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Hannah Jackson, Kathryn Nyman and Les Reid



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A permutation on n elements is called a k -derangement ($k \leq n$) if no k -element subset is mapped to itself. One can form the k -derangement graph on the set of all permutations on n elements by connecting two permutations σ and τ if $\sigma\tau^{-1}$ is a k -derangement. We characterize when such a graph is connected or Eulerian. For n an odd prime power, we determine the independence, clique and chromatic numbers of the 2-derangement graph.

1. Introduction

Permutations which leave no element fixed, known as derangements, were first considered in [de Montmort 1708] and have been extensively studied since. A derangement graph is a graph whose vertices are the elements of the symmetric group S_n and whose edges connect two permutations that differ by a derangement. Derangement graphs have been shown to be connected (for $n > 3$) and Hamiltonian, and their independence number, clique number, and chromatic number have been calculated [Renteln 2007].

Here we consider the generalization of derangements known as k -derangements, which are those permutations in S_n that do not fix any k -element subset of the set being permuted. A k -derangement graph is defined in an analogous manner to a derangement graph. We examine some of the graph-theoretical properties of k -derangement graphs.

2. Preliminaries

Let S_n be the group of permutations on the set $\{1, 2, \dots, n\}$. A permutation $\sigma \in S_n$ maps any k -element subset of $\{1, \dots, n\}$ to a k -element subset of $\{1, \dots, n\}$; in the usual notation,

$$\sigma(\{a_1, \dots, a_k\}) = \{\sigma(a_1), \dots, \sigma(a_k)\}.$$

If $\{a_1, \dots, a_k\} = \{\sigma(a_1), \dots, \sigma(a_k)\}$ (as sets, that is, without regard to order), we

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say that σ fixes the unordered k -tuple $\{a_1, \dots, a_k\}$. (“Unordered k -tuple” is another name for a k -element set.)

If σ does not map *any* of the $\binom{n}{k}$ possible unordered k -tuples to itself, we say that σ is a k -derangement. For example, with $n = 4$, the cyclic permutation $\sigma = (1234)$ is a 2-derangement, because (taking $k = 2$) we have

$$\begin{aligned} (1234)(\{1, 2\}) &= \{(1234)(1), (1234)(2)\} = \{2, 3\}, \\ (1234)(\{1, 3\}) &= \{(1234)(1), (1234)(3)\} = \{2, 4\}, \\ (1234)(\{1, 4\}) &= \{(1234)(1), (1234)(4)\} = \{2, 1\} = \{1, 2\}, \\ (1234)(\{2, 3\}) &= \{(1234)(2), (1234)(3)\} = \{3, 4\}, \\ (1234)(\{2, 4\}) &= \{(1234)(2), (1234)(4)\} = \{3, 1\} = \{1, 3\}, \\ (1234)(\{3, 4\}) &= \{(1234)(3), (1234)(4)\} = \{4, 1\} = \{1, 4\}. \end{aligned}$$

This extends the ordinary notion of a derangement, defined as a permutation $\sigma \in S_n$ such that $\sigma(x) \neq x$ for all $x \in \{1, \dots, n\}$.

The set of k -derangements in S_n is denoted by $\mathcal{D}_{k,n}$, and its cardinality $|\mathcal{D}_{k,n}|$ — the number of k -derangements in S_n — is denoted by $D_k(n)$. As we have seen, (1234) is in $\mathcal{D}_{2,4}$. Specifically,

$$\begin{aligned} \mathcal{D}_{2,4} = \{ &(1234), (1243), (1324), (1342), (1423), (1432), (123)(4), (124)(3), \\ &(132)(4), (134)(2), (142)(3), (143)(2), (234)(1), (243)(1)\}, \end{aligned}$$

and thus $D_2(4) = 14$. The sequence $D_2(n)$ appears as A137482 in the *On-Line Encyclopedia of Integer Sequences*; see [Henshaw 2008]. The number $D_1(n)$ is also simply called the derangement number.

