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Positive solutions to singular third-order boundary value problems on purely discrete time scales

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We study singular discrete third-order boundary value problems with mixed boundary conditions of the form

$$-u^{\Delta\Delta\Delta}(t_{i-2}) + f\big(t_i, u(t_i), u^{\Delta}(t_{i-1}), u^{\Delta\Delta}(t_{i-2})\big) = 0,$$
$$u^{\Delta\Delta}(t_0) = u^{\Delta}(t_{n+1}) = u(t_{n+2}) = 0,$$

over a finite discrete interval $\{t_0, t_1, \ldots, t_n, t_{n+1}, t_{n+2}\}$. We prove the existence of a positive solution by means of the lower and upper solutions method and the Brouwer fixed point theorem in conjunction with perturbation methods to approximate regular problems.

1. Preliminaries

This paper is something of an extension of [Rachůnková and Rachůnek 2006] and [Kunkel 2006; 2008]. Rachůnková and Rachůnek studied a second-order singular boundary value problem for the discrete p-Laplacian, $\phi_p(x) = |x|^{p-2}x$, $p > 1$. In particular, they dealt with the discrete boundary value problem

$$\Delta\big(\phi_p(\Delta u(t-1))\big) + f(t, u(t), \Delta u(t-1)) = 0, \quad t \in [1, T+1],$$
$$\Delta u(0) = u(T+2) = 0,$$

in which $f(t, x_1, x_2)$ was singular in x_1. In [Kunkel 2006] this was extended to the third-order case, but only for $p = 2$; that is, boundary value problem treated was

$$-\Delta\Delta\Delta u(t-2) + f(t, u(t), \Delta u(t-1), \Delta\Delta u(t-2)) = 0, \quad t \in [2, T+1],$$
$$\Delta\Delta u(0) = \Delta u(T+2) = u(T+3) = 0.$$

In [Kunkel 2008], by contrast, the extension was to a second-order singular discrete

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boundary value problem with nonuniform step size:

$$u^{\Delta\Delta}(t_{i-1}) + f(t_i, u(t_i), u^{\Delta}(t_{i-1})) = 0, \quad t_i \in [2, T+1],$$
$$u^{\Delta}(t_0) = u(t_{n+1}) = 0.$$

The analysis in the present paper relies heavily on a lower and upper solutions method in conjunction with an application of the Brouwer fixed point theorem [Zeidler 1986]. We consider only the singular third-order boundary value problem, while letting our function range over a discrete interval with nonuniform step size. We will provide definitions of appropriate lower and upper solutions. The lower and upper solutions will be applied to nonsingular perturbations of our nonlinear problem, ultimately giving rise to our boundary value problem by passing to the limit.

Various forms of the lower and upper solutions method have been used extensively in establishing solutions of boundary value problems for finite difference equations. Examples include [Henderson and Kunkel 2006; Kunkel 2006; Rachůnková and Rachůnek 2006]; we mention especially [Jiang et al. 2005], which deals with singular discrete boundary value problems using the method. Other outstanding works where lower and upper solution methods have been employed to obtain solutions of boundary value problems for finite difference equations include [Agarwal et al. 1999; 2003; 2004; 2005; Agarwal and Wong 1997; Cabada 2011; Henderson and Thompson 2002; Kelley and Peterson 2001; O'Regan and El-Gebeily 2008; Pao 1985; Peterson et al. 2004; Zhang et al. 2002].

Singular discrete boundary value problems also have received a good deal of attention. As representative works, we suggest [Agarwal et al. 1999; 2005; 2008; Agarwal and Wong 1997; Akın-Bohner et al. 2003; Atici et al. 2003; Jódar 1987; Jódar et al. 1992; Naidu and Kailasa Rao 1982; Peterson et al. 2004; Rachůnková and Rachůnek 2009; Yuan et al. 2008; Zheng et al. 2011; Zhang et al. 2002].

We now state the definitions that are used in the remainder of the paper.

