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A permutation on  $n$  elements is called a  $k$ -derangement ( $k \leq n$ ) if no  $k$ -element subset is mapped to itself. One can form the  $k$ -derangement graph on the set of all permutations on  $n$  elements by connecting two permutations  $\sigma$  and  $\tau$  if  $\sigma\tau^{-1}$  is a  $k$ -derangement. We characterize when such a graph is connected or Eulerian. For  $n$  an odd prime power, we determine the independence, clique and chromatic numbers of the 2-derangement graph.

## 1. Introduction

Permutations which leave no element fixed, known as derangements, were first considered in [de Montmort 1708] and have been extensively studied since. A derangement graph is a graph whose vertices are the elements of the symmetric group  $S_n$  and whose edges connect two permutations that differ by a derangement. Derangement graphs have been shown to be connected (for  $n > 3$ ) and Hamiltonian, and their independence number, clique number, and chromatic number have been calculated [Renteln 2007].

Here we consider the generalization of derangements known as  $k$ -derangements, which are those permutations in  $S_n$  that do not fix any  $k$ -element subset of the set being permuted. A  $k$ -derangement graph is defined in an analogous manner to a derangement graph. We examine some of the graph-theoretical properties of  $k$ -derangement graphs.

## 2. Preliminaries

Let  $S_n$  be the group of permutations on the set  $\{1, 2, \dots, n\}$ . A permutation  $\sigma \in S_n$  maps any  $k$ -element subset of  $\{1, \dots, n\}$  to a  $k$ -element subset of  $\{1, \dots, n\}$ ; in the usual notation,

$$\sigma(\{a_1, \dots, a_k\}) = \{\sigma(a_1), \dots, \sigma(a_k)\}.$$

If  $\{a_1, \dots, a_k\} = \{\sigma(a_1), \dots, \sigma(a_k)\}$  (as sets, that is, without regard to order), we

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say that  $\sigma$  fixes the unordered  $k$ -tuple  $\{a_1, \dots, a_k\}$ . (“Unordered  $k$ -tuple” is another name for a  $k$ -element set.)

If  $\sigma$  does not map *any* of the  $\binom{n}{k}$  possible unordered  $k$ -tuples to itself, we say that  $\sigma$  is a  $k$ -derangement. For example, with  $n = 4$ , the cyclic permutation  $\sigma = (1234)$  is a 2-derangement, because (taking  $k = 2$ ) we have

$$\begin{aligned} (1234)(\{1, 2\}) &= \{(1234)(1), (1234)(2)\} = \{2, 3\}, \\ (1234)(\{1, 3\}) &= \{(1234)(1), (1234)(3)\} = \{2, 4\}, \\ (1234)(\{1, 4\}) &= \{(1234)(1), (1234)(4)\} = \{2, 1\} = \{1, 2\}, \\ (1234)(\{2, 3\}) &= \{(1234)(2), (1234)(3)\} = \{3, 4\}, \\ (1234)(\{2, 4\}) &= \{(1234)(2), (1234)(4)\} = \{3, 1\} = \{1, 3\}, \\ (1234)(\{3, 4\}) &= \{(1234)(3), (1234)(4)\} = \{4, 1\} = \{1, 4\}. \end{aligned}$$

This extends the ordinary notion of a derangement, defined as a permutation  $\sigma \in S_n$  such that  $\sigma(x) \neq x$  for all  $x \in \{1, \dots, n\}$ .

The set of  $k$ -derangements in  $S_n$  is denoted by  $\mathcal{D}_{k,n}$ , and its cardinality  $|\mathcal{D}_{k,n}|$  — the number of  $k$ -derangements in  $S_n$  — is denoted by  $D_k(n)$ . As we have seen,  $(1234)$  is in  $\mathcal{D}_{2,4}$ . Specifically,

$$\begin{aligned} \mathcal{D}_{2,4} = \{ &(1234), (1243), (1324), (1342), (1423), (1432), (123)(4), (124)(3), \\ &(132)(4), (134)(2), (142)(3), (143)(2), (234)(1), (243)(1)\}, \end{aligned}$$