The cycle structure of a permutation σ , denoted by C_σ , is the multiset of the lengths of the cycles in its cycle decomposition (e.g., $C_{(12)(3)(45)} = \{2, 2, 1\}$). Note that the cycle structure of $\sigma \in S_n$ is a partition of n . Given a partition r of n , let P_r be the set of all permutations in S_n whose cycle structure is r . For example (as usual, excluding singletons in our notation) $P_{\{2,1,1\}} = \{(12), (13), (14), (23), (24), (34)\}$.

We first note that if the cycle structure of a permutation σ contains a multiset which partitions k , then σ is not a k -derangement. For example, $(12)(34)$ is a 3-derangement in S_4 , but $(12)(3)(4)$ is not, because it fixes the set $\{1, 2, 3\}$, for example. And we see that $\{2, 1\} \subseteq C_{(12)(3)(4)} = \{2, 1, 1\}$ is a partition of 3. Thus we observe that the cycle structure of a permutation determines whether or not it is a k -derangement, and we have the following.

Proposition 1. *A permutation $\sigma \in S_n$ is a k -derangement if and only if the cycle decomposition of σ does not contain a set of cycles whose lengths partition k .*

Proof. If $\{q, r, \dots, s\}$ is a partition of k , and $(a_1 \cdots a_q)(b_1 \cdots b_r) \cdots (c_1 \cdots c_s)$ are cycles of σ , then, for $x = \{a_1, \dots, a_q, b_1, \dots, b_r, c_1, \dots, c_s\}$, $\sigma(x) = x$. Conversely,

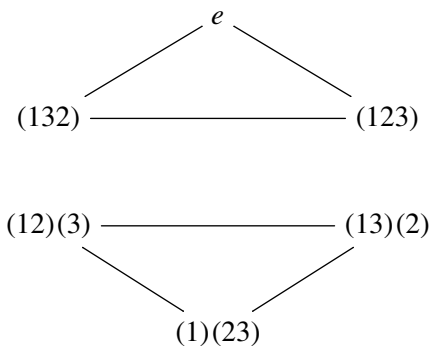


Figure 1. The 2-derangement graph on 6 vertices, $\Gamma_{2,3}$.

if σ has no set of cycles whose lengths partition k , then, given any k -element subset x of $\{1, \dots, n\}$, there is a cycle in σ which contains at least one element in x and contains some element not in x . Hence σ sends an element in x to an element not in x and so $\sigma(x) \neq x$. □

Let $CD_{k,n}$ be the set of cycle structures corresponding to k -derangements in S_n ; for example, $CD_{2,4} = \{\{4\}, \{3, 1\}\}$. Since a cycle structure C_σ is in $CD_{k,n}$ if and only if it is in $CD_{n-k,n}$, we have $\mathcal{D}_{k,n} = \mathcal{D}_{n-k,n}$.

Let G be a group, and let S be a subset of G that is closed under taking inverses. The *Cayley graph* $\Gamma(G, S)$ is the graph whose vertices are the elements of G such that an edge connects two vertices $u, v \in G$ if $su = v$ for some $s \in S$. A *k -derangement graph* is a Cayley graph defined by $\Gamma_{k,n} := \Gamma(S_n, \mathcal{D}_{k,n})$. (Note that $\mathcal{D}_{k,n}$ is symmetric, as the inverse of a k -derangement is a k -derangement, and thus satisfies the requirements for a Cayley graph.) It is worth noting that $\Gamma_{k,n}$ is, by construction, $D_k(n)$ -regular, and that, since $\mathcal{D}_{k,n} = \mathcal{D}_{(n-k),n}$, $\Gamma_{k,n} = \Gamma_{(n-k),n}$. [Figure 1](#) illustrates the 2-derangement graph on 6 vertices, $\Gamma_{2,3}$.

It is possible to consider k -derangements in S_n for any positive k and n . However, if $k = n$, there will be no k -derangements in S_n , since every partition in S_n will have a cycle structure such that the cycle lengths partition k . As such, $\Gamma_{k,n}$ will be the empty (edgeless) graph on n vertices. If $k > n$, then every permutation in S_n is a k -derangement vacuously, and thus $\Gamma_{k,n}$ will be the complete graph on $|S_n|$ vertices. As neither of these cases is particularly interesting, henceforth we will only consider k -derangements where $k < n$.