Definition 1.1. For $0 \le i \le n+2$, let $t_i \in \mathbb{R}$, where $t_0 < t_1 < \cdots < t_{n+1} < t_{n+2}$. Define the discrete intervals

$$\mathbb{T} := [t_0, t_{n+2}] = \{t_0, t_1, \ldots, t_{n+1}, t_{n+2}\},$$
$$\mathbb{T}^{\circ} := [t_2, t_{n+1}] = \{t_2, t_3, \ldots, t_n, t_{n+1}\}.$$

Definition 1.2. For the function $u : \mathbb{T} \to \mathbb{R}$, define the delta derivative [Bohner and Peterson 2001], u^{Δ}, by

$$u^{\Delta}(t_i) := \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i}, \quad t_i \in \mathbb{T}^{\circ} \cup \{t_0, t_{n+1}\}.$$

We make note that $u^{\Delta\Delta}(t_i) = (u^{\Delta})^{\Delta}(t_i)$.

Consider the third-order nonlinear discrete dynamic

$$u^{\Delta\Delta\Delta}(t_{i-2}) + f(t_i, u(t_i), u^{\Delta}(t_{i-1}), u^{\Delta\Delta}(t_{i-2})) = 0, \quad t_i \in \mathbb{T}^{\circ}, \tag{1}$$

with mixed boundary conditions

$$u^{\Delta\Delta}(t_0) = u^{\Delta}(t_{n+1}) = u(t_{n+2}) = 0. \tag{2}$$

Our goal is to prove the existence of a positive solution of problem (1), (2).

Definition 1.3. By a solution of problem (1), (2), we mean a function $u : \mathbb{T}^{\circ} \to \mathbb{R}$ such that u satisfies the discrete dynamic (1) on $\mathbb{T}^{\circ}$ and the boundary conditions (2). If $u(t) > 0$ for $t \in \mathbb{T}^{\circ}$, we say u is a positive solution of the problem (1), (2).

Definition 1.4. Let $\mathfrak{D} \subseteq \mathbb{R}^3$. We say that f is continuous on $\mathbb{T} \times \mathfrak{D}$ if $f(t_i, x, y, z)$ is defined on $t_i \in \mathbb{T}^{\circ}$ and $(x, y, z) \in \mathfrak{D}$, and if $f(t_i, x, y, z)$ is continuous on $\mathfrak{D}$ for each $t_i \in \mathbb{T}^{\circ}$.

We make the following assumptions throughout:

(A) $\mathfrak{D} = (0, \infty) \times \mathbb{R}^2$.

(B) f is continuous on $\mathbb{T}^{\circ} \times \mathfrak{D}$.

(C) $f(t_i, x, y, z)$ has a singularity at $x = 0$; i.e., $\limsup_{x \to 0^+} |f(t_i, x, y, z)| = \infty$ for each $t_i \in \mathbb{T}^{\circ}$ and for some $(y, z) \in \mathbb{R}^2$.

2. Lower and upper solutions method for regular problems

Let us first consider the regular difference equation

$$u^{\Delta\Delta\Delta}(t_{i-2}) + h(t_i, u(t_i), u^{\Delta}(t_{i-1}), u^{\Delta\Delta}(t_{i-2})) = 0, \quad t_i \in \mathbb{T}^{\circ}, \tag{3}$$

where h is continuous on $\mathbb{T}^{\circ} \times \mathbb{R}^3$, along with the boundary conditions (2). We establish a lower and upper solutions method for the regular problem (3), (2).

Definition 2.1. We call $\alpha : \mathbb{T} \to \mathbb{R}$ a lower solution of (3), (2) if

$$\alpha^{\Delta\Delta\Delta}(t_{i-2}) + h(t_i, \alpha(t_i), \alpha^{\Delta}(t_{i-1}), \alpha^{\Delta\Delta\Delta}(t_{i-2})) \geq 0, \quad t_i \in \mathbb{T}^{\circ} \tag{4}$$

and α satisfies boundary conditions

$$\begin{aligned} \alpha^{\Delta\Delta}(t_0) &\leq 0, \\ \alpha^{\Delta}(t_{n+1}) &\geq 0, \\ \alpha(t_{n+2}) &\leq 0. \end{aligned} \tag{5}$$