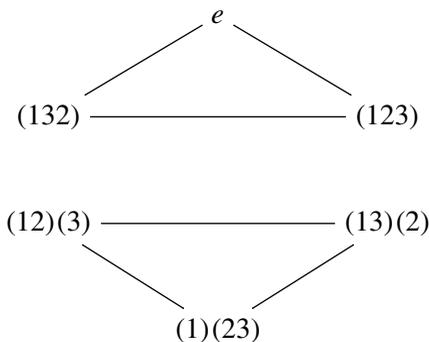
and thus  $D_2(4) = 14$ . The sequence  $D_2(n)$  appears as A137482 in the *On-Line Encyclopedia of Integer Sequences*; see [Henshaw 2008]. The number  $D_1(n)$  is also simply called the derangement number.

The cycle structure of a permutation  $\sigma$ , denoted by  $C_\sigma$ , is the multiset of the lengths of the cycles in its cycle decomposition (e.g.,  $C_{(12)(3)(45)} = \{2, 2, 1\}$ ). Note that the cycle structure of  $\sigma \in S_n$  is a partition of  $n$ . Given a partition  $r$  of  $n$ , let  $P_r$  be the set of all permutations in  $S_n$  whose cycle structure is  $r$ . For example (as usual, excluding singletons in our notation)  $P_{\{2,1,1\}} = \{(12), (13), (14), (23), (24), (34)\}$ .

We first note that if the cycle structure of a permutation  $\sigma$  contains a multiset which partitions  $k$ , then  $\sigma$  is not a  $k$ -derangement. For example,  $(12)(34)$  is a 3-derangement in  $S_4$ , but  $(12)(3)(4)$  is not, because it fixes the set  $\{1, 2, 3\}$ , for example. And we see that  $\{2, 1\} \subseteq C_{(12)(3)(4)} = \{2, 1, 1\}$  is a partition of 3. Thus we observe that the cycle structure of a permutation determines whether or not it is a  $k$ -derangement, and we have the following.

**Proposition 1.** *A permutation  $\sigma \in S_n$  is a  $k$ -derangement if and only if the cycle decomposition of  $\sigma$  does not contain a set of cycles whose lengths partition  $k$ .*

*Proof.* If  $\{q, r, \dots, s\}$  is a partition of  $k$ , and  $(a_1 \cdots a_q)(b_1 \cdots b_r) \cdots (c_1 \cdots c_s)$  are cycles of  $\sigma$ , then, for  $x = \{a_1, \dots, a_q, b_1, \dots, b_r, c_1, \dots, c_s\}$ ,  $\sigma(x) = x$ . Conversely,



**Figure 1.** The 2-derangement graph on 6 vertices,  $\Gamma_{2,3}$ .

if  $\sigma$  has no set of cycles whose lengths partition  $k$ , then, given any  $k$ -element subset  $x$  of  $\{1, \dots, n\}$ , there is a cycle in  $\sigma$  which contains at least one element in  $x$  and contains some element not in  $x$ . Hence  $\sigma$  sends an element in  $x$  to an element not in  $x$  and so  $\sigma(x) \neq x$ .  $\square$

Let  $CD_{k,n}$  be the set of cycle structures corresponding to  $k$ -derangements in  $S_n$ ; for example,  $CD_{2,4} = \{\{4\}, \{3, 1\}\}$ . Since a cycle structure  $C_\sigma$  is in  $CD_{k,n}$  if and only if it is in  $CD_{n-k,n}$ , we have  $\mathcal{D}_{k,n} = \mathcal{D}_{n-k,n}$ .