3. Properties of derangement graphs

[Figure 1](#) shows that $\Gamma_{2,3}$ is not a connected graph, and, since $\Gamma_{2,3} = \Gamma_{1,3}$, we see that $\Gamma_{k,3}$ is disconnected for all $k < n$. But this is an exception rather than the rule, as the following theorem demonstrates.

Theorem 2. For $n > 3$ and $k < n$, $\Gamma_{k,n}$ is connected.

Proof. Every permutation in S_n can be written as the product of adjacent transpositions ($h(h+1)$). These, in turn, can be expressed as products of two k -derangements, so long as $n > 3$, as we will demonstrate. As a result, for $n > 3$, the elements of $\mathcal{D}_{k,n}$ generate S_n , which means that every vertex of $\Gamma_{k,n}$ can be reached by a path from the identity.

We show that the permutation $(1\ 2)$ can be written as the product of two k -derangements and then note that, since it is the form and not the individual labels that are important, any adjacent transposition can be written as the product of two k -derangements. We consider two cases: $k = 1$ and $k \geq 2$.

Case 1: If $k = 1$, then $(1\ 2) = (1\ 2 \cdots n)^2 \cdot (n(n-1) \cdots 1)^2(1\ 2)$. We claim that $(1\ 2 \cdots n)^2$ and $(n(n-1) \cdots 1)^2(1\ 2)$ are each 1-derangements in S_n for all $n > 3$. If n is even, then $(1\ 2 \cdots n)^2 = (1\ 3 \cdots (n-3)(n-1))(2\ 4 \cdots (n-2)n)$, which is a 1-derangement in S_n for all n . Additionally,

$$(n(n-1) \cdots 1)^2(1\ 2) = (1\ n(n-2)(n-4) \cdots 2(n-1)(n-3) \cdots 3),$$

which is also a 1-derangement in S_n for any n .

On the other hand, if n is odd, then

$$(1\ 2 \cdots n)^2 = (1\ 3 \cdots (n-2)n\ 2\ 4 \cdots (n-3)(n-1)),$$

which is a 1-derangement in S_n for all n . And

$$\begin{aligned} (n(n-1) \cdots 1)^2(1\ 2) &= (n(n-2)(n-4) \cdots 3\ 1(n-1)(n-3) \cdots 4\ 2)(1\ 2) \\ &= (1\ n(n-2)(n-4) \cdots 3)(2(n-1)(n-3) \cdots 4), \end{aligned}$$

which is a 1-derangement in S_n so long as $n > 3$. (If $n = 3$, $(312)(12) = (13)(2)$, which is not a 1-derangement.)

Thus, for $n > 3$, we have shown that $(1\ 2)$ can be written as the product of two 1-derangements, and, by extension, every adjacent transposition can be written as the product of two 1-derangements.

Case 2: For $k \geq 2$, $(1\ 2) = (1\ 2 \cdots n)^{-1}(1\ 3\ 4 \cdots n)$. We know $(1\ 2 \cdots n)^{-1}$ is a k -derangement for all k since the inverse of a k -derangement is a k -derangement. And, by the cycle structure, we see that $(1\ 3\ 4 \cdots n) = (1\ 3\ 4 \cdots n)(2)$ is a k -derangement for all k , except $k = 1$ and $k = (n-1)$ (however, since $\Gamma_{1,n} = \Gamma_{(n-1),n}$, Case 1 addresses $(n-1)$ -derangements as well as 1-derangements).

So we have shown that, for $k \geq 2$, $(1\ 2)$ can be written as the product of two k -derangements, and again, by extension, we can write any adjacent transposition as the product of two k -derangements. Thus every vertex is connected by a path to the identity, and $\Gamma_{k,n}$ is connected. \square

It is worth noting that [Theorem 2](#) holds for $n = 2$ as well. Since we are only interested in k -derangements in S_n such that $k < n$, when $n = 2$, k must equal 1, and so $\Gamma_{1,2}$ is the connected graph on two vertices.

Next, we give a characterization in terms of n and k for when a derangement graph is Eulerian. We will require the following result.

