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On closed modular colorings of rooted trees

Bryan Phinezy and Ping Zhang

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Two vertices u and v in a nontrivial connected graph G are twins if u and v have the same neighbors in $V(G) - \{u, v\}$. If u and v are adjacent, they are referred to as true twins, while if u and v are nonadjacent, they are false twins. For a positive integer k, let $c : V(G) \to \mathbb{Z}_k$ be a vertex coloring where adjacent vertices may be assigned the same color. The coloring c induces another vertex coloring $c' : V(G) \to \mathbb{Z}_k$ defined by $c'(v) = \sum_{u \in N[v]} c(u)$ for each $v \in V(G)$, where $N[v]$ is the closed neighborhood of v. Then c is called a closed modular k-coloring if $c'(u) \neq c'(v)$ in $\mathbb{Z}_k$ for all pairs u, v of adjacent vertices that are not true twins. The minimum k for which G has a closed modular k-coloring is the closed modular chromatic number $\overline{\mathrm{mc}}(G)$ of G. A rooted tree T of order at least 3 is even if every vertex of T has an even number of children, while T is odd if every vertex of T has an odd number of children. It is shown that $\overline{\mathrm{mc}}(T) = 2$ for each even rooted tree and $\overline{\mathrm{mc}}(T) \leq 3$ if T is an odd rooted tree having no vertex with exactly one child. Exact values $\overline{\mathrm{mc}}(T)$ are determined for several classes of odd rooted trees T.

1. Introduction

A weighting (or edge labeling with positive integers) of a connected graph G was introduced in [Chartrand et al. 1988] for the purpose of producing a weighted graph whose degrees (obtained by adding the weights of the incident edges of each vertex) were distinct. Such a weighted graph was called *irregular*. This concept could be looked at in another manner, however. In particular, let $\mathbb{N}$ denote the set of positive integers and let E_v denote the set of edges of G incident with a vertex v. An edge coloring $c : E(G) \to \mathbb{N}$, where adjacent edges may be colored the same, is said to be *vertex-distinguishing* if the coloring $c' : V(G) \to \mathbb{N}$ induced by c and defined by $c'(v) = \sum_{e \in E_v} c(e)$ has the property that $c'(x) \neq c'(y)$ for every two distinct vertices x and y of G. The main emphasis of this research dealt with minimizing the largest color assigned to the edges of the graph to produce an irregular graph.

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Vertex-distinguishing colorings have received increased attention during the past 25 years (see [Escuadro et al. 2007]).

Rosa [1967] introduced a vertex labeling that induces an *edge-distinguishing* labeling defined by subtracting labels. In particular, for a graph G of size m, a vertex labeling (an injective function) $f : V(G) \to \{0, 1, \ldots, m\}$ was called a *β-valuation* if the induced edge labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ defined by $f'(uv) = |f(u) - f(v)|$ was bijective. Golomb [1972] called a β-valuation a *graceful labeling* and a graph possessing a graceful labeling a *graceful graph*. It is this terminology that became standard. Much research has been done on graceful graphs. A popular conjecture in graph theory, due to Anton Kotzig and Gerhard Ringel, is the following.

The Graceful Tree Conjecture. *Every nontrivial tree is graceful.*

Jothi [1991] introduced a concept that, in a certain sense, reverses the roles of vertices and edges in graceful labelings (see also [Gallian 1998]). For a connected graph G of order $n \geq 3$, let $f : E(G) \to \mathbb{Z}_n$ be an edge labeling of G that induces a bijective function $f' : V(G) \to \mathbb{Z}_n$ defined by $f'(v) = \sum_{e \in E_v} f(e)$ for each vertex v of G. Such a labeling f is called a *modular edge-graceful labeling*, while a graph possessing such a labeling is called *modular edge-graceful* (see [Jones et al. 2013]). Verifying a conjecture by Gnana Jothi on trees, Jones et al. [2012] showed not only that every tree of order $n \geq 3$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$ but a connected graph of order $n \geq 3$ is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$.

Many of these weighting or labeling concepts were later interpreted as coloring concepts with the resulting vertex-distinguishing labeling becoming a vertex-distinguishing coloring. A *neighbor-distinguishing coloring* is a coloring in which every pair of adjacent vertices are colored differently. Such a coloring is more commonly called a *proper coloring*. The minimum number of colors in a proper vertex coloring of a graph G is its chromatic number $\chi(G)$.

In 2004 a neighbor-distinguishing edge coloring $c : E(G) \to \{1, 2, \ldots, k\}$ of a graph G was introduced (see [Chartrand and Zhang 2009, p. 385]) in which an induced vertex coloring $s : V(G) \to \mathbb{N}$ is defined by $s(v) = \sum_{e \in E_v} c(e)$ for each vertex v of G. The minimum k for which such a neighbor-distinguishing coloring exists is called the *sum distinguishing index*, denoted by $\mathrm{sd}(G)$ of G. This is therefore the proper coloring analogue of the irregular weighting mentioned earlier. Karoński et al. [2004] showed that, if $\chi(G) \leq 3$, then $\mathrm{sd}(G) \leq 3$. Addario-Berry et al. [2005] showed that, for every connected graph G of order at least 3, $\mathrm{sd}(G) \leq 4$. In fact, Karoński et al. [2004] made the following conjecture, which has acquired a name used by many.

The 1-2-3 Conjecture. *If G is a connected graph of order* 3 *or more, then* $\mathrm{sd}(G) \leq 3$.

A number of neighbor-distinguishing vertex colorings different from standard proper colorings have been introduced in the literature (see [Chartrand and Zhang 2009, pp. 379–385], for example). Chartrand et al. [2010] introduced a neighbor-distinguishing vertex coloring of a graph based on sums of colors. For a nontrivial connected graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be colored the same. If k colors are used by c, then c is a *k-coloring* of G. The *color sum* $\sigma(v)$ of a vertex v is defined by $\sigma(v) = \sum_{u \in N(v)} c(u)$ where $N(v)$ denotes the neighborhood of v (the set of vertices adjacent to v). If $\sigma(u) \neq \sigma(v)$ for every two adjacent vertices u and v of G, then c is neighbor-distinguishing and is called a *sigma coloring* of G. The minimum number of colors required in a sigma coloring of a graph G is called the *sigma chromatic number* of G and is denoted by $\sigma(G)$. Chartrand et al. [2010] showed that, for each pair a, b of positive integers with $a \leq b$, there is a connected graph G with $\sigma(G) = a$ and $\chi(G) = b$.