Let  $G$  be a group, and let  $S$  be a subset of  $G$  that is closed under taking inverses. The *Cayley graph*  $\Gamma(G, S)$  is the graph whose vertices are the elements of  $G$  such that an edge connects two vertices  $u, v \in G$  if  $su = v$  for some  $s \in S$ . A  *$k$ -derangement graph* is a Cayley graph defined by  $\Gamma_{k,n} := \Gamma(S_n, \mathcal{D}_{k,n})$ . (Note that  $\mathcal{D}_{k,n}$  is symmetric, as the inverse of a  $k$ -derangement is a  $k$ -derangement, and thus satisfies the requirements for a Cayley graph.) It is worth noting that  $\Gamma_{k,n}$  is, by construction,  $D_k(n)$ -regular, and that, since  $\mathcal{D}_{k,n} = \mathcal{D}_{(n-k),n}$ ,  $\Gamma_{k,n} = \Gamma_{(n-k),n}$ . [Figure 1](#) illustrates the 2-derangement graph on 6 vertices,  $\Gamma_{2,3}$ .

It is possible to consider  $k$ -derangements in  $S_n$  for any positive  $k$  and  $n$ . However, if  $k = n$ , there will be no  $k$ -derangements in  $S_n$ , since every partition in  $S_n$  will have a cycle structure such that the cycle lengths partition  $k$ . As such,  $\Gamma_{k,n}$  will be the empty (edgeless) graph on  $n$  vertices. If  $k > n$ , then every permutation in  $S_n$  is a  $k$ -derangement vacuously, and thus  $\Gamma_{k,n}$  will be the complete graph on  $|S_n|$  vertices. As neither of these cases is particularly interesting, henceforth we will only consider  $k$ -derangements where  $k < n$ .

### 3. Properties of derangement graphs

[Figure 1](#) shows that  $\Gamma_{2,3}$  is not a connected graph, and, since  $\Gamma_{2,3} = \Gamma_{1,3}$ , we see that  $\Gamma_{k,3}$  is disconnected for all  $k < n$ . But this is an exception rather than the rule, as the following theorem demonstrates.

**Theorem 2.** For  $n > 3$  and  $k < n$ ,  $\Gamma_{k,n}$  is connected.

*Proof.* Every permutation in  $S_n$  can be written as the product of adjacent transpositions ( $h(h+1)$ ). These, in turn, can be expressed as products of two  $k$ -derangements, so long as  $n > 3$ , as we will demonstrate. As a result, for  $n > 3$ , the elements of  $\mathcal{D}_{k,n}$  generate  $S_n$ , which means that every vertex of  $\Gamma_{k,n}$  can be reached by a path from the identity.

We show that the permutation  $(1\ 2)$  can be written as the product of two  $k$ -derangements and then note that, since it is the form and not the individual labels that are important, any adjacent transposition can be written as the product of two  $k$ -derangements. We consider two cases:  $k = 1$  and  $k \geq 2$ .

Case 1: If  $k = 1$ , then  $(1\ 2) = (1\ 2 \cdots n)^2 \cdot (n(n-1) \cdots 1)^2(1\ 2)$ . We claim that  $(1\ 2 \cdots n)^2$  and  $(n(n-1) \cdots 1)^2(1\ 2)$  are each 1-derangements in  $S_n$  for all  $n > 3$ . If  $n$  is even, then  $(1\ 2 \cdots n)^2 = (1\ 3 \cdots (n-3)(n-1))(2\ 4 \cdots (n-2)n)$ , which is a 1-derangement in  $S_n$  for all  $n$ . Additionally,

$$(n(n-1) \cdots 1)^2(1\ 2) = (1\ n(n-2)(n-4) \cdots 2(n-1)(n-3) \cdots 3),$$

which is also a 1-derangement in  $S_n$  for any  $n$ .

On the other hand, if  $n$  is odd, then

$$(1\ 2 \cdots n)^2 = (1\ 3 \cdots (n-2)n\ 2\ 4 \cdots (n-3)(n-1)),$$

which is a 1-derangement in  $S_n$  for all  $n$ . And

$$\begin{aligned} (n(n-1) \cdots 1)^2(1\ 2) &= (n(n-2)(n-4) \cdots 3\ 1(n-1)(n-3) \cdots 4\ 2)(1\ 2) \\ &= (1\ n(n-2)(n-4) \cdots 3)(2(n-1)(n-3) \cdots 4), \end{aligned}$$

which is a 1-derangement in  $S_n$  so long as  $n > 3$ . (If  $n = 3$ ,  $(312)(12) = (13)(2)$ , which is not a 1-derangement.)