Lemma 3. *If a cycle structure includes a cycle of length greater than 2, then there are an even number of permutations with that cycle structure.*

Proof. Consider P_r , the set of permutations with a given cycle structure, r . We can pair each $\sigma \in P_r$ with its inverse $\sigma^{-1} \in P_r$, and, so long as $\sigma \neq \sigma^{-1}$ for any $\sigma \in P_r$, $|P_r|$ will be even. Suppose there exists a $\sigma \in P_r$ such that $\sigma = \sigma^{-1}$. Then $\sigma^2 = e$, and so the order of σ is at most 2. The order of a permutation is the least common multiple of the orders of the elements of its cycle structure, so σ must not include a cycle of length greater than 2. This is a contradiction; thus $|P_r|$ is even. \square

Theorem 4. *For $n > 3$ and $k < n$, $\Gamma_{k,n}$ is Eulerian if and only if k is even or k and n are both odd.*

Proof. A graph is Eulerian if and only if it is connected and each vertex has an even degree. In light of [Theorem 2](#) and the previously noted fact that $\Gamma_{k,n}$ is $D_k(n)$ -regular, in order to ascertain if $\Gamma_{k,n}$ is Eulerian, we must determine whether $D_k(n)$ is even or odd.

If k is even, we claim that $D_k(n)$ is the sum of even numbers. Any cycle structure composed entirely of 2- or 1-cycles will partition an even k , and thus any permutation which is in $\mathcal{D}_{k,n}$ for an even k will contain a cycle of length 3 or greater in its cycle decomposition. Now, $\mathcal{D}_{k,n} = P_{r_1} \dot{\cup} P_{r_2} \dot{\cup} \dots \dot{\cup} P_{r_m}$ (disjoint union) such that no r_i partitions k , and, by [Lemma 3](#), $|P_{r_i}|$ is even for all $i \in \{1, \dots, m\}$. Thus, when k is even, $D_k(n)$ is even.

If k and n are both odd, again we see that every permutation in $\mathcal{D}_{k,n}$ will contain a cycle of length 3 or greater in its cycle decomposition, since an odd k can be partitioned by a set of cycles of lengths 1 or 2 if there is at least one 1-cycle. Furthermore, since n is odd, there are no permutations whose cycle structure is composed only of length-2 cycles. Thus, $D_k(n)$ is even.

Finally, we show that, if k is odd and n is even, then $\Gamma_{k,n}$ is not Eulerian. In this case, $P_{\{2,2,\dots,2\}}$ is in $CD_{k,n}$. By choosing pairs of elements for the cycles and dividing by the number of ways to order the cycles, we see that the number of permutations in $P_{\{2,2,\dots,2\}}$ is given by

$$\begin{aligned} \frac{\binom{n}{2} \binom{n-2}{2} \cdots \binom{2}{2}}{\left(\frac{n}{2}\right)!} &= \frac{n(n-1)(n-2) \cdots (3)(2)(1)}{\left(2 \cdot \frac{n}{2}\right) \left(2 \cdot \left(\frac{n}{2} - 1\right)\right) \cdots (6)(4)(2)} \\ &= \frac{n(n-1)(n-2) \cdots (3)(2)(1)}{n(n-2) \cdots (6)(4)(2)} = (n-1)(n-3) \cdots (5)(3)(1). \end{aligned}$$

Since n is even, the product $(n-1)(n-3)\cdots(5)(3)(1)$ is odd. Every other k -derangement in S_n will contain a cycle with length greater than 2, since any combination of 1-cycles or 1- and 2-cycles will partition k . So $D_k(n)$ is the sum of one odd number and even numbers, and so is odd. \square

4. Chromatic, independence and clique numbers for $k = 2$ and n an odd prime power

For the majority of this section, we will think of permutations in terms of the result of their application to the ordering $\{1, 2, 3, \dots, n\}$. Thus, $\{2, 3, 1, 4, 5\}$ represents the permutation which has moved 2 to the first position, 3 to the second, 1 to the third, and left 4 and 5 fixed; that is, the permutation $(132)(4)(5)$ in cycle notation, or the inverse of the permutation $\begin{pmatrix} 12345 \\ 23145 \end{pmatrix}$ in two line notation.