Chartrand et al. [2012] introduced another neighbor-distinguishing vertex coloring that is closely related to colorings discussed above. For a nontrivial connected graph G, let $c : V(G) \to \mathbb{Z}_k$ $(k \geq 2)$ be a vertex coloring where adjacent vertices may be assigned the same color. The coloring c induces another vertex coloring $c' : V(G) \to \mathbb{Z}_k$, where

$$c'(v) = \sum_{u \in N[v]} c(u), \tag{1}$$

where $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v and the sum in (1) is performed in $\mathbb{Z}_k$. A coloring c of G is called a *closed modular k-coloring* if for every pair x, y of adjacent vertices in G either $c'(x) \neq c'(y)$ or $N[x] = N[y]$, in the latter case of which we must have $c'(x) = c'(y)$. Closed modular colorings of graphs were introduced in [Chartrand et al. 2012] and inspired by a domination problem. The minimum k for which G has a closed modular k-coloring is called the *closed modular chromatic number* of G and is denoted by $\overline{\text{mc}}(G)$. Chartrand et al. [2012] observed that the nontrivial complete graphs are the only nontrivial connected graphs G for which $\overline{\text{mc}}(G) = 1$. Two vertices u and v in a connected graph G are *twins* if u and v have the same neighbors in $V(G) - \{u, v\}$. If u and v are adjacent, they are referred to as *true twins*, while if u and v are nonadjacent, they are *false twins*. If u and v are adjacent vertices of a graph G such that $N[u] = N[v]$ (that is, u and v are true twins), then $c'(u) = c'(v)$ for every vertex coloring c of G. The following result appeared in [Chartrand et al. 2012].

Proposition 1.1. *If G is a nontrivial connected graph, then $\overline{\text{mc}}(G)$ exists. Furthermore, if G contains no true twins, then $\overline{\text{mc}}(G) \geq \chi(G)$.*

To illustrate these concepts, consider the bipartite graph G of Figure 1. Since $\chi(G) = 2$ and G has no true twins, it follows that $\overline{\text{mc}}(G) \geq 2$ by Proposition 1.1.

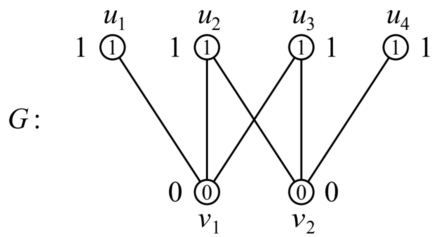

Figure 1. A graph G with $\chi(G) = 2$ and $\overline{\text{mc}}(G) = 3$.

In fact $\overline{\text{mc}}(G) = 3$. Figure 1 shows a closed modular 3-coloring of G (where the color of a vertex is placed within the vertex) together with the color $c'(v)$ for each vertex v of G (where the color $c'(v)$ of a vertex is placed next to the vertex).

For an edge uv of a graph G, the graph G/uv obtained from G by *contracting the edge* uv has the vertex set $V(G)$ in which u and v are identified. If we denote the vertex $u = v$ in G/uv by w, then $V(G/uv) = (V(G) \cup \{w\}) - \{u, v\}$ and the edge set of G/uv is

$$E(G/uv) = \{xy : xy \in E(G),\ x, y \in V(G) - \{u, v\}\} \\ \cup \{wx : ux \in E(G) \text{ or } vx \in E(G), x \in V(G) - \{u, v\}\}.$$

The graph G/uv is referred to as an *elementary contraction* of G. For a nontrivial connected graph G, define the *true twins closure* $\text{TC}(G)$ of G as the graph obtained from G by a sequence of elementary contractions of pairs of true twins in G until no such pair remains. In particular, if G contains no true twins, then $\text{TC}(G) = G$. Thus $\text{TC}(G)$ is a minor of G. Chartrand et al. [2012] showed that $\overline{\text{mc}}(G) = \overline{\text{mc}}(\text{TC}(G))$ for every nontrivial connected graph G. Therefore, it suffices to consider nontrivial connected graphs containing no true twins.

Closed modular chromatic numbers were determined for several classes of regular graphs in [Chartrand et al. 2012]. In particular, it was shown that, for each integer $k \geq 2$, if G is a regular complete k-partite graph such that each of its partite sets has at least $2k + 1$ vertices, then $\overline{\text{mc}}(G) \leq 2\chi(G) - 1$ and this bound is sharp.

In [Phinezy and Zhang 2013], we investigated the closed modular chromatic number for trees and determined it for several classes of trees. For each tree T in these classes, either $\overline{\text{mc}}(T) = 2$ or $\overline{\text{mc}}(T) = 3$. Indeed, this is conjectured to be true in great generality:

Conjecture 1.2 [Phinezy and Zhang 2013]. *If T is a tree of order* 3 *or more*, *then* $\overline{\text{mc}}(T) \leq 3$.

In the paper cited, we showed that Conjecture 1.2 is true if 3 is replaced by 4. In this work, we investigate the closed modular chromatic numbers of rooted trees and confirm Conjecture 1.2 for several classes of rooted trees, including well-studied complete r-ary trees. We refer to [Chartrand et al. 2011] for graph theory notation and terminology not described in this paper. All trees under consideration in this work are rooted trees of order at least 3.

2. Rooted trees

Let T be a rooted tree of order at least 3 having the root v. For each integer i with $0 \le i \le e(v)$, where $e(v)$ is the distance between v and a vertex farthest from v, let

$$V_i = \{x \in V(T) : d(v, x) = i\}.$$

If $x \in V_i$ where $0 \le i \le e(v)$, then x is *at level* i. If $x \in V_i$ $(0 \le i \le e(v) - 1)$ is adjacent to $y \in V_{i+1}$, then x is the *parent* of y and y is a *child* of x. A vertex z is a *descendant* of x (and x is an *ancestor* of z) if the $x - z$ path in T lies below x. In this section, we show that, if T is a rooted tree of order at least 3 such that the numbers of children of all vertices of T have the same parity and no vertex of T has exactly one child, then either $\overline{\mathrm{mc}}(T) = 2$ or $\overline{\mathrm{mc}}(T) = 3$. In order to do this, we first present a result on a special class of trees, which was established in [Phinezy and Zhang 2013]. A *caterpillar* is a tree of order 3 or more, the removal of whose end-vertices produces a path called the *spine* of the caterpillar. Thus every path and star (of order at least 3) and every double star (a tree of diameter 3) is a caterpillar.