Thus, for  $n > 3$ , we have shown that  $(1\ 2)$  can be written as the product of two 1-derangements, and, by extension, every adjacent transposition can be written as the product of two 1-derangements.

Case 2: For  $k \geq 2$ ,  $(1\ 2) = (1\ 2 \cdots n)^{-1}(1\ 3\ 4 \cdots n)$ . We know  $(1\ 2 \cdots n)^{-1}$  is a  $k$ -derangement for all  $k$  since the inverse of a  $k$ -derangement is a  $k$ -derangement. And, by the cycle structure, we see that  $(1\ 3\ 4 \cdots n) = (1\ 3\ 4 \cdots n)(2)$  is a  $k$ -derangement for all  $k$ , except  $k = 1$  and  $k = (n-1)$  (however, since  $\Gamma_{1,n} = \Gamma_{(n-1),n}$ , Case 1 addresses  $(n-1)$ -derangements as well as 1-derangements).

So we have shown that, for  $k \geq 2$ ,  $(1\ 2)$  can be written as the product of two  $k$ -derangements, and again, by extension, we can write any adjacent transposition as the product of two  $k$ -derangements. Thus every vertex is connected by a path to the identity, and  $\Gamma_{k,n}$  is connected.  $\square$

It is worth noting that [Theorem 2](#) holds for  $n = 2$  as well. Since we are only interested in  $k$ -derangements in  $S_n$  such that  $k < n$ , when  $n = 2$ ,  $k$  must equal 1, and so  $\Gamma_{1,2}$  is the connected graph on two vertices.

Next, we give a characterization in terms of  $n$  and  $k$  for when a derangement graph is Eulerian. We will require the following result.

**Lemma 3.** *If a cycle structure includes a cycle of length greater than 2, then there are an even number of permutations with that cycle structure.*

*Proof.* Consider  $P_r$ , the set of permutations with a given cycle structure,  $r$ . We can pair each  $\sigma \in P_r$  with its inverse  $\sigma^{-1} \in P_r$ , and, so long as  $\sigma \neq \sigma^{-1}$  for any  $\sigma \in P_r$ ,  $|P_r|$  will be even. Suppose there exists a  $\sigma \in P_r$  such that  $\sigma = \sigma^{-1}$ . Then  $\sigma^2 = e$ , and so the order of  $\sigma$  is at most 2. The order of a permutation is the least common multiple of the orders of the elements of its cycle structure, so  $\sigma$  must not include a cycle of length greater than 2. This is a contradiction; thus  $|P_r|$  is even.  $\square$

**Theorem 4.** *For  $n > 3$  and  $k < n$ ,  $\Gamma_{k,n}$  is Eulerian if and only if  $k$  is even or  $k$  and  $n$  are both odd.*

*Proof.* A graph is Eulerian if and only if it is connected and each vertex has an even degree. In light of [Theorem 2](#) and the previously noted fact that  $\Gamma_{k,n}$  is  $D_k(n)$ -regular, in order to ascertain if  $\Gamma_{k,n}$  is Eulerian, we must determine whether  $D_k(n)$  is even or odd.

If  $k$  is even, we claim that  $D_k(n)$  is the sum of even numbers. Any cycle structure composed entirely of 2- or 1-cycles will partition an even  $k$ , and thus any permutation which is in  $\mathcal{D}_{k,n}$  for an even  $k$  will contain a cycle of length 3 or greater in its cycle decomposition. Now,  $\mathcal{D}_{k,n} = P_{r_1} \dot{\cup} P_{r_2} \dot{\cup} \dots \dot{\cup} P_{r_m}$  (disjoint union) such that no  $r_i$  partitions  $k$ , and, by [Lemma 3](#),  $|P_{r_i}|$  is even for all  $i \in \{1, \dots, m\}$ . Thus, when  $k$  is even,  $D_k(n)$  is even.