We note that in order for vu^{-1} (or, equivalently, $v^{-1}u$) to be a k -derangement, it is necessary and sufficient that no unordered k -tuple of elements be sent to the same unordered k -tuple of positions by both u and v . For example, the permutations $u = \{2, 3, 1, 4, 5\}$ and $v = \{4, 1, 3, 5, 2\}$ both send the pair $\{1, 3\}$ to the second and third positions. Thus $(vu^{-1})(\{2, 3\}) = \{2, 3\}$, and so vu^{-1} is not a 2-derangement and there is no edge between u and v in the 2-derangement graph. More formally, suppose u and v both send the k -tuple $M' = \{a'_1, a'_2, \dots, a'_k\}$ to positions $M = \{a_1, a_2, \dots, a_k\}$. Then, $(vu^{-1})(M) = v(M') = M$. Thus, vu^{-1} is not a k -derangement.

On the other hand, if u and v send no k -tuple to the same positions we claim vu^{-1} is a k -derangement. Consider an arbitrary k -tuple, $M = \{a_1, a_2, \dots, a_k\}$, and suppose u maps the k -tuple $M' = \{a'_1, a'_2, \dots, a'_k\}$ to the positions given in M . Then $(vu^{-1})(M) = v(M') \neq M$ since v cannot send the k -tuple M' to the same positions as u does. Thus, vu^{-1} is a k -derangement.

In [Theorem 6](#), we find the clique number of the 2-derangement graph, $\omega(\Gamma_{2,n})$, for n an odd prime power, by constructing a clique of maximal size. Before establishing this clique number, we note an upper bound on the clique number of a general k -derangement graph.

Lemma 5. For $k < n$, $\omega(\Gamma_{k,n}) \leq \binom{n}{k}$.

Proof. The clique number of the k -derangement graph, $\omega(\Gamma_{k,n})$, cannot be greater than $\binom{n}{k}$, since there are only $\binom{n}{k}$ subsets of size k and hence at most $\binom{n}{k}$ different unordered k -tuples of positions for an arbitrary k -tuple of elements to be sent under a permutation. \square

Theorem 6. If n is an odd prime power, then $\omega(\Gamma_{2,n}) = \binom{n}{2}$.

Proof. We will explicitly construct a clique with $\binom{n}{2}$ elements. Let $n = p^r$, with p a prime greater than 2, and let \mathbb{F}_{p^r} denote the field with p^r elements. Rather than

letting S_n act on $\{1, \dots, n\}$, we will let it act on \mathbb{F}_{p^r} and construct $\Gamma_{2,n}$ accordingly. Let $v = (x_1, \dots, x_n)$ be an ordered n -tuple whose entries are the elements of \mathbb{F}_{p^r} in some order. Given any function $\phi : \mathbb{F}_{p^r} \rightarrow \mathbb{F}_{p^r}$, we define $\phi(v) = (\phi(x_1), \dots, \phi(x_n))$. Partition the nonzero elements of \mathbb{F}_{p^r} by pairing each element with its (additive) inverse, and let T be a set obtained by choosing exactly one element from each pair, giving $|T| = (p^r - 1)/2$.

Define $f_{s,\alpha}(x) = sx + \alpha$, and consider the set $X = \{f_{s,\alpha}(v) \mid s \in T \text{ and } \alpha \in \mathbb{F}_{p^r}\}$. Since $s \neq 0$, $f_{s,\alpha}$ is a bijection and $f_{s,\alpha}(v)$ is a permutation of the elements of \mathbb{F}_{p^r} . We claim that X is a clique in $\Gamma_{2,n}$. Suppose not; that is, suppose there are $s, t \in T$ and $\alpha, \beta \in \mathbb{F}_{p^r}$, $(s, \alpha) \neq (t, \beta)$, such that $f_{s,\alpha}(v)$ is not a 2-derangement of $f_{t,\beta}(v)$. In that case there exist $x, y \in \mathbb{F}_{p^r}$, $x \neq y$, such that either $f_{s,\alpha}(x) = f_{t,\beta}(x)$ and $f_{s,\alpha}(y) = f_{t,\beta}(y)$ or $f_{s,\alpha}(x) = f_{t,\beta}(y)$ and $f_{s,\alpha}(y) = f_{t,\beta}(x)$. In the first case, subtracting the two equations and rewriting yields $(s - t)(x - y) = 0$. If $s = t$, then $\alpha = \beta$, giving a contradiction. If $s \neq t$, then $x = y$ and again we have a contradiction. In the second case, subtracting and rewriting yields $(s + t)(x - y) = 0$ and, since $s + t \neq 0$ for $s, t \in T$, $x = y$ and this also give a contradiction. Thus, X is a clique of size $p^r(p^r - 1)/2 = \binom{p^r}{2}$. \square

