEREBEE TUNNO AND ASHTON ERWIN	53
Knots in the canonical book representation of complete graphs DANA ROWLAND AND ANDREA POLITANO	65
On closed modular colorings of rooted trees BRYAN PHINEZY AND PING ZHANG	83
Iterations of quadratic polynomials over finite fields WILLIAM WORDEN	99
Positive solutions to singular third-order boundary value problems on purely discrete time scales COURTNEY DEHOET, CURTIS KUNKEL AND ASHLEY MARTIN	113



1944-4176(2013)6:1;1-A