If  $k$  and  $n$  are both odd, again we see that every permutation in  $\mathcal{D}_{k,n}$  will contain a cycle of length 3 or greater in its cycle decomposition, since an odd  $k$  can be partitioned by a set of cycles of lengths 1 or 2 if there is at least one 1-cycle. Furthermore, since  $n$  is odd, there are no permutations whose cycle structure is composed only of length-2 cycles. Thus,  $D_k(n)$  is even.

Finally, we show that, if  $k$  is odd and  $n$  is even, then  $\Gamma_{k,n}$  is not Eulerian. In this case,  $P_{\{2,2,\dots,2\}}$  is in  $CD_{k,n}$ . By choosing pairs of elements for the cycles and dividing by the number of ways to order the cycles, we see that the number of permutations in  $P_{\{2,2,\dots,2\}}$  is given by

$$\begin{aligned} \frac{\binom{n}{2} \binom{n-2}{2} \cdots \binom{2}{2}}{\left(\frac{n}{2}\right)!} &= \frac{n(n-1)(n-2) \cdots (3)(2)(1)}{\left(2 \cdot \frac{n}{2}\right) \left(2 \cdot \left(\frac{n}{2} - 1\right)\right) \cdots (6)(4)(2)} \\ &= \frac{n(n-1)(n-2) \cdots (3)(2)(1)}{n(n-2) \cdots (6)(4)(2)} = (n-1)(n-3) \cdots (5)(3)(1). \end{aligned}$$

Since  $n$  is even, the product  $(n-1)(n-3)\cdots(5)(3)(1)$  is odd. Every other  $k$ -derangement in  $S_n$  will contain a cycle with length greater than 2, since any combination of 1-cycles or 1- and 2-cycles will partition  $k$ . So  $D_k(n)$  is the sum of one odd number and even numbers, and so is odd.  $\square$

#### 4. Chromatic, independence and clique numbers for $k = 2$ and $n$ an odd prime power

For the majority of this section, we will think of permutations in terms of the result of their application to the ordering  $\{1, 2, 3, \dots, n\}$ . Thus,  $\{2, 3, 1, 4, 5\}$  represents the permutation which has moved 2 to the first position, 3 to the second, 1 to the third, and left 4 and 5 fixed; that is, the permutation  $(132)(4)(5)$  in cycle notation, or the inverse of the permutation  $\begin{pmatrix} 12345 \\ 23145 \end{pmatrix}$  in two line notation.

We note that in order for  $vu^{-1}$  (or, equivalently,  $v^{-1}u$ ) to be a  $k$ -derangement, it is necessary and sufficient that no unordered  $k$ -tuple of elements be sent to the same unordered  $k$ -tuple of positions by both  $u$  and  $v$ . For example, the permutations  $u = \{2, 3, 1, 4, 5\}$  and  $v = \{4, 1, 3, 5, 2\}$  both send the pair  $\{1, 3\}$  to the second and third positions. Thus  $(vu^{-1})(\{2, 3\}) = \{2, 3\}$ , and so  $vu^{-1}$  is not a 2-derangement and there is no edge between  $u$  and  $v$  in the 2-derangement graph. More formally, suppose  $u$  and  $v$  both send the  $k$ -tuple  $M' = \{a'_1, a'_2, \dots, a'_k\}$  to positions  $M = \{a_1, a_2, \dots, a_k\}$ . Then,  $(vu^{-1})(M) = v(M') = M$ . Thus,  $vu^{-1}$  is not a  $k$ -derangement.

On the other hand, if  $u$  and  $v$  send no  $k$ -tuple to the same positions we claim  $vu^{-1}$  is a  $k$ -derangement. Consider an arbitrary  $k$ -tuple,  $M = \{a_1, a_2, \dots, a_k\}$ , and suppose  $u$  maps the  $k$ -tuple  $M' = \{a'_1, a'_2, \dots, a'_k\}$  to the positions given in  $M$ . Then  $(vu^{-1})(M) = v(M') \neq M$  since  $v$  cannot send the  $k$ -tuple  $M'$  to the same positions as  $u$  does. Thus,  $vu^{-1}$  is a  $k$ -derangement.