The next example illustrates the construction when $n = 7$.

Example 7. We build a clique of size $\binom{7}{2}$ in the derangement graph $\Gamma_{2,7}$ consisting of $\frac{7-1}{2}$ blocks, each of which contains 7 permutations. We let $v = (1, 2, 3, 4, 5, 6, 7)$ (writing 7 instead of 0) and take $T = \{1, 4, 5\}$. Then

$$\begin{aligned} f_{1,0}(v) &= (1, 2, 3, 4, 5, 6, 7), & f_{4,0}(v) &= (4, 1, 5, 2, 6, 3, 7), \\ f_{5,0}(v) &= (5, 3, 1, 6, 4, 2, 7). \end{aligned}$$

Increasing α from 0 cyclically permutes the 7-tuples. Block 1 consists of the arrangements $\{f_{1,\alpha}(v) \mid \alpha \in \mathbb{F}_7\}$, that is, the arrangement $(1, 2, 3, 4, 5, 6, 7)$ and the remaining 6 rotations of this arrangement (e.g., $(2, 3, 4, 5, 6, 7, 1)$, $(3, 4, 5, 6, 7, 1, 2)$, etc.). Block 2 consists of the arrangement $f_{4,0}(v)$ along with all of its rotations. Finally, block 3 consists of $f_{5,0}(v)$ and its rotations. To see that these permutations form a clique, consider, for example, the pair $\{1, 2\}$. These elements are one position apart in block 1, two positions apart in block 2 and three positions apart in block 3 (counting the shortest distance between them either forwards or backwards). So the pair $\{1, 2\}$ cannot occupy the same positions in two permutations which appear in different blocks. Furthermore, within a block, the rotations insure that the pair never occupies the same positions.

Remark 8. Cliques achieving the upper bound of [Lemma 5](#) are known as *sharply k -homogeneous sets* of permutations. A corollary in [\[Nomura 1985\]](#) shows that, for $2k \leq n$, the existence of such a k -homogeneous set implies $n + 1 \equiv 0 \pmod k$. Thus [Theorem 6](#) cannot be extended to even n , and we have the following.

Corollary 9. For n even and $n \geq 4$, $\omega(\Gamma_{2,n}) < \binom{n}{2}$.

A computer search confirms that $\omega(\Gamma_{2,4}) = 5 < \binom{4}{2}$.

Next we turn to the independence number $\alpha(\Gamma_{k,n})$ and the chromatic number $\chi(\Gamma_{k,n})$ of the k -derangement graph. We will require the following lemma which has been adapted from Frankl and Deza's lemma [1977] and applied to k -tuples of elements.

Lemma 10. For $k < n$, $\alpha(\Gamma_{k,n})\omega(\Gamma_{k,n}) \leq n!$.