Theorem 2.1. *If T is a caterpillar of order at least* 3, *then* $\overline{\mathrm{mc}}(T) \le 3$.

Theorem 2.2. *Let T be a rooted tree of order at least* 3.

(a) *If each vertex of T has an even number of children*, *then* $\overline{\mathrm{mc}}(T) = 2$.

(b) *If each vertex of T has either no child or an odd number of children and no vertex has exactly one child*, *then* $\overline{\mathrm{mc}}(T) \le 3$.

Proof. Suppose that v is the root of T. For each integer i with $0 \le i \le e(v)$, let

$$V_i = \{x \in V(T) : d(v, x) = i\}.$$

To verify (a), define the coloring $c : V(G) \to \mathbb{Z}_2$ by

$$c(x) = \begin{cases} 1 & \text{if } x \in V_i \text{ where } i \equiv 0, 1 \pmod 4, \\ 0 & \text{if } x \in V_i \text{ where } i \equiv 2, 3 \pmod 4. \end{cases} \tag{2}$$

Then $c'(x) = 1$ if $x \in V_i$ and i is even and $c'(x) = 0$ if $x \in V_i$ and i is odd. Thus c is a closed modular 2-coloring and so $\overline{\mathrm{mc}}(T) = 2$ if each vertex of T has an even number of children.

To verify (b), we proceed by strong induction. If T is a star, then $\overline{\mathrm{mc}}(T) \le 3$ by Theorem 2.1. Assume for an integer $n \ge 4$ that, if each vertex of a tree of order at most n has either no child or an odd number of children and no vertex has exactly one child, then the closed modular chromatic number of the tree is at most 3. Let T be a tree of order $n+1$ such that each vertex of T has either no child or an odd number of children and no vertex has exactly one child. We may assume that T is not a star. Let x be a peripheral vertex of T; then x is an end-vertex of T. Suppose that x is a child of the vertex y in T. Since each vertex of T has either no child or an odd number of children and no vertex of T has exactly one child, it follows that y has an odd number $r \geq 3$ of children; say $x = x_1, x_2, \ldots, x_r$ are children of y. Then each child of y is an end-vertex of T. Let $X = \{x = x_1, x_2, \ldots, x_r\}$. Consider $T^* = T - X$ which is a tree of order less than $n+1$ such that each vertex of T^* has either no child or an odd number of children and no vertex of T^* has exactly one child. By the induction hypothesis, T^* has a closed modular 3-coloring $c : V(T^*) \to \mathbb{Z}_3$. Next, we show that T has a closed modular 3-coloring $c_T : V(T) \to \mathbb{Z}_3$ such that $c_T(u) = c(u)$ and $c'_T(u) = c'(u)$ for each $u \in V(T^*)$. Since r is odd, $r \equiv 1, 3, 5 \pmod 6$. We consider these three cases.

Case 1: $r \equiv 1 \pmod 6$. In this case, $r \geq 7$. We define c_T on X such that $c'_T(y) = c'(y)$. If $c(y) \neq c'(y)$, then c_T assigns the color 0 to x_i for $1 \leq i \leq r$. Hence $c'_T(x_i) = c(y) \neq c'(y)$ for $1 \leq i \leq r$. If $c(y) = c'(y)$, then c_T assigns the color 2 to x_1 and x_2 and the color 1 to x_i for $3 \leq i \leq r$. Hence $c'_T(x_i) = c'(y) + 2 \neq c'(y)$ for $i = 1, 2$ and $c'_T(x_i) = c'(y) + 1 \neq c'(y)$ for $3 \leq i \leq r$.

Case 2: $r \equiv 3 \pmod 6$. We define c_T on X such that $c'_T(y) = c'(y)$. If $c(y) \neq c'(y)$, then c_T assigns the color 0 to x_i for $1 \leq i \leq r$. Hence $c'_T(x_i) = c(y) \neq c'(y)$ for $1 \leq i \leq r$. If $c(y) = c'(y)$, then c_T assigns the color 1 to x_i for $1 \leq i \leq r$. Hence $c'_T(x_i) = c'(y) + 1 \neq c'(y)$ for $1 \leq i \leq r$.

Case 3: $r \equiv 5 \pmod 6$. We define c_T on X such that $c'_T(y) = c'(y)$. If $c(y) \neq c'(y)$, then c_T assigns the color 0 to x_i for $1 \leq i \leq r$. Hence $c'_T(x_i) = c(y) \neq c'(y)$ for $1 \leq i \leq r$. If $c(y) = c'(y)$, then c_T assigns the color 2 to x_1 and assigns the color 1 to x_i for $2 \leq i \leq r$. Hence $c'_T(x_1) = c'(y) + 2$ and $c'_T(x_i) = c'(y) + 1$ for $2 \leq i \leq r$.

In each case, c_T is a closed modular 3-coloring of T and so $\overline{\mathrm{mc}}(T) \leq 3$. □

Theorem 2.2 provides the closed modular chromatic numbers for a well-known class of rooted trees. A rooted tree T is a *complete r-ary tree* for some integer $r \geq 2$ if every vertex of T has either r children or no child. The following is a consequence of Theorem 2.2.

Corollary 2.3. *For an integer $r \geq 2$, let T be a complete r-ary tree.*

(a) *If r is even, then* $\overline{\mathrm{mc}}(T) = 2$.

(b) *If r is odd, then* $\overline{\mathrm{mc}}(T) \leq 3$.

In the view of Theorem 2.2, it would be useful to introduce an additional terminology. A rooted tree T of order at least 3 is *even* if every vertex of T has an even number of children, while T is *odd* if every vertex of T has an odd number of children. It then follows by Theorem 2.2 that $\overline{\mathrm{mc}}(T) = 2$ if T is an even rooted tree and $\overline{\mathrm{mc}}(T) \leq 3$ if T is an odd rooted tree and no vertex of T has exactly one child.

3. Odd rooted trees

In this section we investigate the closed modular colorings of odd rooted trees of order at least 3. We will see that, if the locations of leaves of an odd rooted tree T are given, then in some cases it is possible to determine the exact value of $\overline{\mathrm{mc}}(T)$. For each integer $p \in \{0, 1, 2, 3, 4, 5\}$, an odd rooted tree T of order at least 3 having root v is said to be of *type* p if $d(v, u) \equiv p \pmod 6$ for every leaf u in T. We now determine all odd rooted trees of type p were $0 \leq p \leq 5$ that have closed modular chromatic number 2.