In [Theorem 6](#), we find the clique number of the 2-derangement graph,  $\omega(\Gamma_{2,n})$ , for  $n$  an odd prime power, by constructing a clique of maximal size. Before establishing this clique number, we note an upper bound on the clique number of a general  $k$ -derangement graph.

**Lemma 5.** For  $k < n$ ,  $\omega(\Gamma_{k,n}) \leq \binom{n}{k}$ .

*Proof.* The clique number of the  $k$ -derangement graph,  $\omega(\Gamma_{k,n})$ , cannot be greater than  $\binom{n}{k}$ , since there are only  $\binom{n}{k}$  subsets of size  $k$  and hence at most  $\binom{n}{k}$  different unordered  $k$ -tuples of positions for an arbitrary  $k$ -tuple of elements to be sent under a permutation.  $\square$

**Theorem 6.** If  $n$  is an odd prime power, then  $\omega(\Gamma_{2,n}) = \binom{n}{2}$ .

*Proof.* We will explicitly construct a clique with  $\binom{n}{2}$  elements. Let  $n = p^r$ , with  $p$  a prime greater than 2, and let  $\mathbb{F}_{p^r}$  denote the field with  $p^r$  elements. Rather than

letting  $S_n$  act on  $\{1, \dots, n\}$ , we will let it act on  $\mathbb{F}_{p^r}$  and construct  $\Gamma_{2,n}$  accordingly. Let  $v = (x_1, \dots, x_n)$  be an ordered  $n$ -tuple whose entries are the elements of  $\mathbb{F}_{p^r}$  in some order. Given any function  $\phi : \mathbb{F}_{p^r} \rightarrow \mathbb{F}_{p^r}$ , we define  $\phi(v) = (\phi(x_1), \dots, \phi(x_n))$ . Partition the nonzero elements of  $\mathbb{F}_{p^r}$  by pairing each element with its (additive) inverse, and let  $T$  be a set obtained by choosing exactly one element from each pair, giving  $|T| = (p^r - 1)/2$ .

Define  $f_{s,\alpha}(x) = sx + \alpha$ , and consider the set  $X = \{f_{s,\alpha}(v) \mid s \in T \text{ and } \alpha \in \mathbb{F}_{p^r}\}$ . Since  $s \neq 0$ ,  $f_{s,\alpha}$  is a bijection and  $f_{s,\alpha}(v)$  is a permutation of the elements of  $\mathbb{F}_{p^r}$ . We claim that  $X$  is a clique in  $\Gamma_{2,n}$ . Suppose not; that is, suppose there are  $s, t \in T$  and  $\alpha, \beta \in \mathbb{F}_{p^r}$ ,  $(s, \alpha) \neq (t, \beta)$ , such that  $f_{s,\alpha}(v)$  is not a 2-derangement of  $f_{t,\beta}(v)$ . In that case there exist  $x, y \in \mathbb{F}_{p^r}$ ,  $x \neq y$ , such that either  $f_{s,\alpha}(x) = f_{t,\beta}(x)$  and  $f_{s,\alpha}(y) = f_{t,\beta}(y)$  or  $f_{s,\alpha}(x) = f_{t,\beta}(y)$  and  $f_{s,\alpha}(y) = f_{t,\beta}(x)$ . In the first case, subtracting the two equations and rewriting yields  $(s - t)(x - y) = 0$ . If  $s = t$ , then  $\alpha = \beta$ , giving a contradiction. If  $s \neq t$ , then  $x = y$  and again we have a contradiction. In the second case, subtracting and rewriting yields  $(s + t)(x - y) = 0$  and, since  $s + t \neq 0$  for  $s, t \in T$ ,  $x = y$  and this also give a contradiction. Thus,  $X$  is a clique of size  $p^r(p^r - 1)/2 = \binom{p^r}{2}$ .  $\square$

The next example illustrates the construction when  $n = 7$ .