Proof. Let \mathcal{P} be a set of permutations in S_n , every pair of which has at least one unordered k -tuple of elements in the same unordered k -tuple of positions. That is, for any $u, v \in \mathcal{P}$, there exists a set $M = \{a_1, \dots, a_k\} \subseteq \{1, \dots, n\}$ such that $(v^{-1}u)(M) = M$. Note that \mathcal{P} is an independent set in the k -derangement graph. Let \mathcal{Q} be a set of permutations in S_n such that each pair of permutations has no k -tuple of elements in the same positions; that is, \mathcal{Q} is a clique in the k -derangement graph. We claim that products of the form PQ with $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ give distinct permutations of n . Suppose, for the sake of contradiction, that $P_1Q_1 = P_2Q_2$ for $P_1, P_2 \in \mathcal{P}$ and $Q_1, Q_2 \in \mathcal{Q}$ with $P_1 \neq P_2$ and $Q_1 \neq Q_2$. This implies that $P_1^{-1}P_2 = Q_1Q_2^{-1}$. Now, since P_1 and P_2 are in \mathcal{P} , there is a k -tuple of elements $M = \{a_1, \dots, a_k\}$ such that $(P_1^{-1}P_2)(M) = M$. However, this implies $(Q_1Q_2^{-1})(M) = M$. But we know that the permutations in \mathcal{Q} agree on no k -tuples, and so we must have $Q_1 = Q_2$ and, hence, $P_1 = P_2$. Finally, since each product gives a unique permutation of n , there can be no more than $n!$ such products. \square

Theorem 11. For $k < n$, $\alpha(\Gamma_{k,n}) \geq k!(n-k)!$ and $\chi(\Gamma_{k,n}) \leq \binom{n}{k}$.

Proof. Consider H , the set of all permutations in S_n that send $\{1, 2, \dots, k\}$ to itself (and hence $\{k+1, \dots, n\}$ to itself). It is clear that H is a subgroup of S_n isomorphic to $S_k \times S_{n-k}$ and that $|H| = k!(n-k)!$. Since the unordered k -tuple $\{1, 2, \dots, k\}$ is fixed, none of these are k -derangements of each other, so H is an independent set and $\alpha(\Gamma_{k,n}) \geq k!(n-k)!$.

The cosets of H partition S_n , and each forms an independent set, since $\tau_1, \tau_2 \in \sigma H$ implies that $\tau_1^{-1}\tau_2 \in H$ is not a k -derangement and hence the vertices associated to τ_1 and τ_2 are not connected by an edge. Giving each of the $\frac{n!}{k!(n-k)!} = \binom{n}{k}$ cosets a different color results in a valid coloring of $\Gamma_{k,n}$, so $\chi(\Gamma_{k,n}) \leq \binom{n}{k}$. \square

Corollary 12. For n an odd prime power, $\alpha(\Gamma_{2,n}) = 2(n-2)!$ and $\chi(\Gamma_{2,n}) = \binom{n}{2}$.

Proof. By Lemma 10 and Theorem 6, we have $\binom{n}{2} \cdot \alpha(\Gamma_{2,n}) \leq n!$. Thus

$$\alpha(\Gamma_{2,n}) \leq n! \cdot \frac{2(n-2)!}{n!} = 2(n-2)!$$

and Theorem 11 gives the reverse inequality. For any graph G , $\chi(G) \geq \omega(G)$, so, by Theorem 6, $\chi(\Gamma_{2,n}) \geq \binom{n}{2}$ and again Theorem 11 gives the reverse inequality. \square

5. Further questions

In the last section, we showed that the clique number of the 2-derangement graph is equal to $\binom{n}{2}$ when n is an odd prime power and strictly less than that if n is even (and at least 4). The clique construction of [Theorem 6](#) fails to work when n is odd and not a prime power since there is no field of that cardinality. We believe that in this case the clique number is strictly smaller than $\binom{n}{2}$. For arbitrary k , we have some faint hope that the bounds given in [Theorem 11](#) for $\alpha(\Gamma_{k,n})$ and $\chi(\Gamma_{k,n})$ are actually equalities, but the situation for $\omega(\Gamma_{k,n})$ remains unclear.

In another direction, the numerical evidence is overwhelming that the derangement graphs are Hamiltonian. We hope to explore these and other questions in future work.

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hljacks@sy.edu

Mathematics Department, Syracuse University, 215 Carnegie,
Syracuse, NY 13244, United States

knyman@willamette.edu

Mathematics Department, Willamette University,
900 State Street, Salem, OR 97301, United States

lesreid@missouristate.edu

Mathematics Department, Missouri State University, 901
South National Avenue, Springfield, MO 65897, United States

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