Theorem 3.1. *For each integer $p \in \{0, 1, 2, 3, 4, 5\}$, let T be an odd rooted tree of order at least 3 that is of type p. Then $\overline{\mathrm{mc}}(T) = 2$ if and only if $p \neq 1$.*

Proof. Suppose that v is the root of T. For each integer i with $0 \leq i \leq e(v)$, let $V_i = \{x \in V(T) : d(v, x) = i\}$. First, suppose that $0 \leq p \leq 5$ and $p \neq 1$. We show $\overline{\mathrm{mc}}(T) = 2$. Since $\chi(T) = 2$ for every nontrivial tree T, it suffices to construct a closed modular 2-coloring $c : V(T) \to \mathbb{Z}_2$ of T. We consider three cases, according to the values of p.

Case 1: $p = 0$. In this case, a coloring $c : V(T) \to \mathbb{Z}_2$ is defined by

$$c(x) = \begin{cases} 0 & \text{if } x \in V_i \text{ and } i \equiv 0, 1, 5 \pmod 6, \\ 1 & \text{if } x \in V_i \text{ and } i \equiv 2, 3, 4 \pmod 6. \end{cases}$$

Then the induced coloring $c' : V(T) \to \mathbb{Z}_2$ is defined as

$$c'(x) = \begin{cases} 0 & \text{if } x \in V_i \text{ and } i \text{ is even}, \\ 1 & \text{if } x \in V_i \text{ and } i \text{ is odd}. \end{cases} \tag{3}$$

Case 2: $p \equiv 2, 3, 4 \pmod 6$. In this case, a coloring $c : V(T) \to \mathbb{Z}_2$ is defined by

$$c(x) = \begin{cases} 1 & \text{if } x \in V_i \text{ and } i \equiv 0, 1, 2 \pmod 6, \\ 0 & \text{if } x \in V_i \text{ and } i \equiv 3, 4, 5 \pmod 6. \end{cases}$$

Then the induced coloring $c' : V(T) \to \mathbb{Z}_2$ is defined as in (3).

Case 3: $p \equiv 5 \pmod 6$. In this case, a coloring $c : V(T) \to \mathbb{Z}_2$ is defined by

$$c(x) = \begin{cases} 0 & \text{if } x \in V_i \text{ and } i \equiv 0, 4, 5 \pmod 6, \\ 1 & \text{if } x \in V_i \text{ and } i \equiv 1, 2, 3 \pmod 6. \end{cases}$$

Then the induced coloring $c' : V(T) \to \mathbb{Z}_2$ is defined as

$$c'(x) = \begin{cases} 1 & \text{if } x \in V_i \text{ and } i \text{ is even}, \\ 0 & \text{if } x \in V_i \text{ and } i \text{ is odd}. \end{cases}$$

Thus c is a closed modular 2-coloring of T and so $\overline{\mathrm{mc}}(T) = 2$.

For the converse, suppose that T is an odd rooted tree of order at least 3 that is of type 1. Thus, if u is a leaf of T, then $u \in V_k$ for some integer k, where then $1 \le k \le e(v)$ and $k \equiv 1 \pmod 6$. We show that $\overline{\mathrm{mc}}(T) \neq 2$. Assume, to the contrary, that there is a closed modular 2-coloring $c : V(T) \to \mathbb{Z}_2$ of T. Then $c'(v) = 0$ or $c'(v) = 1$. We consider these two cases.

Case 1: $c'(v) = 0$. Thus $c'(x) = 0$ if $x \in V_i$ and i is even and $c'(x) = 1$ if $x \in V_i$ and i is odd. Since $c(v) \in \{0, 1\}$, there are two subcases.

Subcase 1.1: $c(v) = 0$. Since $c'(v) = 0$ and $c(v) = 0$, there is $v_1 \in V_1$ such that $c(v_1) = 0$. Since $c'(v_1) = 1$ and $c(v) = c(v_1) = 0$, there is $v_2 \in V_2$ such that $c(v_2) = 1$. Since $c'(v_2) = 0$, $c(v_1) = 0$ and $c(v_2) = 1$, there is $v_3 \in V_3$ such that $c(v_3) = 1$. Observe for each $i \ge 3$ that $c(v_i)$ is uniquely determined by $c'(v_{i-1})$, $c(v_{i-2})$ and $c(v_{i-1})$. Repeating this procedure, we obtain a path $P_k = (v_1, v_2, \ldots, v_k)$ in T such that (1) v_k is a leaf of T, $d(v, v_i) = i$ for $1 \le i \le k$ and $k \equiv 1 \pmod 6$ and (2) the color sequence $s_c = (c(v_1), c(v_2), \ldots, c(v_k))$ of the coloring c on the path P_k is

$$s_c = (0, \underline{1, 1, 1, 0, 0, 0}, \underline{1, 1, 1, 0, 0, 0}, \ldots, \underline{1, 1, 1, 0, 0, 0}).$$

Hence $(c(v_{k-2}), c(v_{k-1}), c(v_k)) = (0, 0, 0)$. However, then $c'(v_{k-1}) = c'(v_k) = 0$, which is a contradiction.

Subcase 1.2: $c(v) = 1$. By the same argument as in Subcase 1.1, we conclude that there must be a path $P_k = (v_1, v_2, \ldots, v_k)$ in T such that (1) v_k is a leaf of T, $d(v, v_i) = i$ for $1 \le i \le k$ and $k \equiv 1 \pmod 6$ and (2) the color sequence $s_c = (c(v_1), c(v_2), \ldots, c(v_k))$ of the coloring c on the path P_k is

$$s_c = (1, \underline{1, 0, 0, 0, 1, 1}, \underline{1, 0, 0, 0, 1, 1}, \ldots, \underline{1, 0, 0, 0, 1, 1}).$$

Hence $(c(v_{k-2}), c(v_{k-1}), c(v_k)) = (0, 1, 1)$. However, then $c'(v_{k-1}) = c'(v_k) = 0$, which is a contradiction.

Case 2: $c'(v) = 1$. Thus $c'(x) = 1$ if $x \in V_i$ and i is even and $c'(x) = 0$ if $x \in V_i$ and i is odd. Since $c(v) \in \{0, 1\}$, there are two subcases.