**Example 7.** We build a clique of size  $\binom{7}{2}$  in the derangement graph  $\Gamma_{2,7}$  consisting of  $\frac{7-1}{2}$  blocks, each of which contains 7 permutations. We let  $v = (1, 2, 3, 4, 5, 6, 7)$  (writing 7 instead of 0) and take  $T = \{1, 4, 5\}$ . Then

$$\begin{aligned} f_{1,0}(v) &= (1, 2, 3, 4, 5, 6, 7), & f_{4,0}(v) &= (4, 1, 5, 2, 6, 3, 7), \\ f_{5,0}(v) &= (5, 3, 1, 6, 4, 2, 7). \end{aligned}$$

Increasing  $\alpha$  from 0 cyclically permutes the 7-tuples. Block 1 consists of the arrangements  $\{f_{1,\alpha}(v) \mid \alpha \in \mathbb{F}_7\}$ , that is, the arrangement  $(1, 2, 3, 4, 5, 6, 7)$  and the remaining 6 rotations of this arrangement (e.g.,  $(2, 3, 4, 5, 6, 7, 1)$ ,  $(3, 4, 5, 6, 7, 1, 2)$ , etc.). Block 2 consists of the arrangement  $f_{4,0}(v)$  along with all of its rotations. Finally, block 3 consists of  $f_{5,0}(v)$  and its rotations. To see that these permutations form a clique, consider, for example, the pair  $\{1, 2\}$ . These elements are one position apart in block 1, two positions apart in block 2 and three positions apart in block 3 (counting the shortest distance between them either forwards or backwards). So the pair  $\{1, 2\}$  cannot occupy the same positions in two permutations which appear in different blocks. Furthermore, within a block, the rotations insure that the pair never occupies the same positions.

**Remark 8.** Cliques achieving the upper bound of Lemma 5 are known as *sharply  $k$ -homogeneous sets* of permutations. A corollary in [Nomura 1985] shows that, for  $2k \leq n$ , the existence of such a  $k$ -homogeneous set implies  $n + 1 \equiv 0 \pmod k$ . Thus Theorem 6 cannot be extended to even  $n$ , and we have the following.

**Corollary 9.** For  $n$  even and  $n \geq 4$ ,  $\omega(\Gamma_{2,n}) < \binom{n}{2}$ .

A computer search confirms that  $\omega(\Gamma_{2,4}) = 5 < \binom{4}{2}$ .

Next we turn to the independence number  $\alpha(\Gamma_{k,n})$  and the chromatic number  $\chi(\Gamma_{k,n})$  of the  $k$ -derangement graph. We will require the following lemma which has been adapted from Frankl and Deza's lemma [1977] and applied to  $k$ -tuples of elements.

**Lemma 10.** For  $k < n$ ,  $\alpha(\Gamma_{k,n})\omega(\Gamma_{k,n}) \leq n!$ .

*Proof.* Let  $\mathcal{P}$  be a set of permutations in  $S_n$ , every pair of which has at least one unordered  $k$ -tuple of elements in the same unordered  $k$ -tuple of positions. That is, for any  $u, v \in \mathcal{P}$ , there exists a set  $M = \{a_1, \dots, a_k\} \subseteq \{1, \dots, n\}$  such that  $(v^{-1}u)(M) = M$ . Note that  $\mathcal{P}$  is an independent set in the  $k$ -derangement graph. Let  $\mathcal{Q}$  be a set of permutations in  $S_n$  such that each pair of permutations has no  $k$ -tuple of elements in the same positions; that is,  $\mathcal{Q}$  is a clique in the  $k$ -derangement graph. We claim that products of the form  $PQ$  with  $P \in \mathcal{P}$  and  $Q \in \mathcal{Q}$  give distinct permutations of  $n$ . Suppose, for the sake of contradiction, that  $P_1Q_1 = P_2Q_2$  for  $P_1, P_2 \in \mathcal{P}$  and  $Q_1, Q_2 \in \mathcal{Q}$  with  $P_1 \neq P_2$  and  $Q_1 \neq Q_2$ . This implies that  $P_1^{-1}P_2 = Q_1Q_2^{-1}$ . Now, since  $P_1$  and  $P_2$  are in  $\mathcal{P}$ , there is a  $k$ -tuple of elements  $M = \{a_1, \dots, a_k\}$  such that  $(P_1^{-1}P_2)(M) = M$ . However, this implies  $(Q_1Q_2^{-1})(M) = M$ . But we know that the permutations in  $\mathcal{Q}$  agree on no  $k$ -tuples, and so we must have  $Q_1 = Q_2$  and, hence,  $P_1 = P_2$ . Finally, since each product gives a unique permutation of  $n$ , there can be no more than  $n!$  such products.  $\square$