Subcase 2.1: $c(v) = 0$. Since $c'(v) = 1$ and $c(v) = 0$, there is $v_1 \in V_1$ such that $c(v_1) = 1$. Since $c'(v_1) = 0$, $c(v) = 0$ and $c(v_1) = 1$, there is $v_2 \in V_2$ such that $c(v_2) = 1$. Since $c'(v_2) = 1$, $c(v_1) = c(v_2) = 1$, there is $v_3 \in V_3$ such that $c(v_3) = 1$. Observe for each $i \ge 3$ that $c(v_i)$ is uniquely determined by $c'(v_{i-1})$, $c(v_{i-2})$ and $c(v_{i-1})$. Repeating this procedure, we obtain a path $P_k = (v_1, v_2, \ldots, v_k)$ in T such that (1) v_k is a leaf of T, $d(v, v_i) = i$ for $1 \le i \le k$ and $k \equiv 1 \pmod 6$ and (2) the color sequence $s_c = (c(v_1), c(v_2), \ldots, c(v_k))$ of the coloring c on the path P_k is

$$s_c = (1, \underline{1, 1, 0, 0, 0, 1}, \underline{1, 1, 0, 0, 0, 1}, \ldots, \underline{1, 1, 0, 0, 0, 1}).$$

Hence $(c(v_{k-2}), c(v_{k-1}), c(v_k)) = (0, 0, 1)$. However, then $c'(v_{k-1}) = c'(v_k) = 1$, which is a contradiction.

Subcase 2.2: $c(v) = 1$. By the same argument as in Subcase 2.1, we conclude that there must be a path $P_k = (v_1, v_2, \ldots, v_k)$ in T such that (1) v_k is a leaf of T, $d(v, v_i) = i$ for $1 \le i \le k$ and $k \equiv 1 \pmod 6$ and (2) the color sequence $s_c = (c(v_1), c(v_2), \ldots, c(v_k))$ of the coloring c on the path P_k is

$$s_c = (0, \underline{1, 0, 1, 0, 1, 0}, \underline{1, 0, 1, 0, 1, 0}, \ldots, \underline{1, 0, 1, 0, 1, 0}).$$

Hence $(c(v_{k-2}), c(v_{k-1}), c(v_k)) = (0, 1, 0)$. However, then $c'(v_{k-1}) = c'(v_k) = 1$, which is a contradiction. □

By Theorem 3.1, if T is an odd rooted tree of order at least 3 that is of type 1, then $\overline{\mathrm{mc}}(T) \ge 3$. On the other hand, every odd rooted tree of order at least 3 we have encountered that is of type 1 has closed modular chromatic number 3. Furthermore, the following is a consequence of Theorems 2.2 and 3.1.

Corollary 3.2. *If T is an odd rooted tree of order at least 3 that is of type 1 such that no vertex has exactly one child, then $\overline{\mathrm{mc}}(T) = 3$.*

By Theorem 3.1, if p is an integer with $0 \le p \le 5$ and $p \ne 1$, then every odd rooted tree of order at least 3 that is of type p has closed modular chromatic number 2. This gives rise to the question:

> If $S \subseteq \{0, 1, 2, 3, 4, 5\}$ where $|S| \ge 2$ and $1 \notin S$ and T is an odd rooted tree of order at least 3 having root v such that, for every leaf u in T, $d(v, u) \equiv p \pmod 6$ for some $p \in S$, then is it necessary that $\overline{\mathrm{mc}}(T) = 2$?

The answer to this question is no, as we show next. First, it will be convenient to introduce an additional definition. For a nonempty subset $S \subseteq \{0, 2, 3, 4, 5\}$, an odd rooted tree T having root v is said to be of *type* S if, for every leaf u in T, $d(v, u) \equiv p \pmod 6$ for some $p \in S$ and, for each $p \in S$, there is at least one leaf u in T such that $d(v, u) \equiv p \pmod 6$. In particular, if $S = \{p\}$ where $p \in \{0, 2, 3, 4, 5\}$, then T is of type p. We first consider odd rooted trees of type S, where $S = \{2, 5\}$ or $S = \{0, 3\}$. In the next two results, we show that if $S = \{2, 5\}$ or $S = \{0, 3\}$, then it is possible for an odd rooted tree T of type S to have $\overline{\mathrm{mc}}(T) = 3$.

Theorem 3.3. *For $S = \{2, 5\}$, there are odd rooted trees of type S such that $\overline{\mathrm{mc}}(T) = 3$.*

Proof. Consider the tree T in Figure 2, each of whose leaves are at level 2 or at level 5. We show $\overline{\mathrm{mc}}(T) = 3$. For each integer i with $0 \le i \le 5$, let $V_i = \{x \in V(T) : d(v, x) = i\}$. Thus, if x is a leaf of T, then $x \in V_2$ or $x \in V_5$. By Corollary 2.3, $\overline{\mathrm{mc}}(T) \le 3$. It remains to show that $\overline{\mathrm{mc}}(T) \ne 2$. Assume, to the contrary, that there is a closed modular 2-coloring $c : V(T) \to \mathbb{Z}_2$ of T. Thus $c(v) = 0$ or $c(v) = 1$. We consider these two cases.

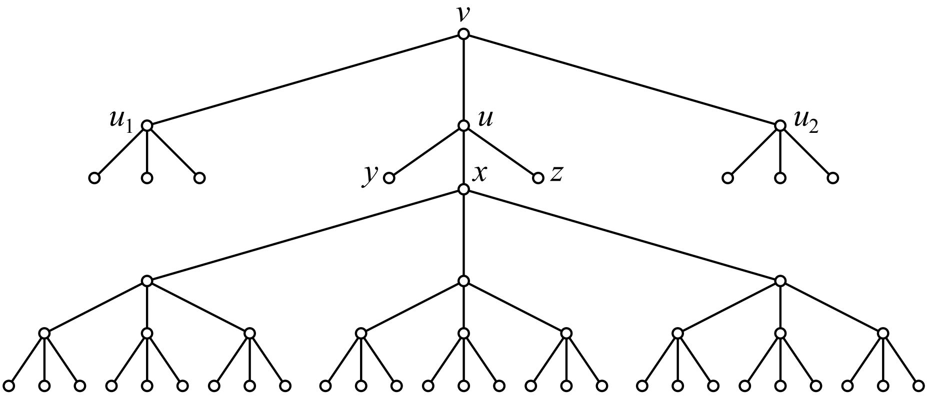

Figure 2. A tree T with $\overline{\mathrm{mc}}(T) = 3$.