**Theorem 11.** For  $k < n$ ,  $\alpha(\Gamma_{k,n}) \geq k!(n-k)!$  and  $\chi(\Gamma_{k,n}) \leq \binom{n}{k}$ .

*Proof.* Consider  $H$ , the set of all permutations in  $S_n$  that send  $\{1, 2, \dots, k\}$  to itself (and hence  $\{k+1, \dots, n\}$  to itself). It is clear that  $H$  is a subgroup of  $S_n$  isomorphic to  $S_k \times S_{n-k}$  and that  $|H| = k!(n-k)!$ . Since the unordered  $k$ -tuple  $\{1, 2, \dots, k\}$  is fixed, none of these are  $k$ -derangements of each other, so  $H$  is an independent set and  $\alpha(\Gamma_{k,n}) \geq k!(n-k)!$ .

The cosets of  $H$  partition  $S_n$ , and each forms an independent set, since  $\tau_1, \tau_2 \in \sigma H$  implies that  $\tau_1^{-1}\tau_2 \in H$  is not a  $k$ -derangement and hence the vertices associated to  $\tau_1$  and  $\tau_2$  are not connected by an edge. Giving each of the  $\frac{n!}{k!(n-k)!} = \binom{n}{k}$  cosets a different color results in a valid coloring of  $\Gamma_{k,n}$ , so  $\chi(\Gamma_{k,n}) \leq \binom{n}{k}$ .  $\square$

**Corollary 12.** For  $n$  an odd prime power,  $\alpha(\Gamma_{2,n}) = 2(n-2)!$  and  $\chi(\Gamma_{2,n}) = \binom{n}{2}$ .

*Proof.* By Lemma 10 and Theorem 6, we have  $\binom{n}{2} \cdot \alpha(\Gamma_{2,n}) \leq n!$ . Thus

$$\alpha(\Gamma_{2,n}) \leq n! \cdot \frac{2(n-2)!}{n!} = 2(n-2)!$$

and Theorem 11 gives the reverse inequality. For any graph  $G$ ,  $\chi(G) \geq \omega(G)$ , so, by Theorem 6,  $\chi(\Gamma_{2,n}) \geq \binom{n}{2}$  and again Theorem 11 gives the reverse inequality.  $\square$

## 5. Further questions

In the last section, we showed that the clique number of the 2-derangement graph is equal to  $\binom{n}{2}$  when  $n$  is an odd prime power and strictly less than that if  $n$  is even (and at least 4). The clique construction of [Theorem 6](#) fails to work when  $n$  is odd and not a prime power since there is no field of that cardinality. We believe that in this case the clique number is strictly smaller than  $\binom{n}{2}$ . For arbitrary  $k$ , we have some faint hope that the bounds given in [Theorem 11](#) for  $\alpha(\Gamma_{k,n})$  and  $\chi(\Gamma_{k,n})$  are actually equalities, but the situation for  $\omega(\Gamma_{k,n})$  remains unclear.

In another direction, the numerical evidence is overwhelming that the derangement graphs are Hamiltonian. We hope to explore these and other questions in future work.

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We would like to acknowledge Sam Tencer's contribution to this investigation. We are also grateful to an anonymous referee for spotting the connection between cliques attaining the bound of [Lemma 5](#) and sharply  $k$ -homogeneous sets of permutations.

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