Case 1: $c(v) = 0$. Since either $c'(v) = 0$ or $c'(v) = 1$, there are two subcases.

Subcase 1.1: $c'(v) = 0$. Thus $c'(w) = 0$ if $w \in V_i$ and i is even and $c'(w) = 1$ if $w \in V_i$ and i is odd. Furthermore, $c(u_1) = 0$ or $c(u_1) = 1$. First, assume that $c(u_1) = 0$. Since $c'(u_1) = 1$ and $c(v) = 0$, there is a child w of u_1 such that $c(w) = 1$. However, then $c'(w) = c'(u_1) = 1$, a contradiction. Next, assume that $c(u_1) = 1$. Since $c'(u_1) = 1$ and $c(v) = 0$, there is a child w of u_1 such that $c(w) = 0$. However, then $c'(w) = c'(u_1) = 1$, a contradiction.

Subcase 1.2: $c'(v) = 1$. Thus $c'(w) = 0$ if $w \in V_i$ and i is odd and $c'(w) = 1$ if $w \in V_i$ and i is even. Furthermore, $c(u_1) = 1$ or $c(u_1) = 0$. First, assume that $c(u_1) = 1$. Since $c'(u_1) = 0$ and $c(v) = 0$, there is a child w of u_1 such that $c(w) = 1$. However, then $c'(w) = c'(u_1) = 0$, a contradiction. Next, assume that $c(u_1) = 0$. Since $c'(u_1) = 0$ and $c(v) = c(u_1) = 0$, there is a child w of u_1 such that $c(w) = 0$. However, then $c'(w) = c'(u_1) = 0$, a contradiction.

Case 2: $c(v) = 1$. Since either $c'(v) = 0$ or $c'(v) = 1$, there are two subcases.

Subcase 2.1: $c'(v) = 0$. Then $c(u) = 1$ or $c(u) = 0$. First, assume that $c(u) = 1$. Since $c'(u) = 1$ and $c(v) = 1$, there is a child w of u such that $c(w) = 1$. We claim that $c(y) \neq 0$ and $c(z) \neq 0$, for otherwise, say $c(y) = 0$. Then $c'(u) = c'(y) = 1$, a contradiction. Thus $c(y) = c(z) = 1$, as claimed, which implies that $c(x) = 1$. Since $c'(x) = 0$, there is a child w of x such that $c(w) = 0$. Since $c'(w) = 1$, there is a child w_1 of w such that $c(w_1) = 0$. Since $c'(w_1) = 0$, there is a child w_2 of w_1 such that $c(w_2) = 0$. However, then $c'(w_1) = c'(w_2) = 0$, a contradiction. Next, assume that $c(u) = 0$. We saw that $c(y) \neq 1$ and $c(z) \neq 1$ and so $c(y) = c(z) = 0$. Since $c'(u) = 1$, it follows that $c(x) = 0$. Since $c'(x) = 1$, there is a child w of x such that $c(w) = 0$. Since $c'(w) = 1$, there is a child w_1 of w such that $c(w_1) = 1$.

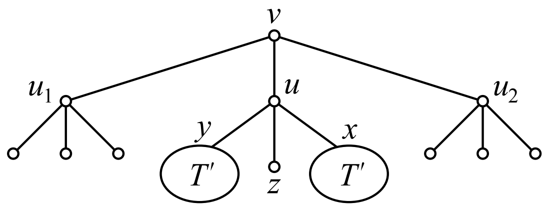

Figure 3. A tree T^* of type $S = \{2, 5\}$ with $\overline{\text{mc}}(T^*) = 2$.

Since $c'(w_1) = 0$, there is a child w_2 of w_1 such that $c(w_2) = 1$. However, then $c'(w_1) = c'(w_2) = 0$, a contradiction.

Subcase 2.2: $c'(v) = 1$. Then $c(u) = 1$ or $c(u) = 0$. We consider these two possibilities.

Subcase 2.2.1: $c(u) = 1$. Now either $c(x) = 0$ or $c(x) = 1$. First assume that $c(x) = 0$. Since $c'(x) = 1$ and $c(u) = 1$, there is a child w of x such that $c(w) = 0$. Since $c'(w) = 0$ and $c(x) = 0$, there is a child w_1 of w such that $c(w_1) = 0$. Since $c'(w_1) = 1$ and $c(w_1) = 0$, there is a child w_2 of w_1 such that $c(w_2) = 1$. However, then $c'(w_1) = c'(w_2) = 1$, a contradiction. Next, assume that $c(x) = 1$. Since $c(u) = 1$ and $c'(u) = 0$, one of y and z must be colored 1, say $c(y) = 1$. However, then $c'(y) = c'(u) = 0$, a contradiction.

Subcase 2.2.2: $c(u) = 0$. Now either $c(x) = 0$ or $c(x) = 1$. First assume that $c(x) = 0$. Since $c'(u) = 0$, exactly one of y and z is colored 1, say $c(y) = 1$ and $c(z) = 0$. However, then $c'(z) = c'(u) = 0$, a contradiction. Next, assume that $c(x) = 1$. Since $c'(x) = c(x) = 1$, there is a child w of x such that $c(w) = 0$. Since $c'(w) = 0$, $c(x) = 1$ and $c(w) = 0$, there is a child w_1 of w such that $c(w_1) = 1$. Since $c'(w_1) = 1$, there is a child w_2 of w_1 such that $c(w_2) = 0$. However, then $c'(w_1) = c'(w_2) = 1$, a contradiction. □

By Theorem 3.3, despite the fact that every odd rooted tree of type 2 or type 5 has closed modular chromatic number 2, there are odd rooted trees T of type $S = \{2, 5\}$ for which $\overline{\text{mc}}(T) = 3$. On the other hand, there are odd rooted trees of type $S = \{2, 5\}$ having closed modular chromatic number 2. For example, we start with the tree T in Figure 2. Let T' be the subtree of T whose vertex set consists of x and all descendants of x. Then the tree T^* is constructed from T of Figure 2 by replacing y with a copy of T' (see Figure 3). The coloring $c : V(T) \to \mathbb{Z}_2$ defined by assigning the color 0 to each vertex in $\{u, u_1, u_2, x, y\}$ and assigning the color 1 to the remaining vertices of T^* is a closed modular 2-coloring. Therefore, $\overline{\text{mc}}(T^*) = 2$.

Theorem 3.4. *For $S = \{0, 3\}$, there are odd trees T of type S such that $\overline{\text{mc}}(T) = 3$.*

Proof. Consider the tree T of Figure 4, each of whose leaves are at level 3 or at level 6. We show that $\overline{\text{mc}}(T) = 3$. For each integer i with $0 \le i \le 6$, let

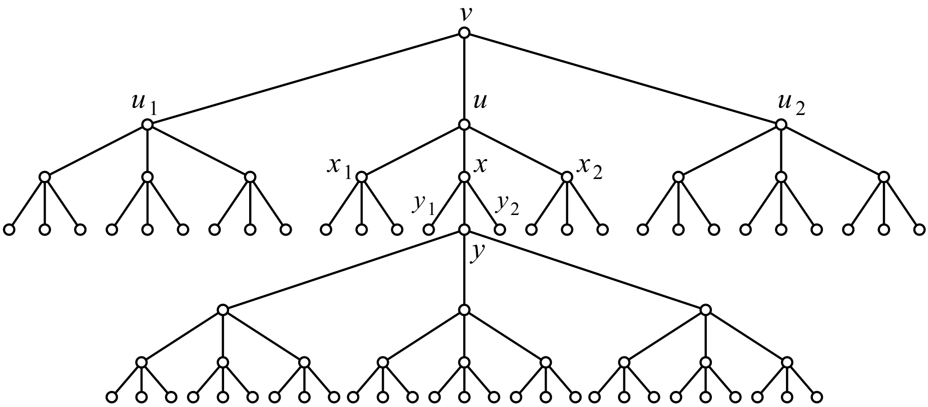

Figure 4. A tree T with $\overline{\mathrm{mc}}(T) = 3$.

$V_i = \{x \in V(T) : d(v, x) = i\}$. If x is a leaf of T, then $x \in V_3$ or $x \in V_6$. By Corollary 2.3, $\overline{\mathrm{mc}}(T) \leq 3$. Thus it remains to show that $\overline{\mathrm{mc}}(T) \neq 2$. Assume, to the contrary, that there is a closed modular 2-coloring $c : V(T) \to \mathbb{Z}_2$ of T. Thus $c(v) = 0$ or $c(v) = 1$. We consider these two cases.

Case 1: $c(v) = 0$. Since either $c'(v) = 0$ or $c'(v) = 1$, there are two subcases.

Subcase 1.1: $c'(v) = 0$. In this case, there is a child of v that is colored 0. First, assume that $c(u_1) = 0$. Since $c'(u_1) = 1$, there is a child w of u_1 such that $c(w) = 1$. Since $c'(w) = 0$ and $c(w) = 1$, there is a child w_1 of w such that $c(w_1) = 1$. However, then $c'(w_1) = c'(w) = 0$, a contradiction. Thus $c(u_1) = 1$ and, similarly, $c(u_2) = 1$. This implies that $c(u)$ must be 0. Note that $c(x_1) = 1$ or $c(x_1) = 0$. If $c(x_1) = 1$, then there is a child w of x_1 such that $c(w) = 1$. However, then $c'(x_1) = c'(w) = 0$, a contradiction. If $c(x_1) = 0$, then there is a child w of x_1 such that $c(w) = 0$. However, then $c'(x_1) = c'(w) = 0$, a contradiction.

Subcase 1.2: $c'(v) = 1$. Since $c(v) = 0$, either exactly one or exactly three children of v must be colored 1. First, suppose that $c(u_1) = 0$. Then there is a child w of u_1 such that $c(w) = 0$. Since $c'(w) = 1$, there is a child w_1 of w such that $c(w_1) = 1$. However, then $c'(w_1) = c'(w) = 1$, a contradiction. Thus $c(u_1) = 1$ and, similarly, $c(u_2) = 1$. This implies that $c(u)$ must be 1. Note that $c(x) = 1$ or $c(x) = 0$. We consider these two subcases.

Subcase 1.2.1: $c(x) = 1$. In this case, either exactly one or exactly three children of x must be colored 1. Since $c'(x) = 1$, it follows $c(y_1) = 1$ (for otherwise, $c'(y_1) = 1$). Similarly $c(y_2) = 1$. Thus $c(y)$ must be 1. Since $c'(y) = 0$, there is a child w of y such that $c(w) = 0$. Since $c'(w) = 1$, there is a child w_1 of w such that $c(w_1) = 0$. This in turn implies that there is a child w_2 of w_1 such that $c(w_2) = 0$. However, then $c'(w_1) = c'(w_2) = 0$, a contradiction.

Subcase 1.2.2: $c(x)=0$. Since $c'(x)=1$, it follows that $c(y_1)=c(y_2)=0$, which implies that $c(y)=0$. Since $c'(y)=c(y)=0$, there is a child w of y such that $c(w)=0$. Since $c'(w)=1$, there is a child w_1 of w such that $c(w_1)=1$. This in turn implies that there is a child w_2 of w_1 such that $c(w_2)=1$. However, then $c'(w_1)=c'(w_2)=0$, a contradiction.

Case 2: $c(v)=1$. Since either $c'(v)=0$ or $c'(v)=1$, there are two subcases.

Subcase 2.1: $c'(v)=0$. Note that $c(u_1)=0$ or $c(u_1)=1$. First, assume that $c(u_1)=0$. Since $c'(u_1)=1$, there is a child w of u_1 such that $c(w)=0$. Since $c'(w)=0$, there is a child w_1 of w such that $c(w_1)=0$. However, then $c'(w_1)=c'(w)=0$, a contradiction. Thus $c(u_1)=1$ and, similarly, $c(u_2)=1$. This implies that $c(u)$ must be 1. Note that $c(x)=0$ or $c(x)=1$. There are two subcases.

Subcase 2.1.1: $c(x)=0$. If $c(y_1)=0$, then $c'(y)=c'(x)=0$, a contradiction. Thus $c(y_1)=1$ and, similarly, $c(y_2)=1$. This implies that $c(y)$ must be 1. Since $c'(y)=1$, there is a child w of y such that $c(w)=0$. Since $c'(w)=0$ and $c(y)=1$, there is a child w_1 of w such that $c(w_1)=1$. This in turn implies that there is a child w_2 of w_1 such that $c(w_2)=0$. However, then $c'(w_1)=c'(w_2)=1$, a contradiction.

Subcase 2.1.2: $c(x)=1$. If $c(y_1)=1$, then $c'(y)=c'(x)=0$, a contradiction. Thus $c(y_1)=0$ and, similarly, $c(y_2)=0$. This implies that $c(y)$ must be 0. Since $c'(y)=1$ and $c(x)=1$, there is a child w of y such that $c(w)=0$. Since $c'(w)=0$, there is a child w_1 of w such that $c(w_1)=0$. This in turn implies that there is a child w_2 of w_1 such that $c(w_2)=0$. However, then $c'(w_1)=c'(w_2)=1$, a contradiction.

Subcase 2.2: $c'(v)=1$. Note that $c(u_1)=0$ or $c(u_1)=1$. First, assume that $c(u_1)=0$. Since $c'(u_1)=c(u_1)=0$ and $c(v)=1$, there is a child w of u_1 such that $c(w)=1$. Since $c'(w)=1$, there is a child w_1 of w such that $c(w_1)=0$. However, then $c'(w_1)=c'(w)=1$, a contradiction. Thus $c(u_1)=1$ and, similarly, $c(u_2)=1$. This implies that $c(u)$ must be 0. Note that $c(x_1)=0$ or $c(x_1)=1$. Furthermore, $c'(x_1)=1$. If $c(x_1)=0$, then there is a child w of x_1 such that $c(w)=1$. However, then $c'(x_1)=c'(w)=1$, a contradiction. If $c(x_1)=1$, then there is a child w of x_1 such that $c(w)=0$. However, then $c'(x_1)=c'(w)=1$, a contradiction. □

As with the case when $S=\{2,5\}$, there are odd rooted trees of type $S=\{0,3\}$ having closed modular chromatic number 2. For example, we start with the tree T in Figure 4. Let T' be the subtree of T whose vertex set consists of y and all descendants of y. Then the tree T^* is constructed from T of Figure 4 by replacing y_1 with a copy of T' as we did in the case when $S=\{2,5\}$ (see Figure 3). The coloring $c: V(T)\to \mathbb{Z}_2$ defined by assigning the color 0 to each vertex in $\{v, y, y_1\}\cup V_5$ and assigning the color 1 to the remaining vertices of T^* is a closed modular 2-coloring. Therefore, $\overline{\mathrm{mc}}(T^*)=2$.

Next, we show that, if S is a nonempty subset of $\{0, 2, 3, 4, 5\}$ such that S contains at most one of 2 and 5 and at most one of 0 and 3, then every odd rooted tree of type S has closed modular chromatic number 2.

Theorem 3.5. *Let S be a nonempty subset of $\{0, 2, 3, 4, 5\}$ such that S contains at most one of 2 and 5 and at most one of 0 and 3. If T is an odd rooted tree of order at least 3 that is of type S, then $\overline{\mathrm{mc}}(T) = 2$.*

Proof. By Theorem 3.1, we may assume that $|S| \ge 2$. Since $|S \cap \{2, 5\}| \le 1$ and $|S \cap \{0, 3\}| \le 1$, it follows that $|S| \le 3$. Thus we consider two cases, according to whether $|S| = 3$ or $|S| = 2$.

Case 1: $|S| = 3$. Then S is one of the sets $\{0, 2, 4\}$, $\{0, 4, 5\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$. Since $\chi(T) = 2$ for every nontrivial tree T, it suffices to show that there is a closed modular 2-coloring by Proposition 1.1. For each integer i with $0 \le i \le e(v)$, let

$$V_i = \{x \in V(T) : d(v, x) = i\}.$$

First, suppose that $S = \{0, 2, 4\}$. Define a coloring $c : V(T) \to \mathbb{Z}_2$ by

$$c(x) = \begin{cases} 0 & \text{if } x \in V_i \text{ and } i \text{ is odd,} \\ 1 & \text{if } x \in V_i \text{ and } i \text{ is even.} \end{cases}$$

Then $c'(x) = c(x)$ for each $x \in V(T)$. Next, suppose that S is one of $\{0, 4, 5\}$ and $\{2, 3, 4\}$. If $S = \{0, 4, 5\}$, then define a coloring $c : V(T) \to \mathbb{Z}_2$ by

$$c(x) = \begin{cases} 0 & \text{if } x \in V_i \text{ where } i \equiv 0, 1, 5 \pmod 6, \\ 1 & \text{if } x \in V_i \text{ where } i \equiv 2, 3, 4 \pmod 6. \end{cases}$$

If $S = \{2, 3, 4\}$, then define a coloring $c : V(T) \to \mathbb{Z}_2$ by

$$c(x) = \begin{cases} 0 & \text{if } x \in V_i \text{ where } i \equiv 3, 4, 5 \pmod 6, \\ 1 & \text{if } x \in V_i \text{ where } i \equiv 0, 1, 2 \pmod 6. \end{cases}$$

In either case, $c'(x) = 0$ if $x \in V_i$ and i is even and $c'(x) = 1$ if $x \in V_i$ and i is odd. Finally, suppose that $S = \{3, 4, 5\}$. Define a coloring $c : V(T) \to \mathbb{Z}_2$ by

$$c(x) = \begin{cases} 0 & \text{if } x \in V_i \text{ where } i \equiv 0, 4, 5 \pmod 6, \\ 1 & \text{if } x \in V_i \text{ where } i \equiv 1, 2, 3 \pmod 6. \end{cases}$$

Then $c'(x) = 0$ if $x \in V_i$ and i is odd and $c'(x) = 1$ if $x \in V_i$ and i is even. In each case, c is a closed modular 2-coloring of T and so $\overline{\mathrm{mc}}(T) = 2$.

Case 2: $|S| = 2$. Then S is a 2-element subset of one of the sets $\{0, 2, 4\}$, $\{0, 4, 5\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$ in Case 1. Observe that the closed modular 2-colorings described in Case 1 will provide closed modular 2-colorings for this case. For example, if S is a 2-element subset of $S' = \{0, 2, 4\}$, then a closed modular 2-coloring of a tree of type S' described in Case 1 provides a closed modular 2-coloring of T. Therefore, $\overline{\mathrm{mc}}(T) = 2$ in this case as well. □

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