Definition 2.2. We call $\beta : \mathbb{T} \to \mathbb{R}$ an upper solution of (3), (2) if

$$\beta^{\Delta\Delta\Delta}(t_{i-2}) + h(t_i, \beta(t_i), \beta^{\Delta}(t_{i-1}), \beta^{\Delta\Delta}(t_{i-2})) \le 0, \quad t_i \in \mathbb{T}^{\circ} \tag{6}$$

and β satisfies boundary conditions

$$\begin{aligned} \beta^{\Delta\Delta}(t_0) &\ge 0, \\ \beta^{\Delta}(t_{n+1}) &\le 0, \\ \beta(t_{n+2}) &\ge 0. \end{aligned} \tag{7}$$

Theorem 2.1 (lower and upper solutions method). *Let α and β be lower and upper solutions of* (3), (2), *respectively, with $\alpha \le \beta$ on $\mathbb{T}^{\circ}$. Let $h(t_i, x, y, z)$ be continuous on $\mathbb{T}^{\circ} \times \mathbb{R}^3$ and nonincreasing in its z variable. Then* (3), (2) *has a solution u satisfying*

$$\alpha(t) \le u(t) \le \beta(t), \quad t \in \mathbb{T}.$$

Proof. We proceed through a sequence of steps involving modifications of the function h.

Step 1. For $t_i \in \mathbb{T}^{\circ}$, $(x, y, z) \in \mathbb{R}^3$, define

$$\tilde{h}\left(t_i, x, y, \frac{y-z}{t_{i-1}-t_{i-2}}\right) = \begin{cases} h\left(t_i, \beta(t_i), \beta^{\Delta}(t_{i-1}), \dfrac{\beta^{\Delta}(t_{i-1})-\sigma(t_{i-1}, z)}{t_{i-1}-t_{i-2}}\right) + \dfrac{\beta^{\Delta}(t_{i-1})-y}{\beta^{\Delta}(t_{i-1})-y+1}, & y < \beta^{\Delta}(t_{i-1}), \\ h\left(t_i, x, y, \dfrac{y-\sigma(t_{i-2}, z)}{t_{i-1}-t_{i-2}}\right), & \beta^{\Delta}(t_{i-1}) \le y \le \alpha^{\Delta}(t_{i-1}), \\ h\left(t_i, \alpha(t_i), \alpha^{\Delta}(t_{i-1}), \dfrac{\alpha^{\Delta}(t_{i-1})-\sigma(t_{i-1}, z)}{t_{i-1}-t_{i-2}}\right) + \dfrac{y-\alpha^{\Delta}(t_{i-1})}{y-\alpha^{\Delta}(t_{i-1})+1}, & y > \alpha^{\Delta}(t_{i-1}), \end{cases}$$

where

$$\sigma(t_{i-2}, z) = \begin{cases} \alpha^{\Delta}(t_{i-2}), & z > \alpha^{\Delta}(t_{i-2}), \\ z, & \beta^{\Delta}(t_{i-2}) \le z \le \alpha^{\Delta}(t_{i-2}), \\ \beta^{\Delta}(t_{i-2}), & z < \beta^{\Delta}(t_{i-2}). \end{cases}$$

By its construction, $\tilde{h}$ is continuous on $\mathbb{T}^{\circ} \times \mathbb{R}^3$ and there exists $M > 0$ so that

$$|\tilde{h}(t_i, x, y, z)| \le M, \quad t_i \in \mathbb{T}^{\circ}, \ (x, y, z) \in \mathbb{R}^3.$$

We now study the auxiliary equation

$$u^{\Delta\Delta\Delta}(t_{i-2}) + \tilde{h}(t_i, u(t_i), u^{\Delta}(t_{i-1}), u^{\Delta\Delta}(t_{i-2})) = 0, \quad t_i \in \mathbb{T}^{\circ}, \tag{8}$$

with boundary conditions (2). Our immediate goal is to prove the existence of a solution of (8), (2).

Step 2. The Brouwer fixed point theorem states that, for

$$K = \{(x_1), \dots, (x_n) : c_i \le x_i \le d_i, i = 1, \dots, n\},$$

if $T : K \to K$ is continuous, then T has a fixed point in K. To this end, define

$$E = \{u : \mathbb{T} \to \mathbb{R} : u^{\Delta\Delta}(t_0) = u^{\Delta}(t_{n+1}) = u(t_{n+2}) = 0\}$$

and also define

$$\|u\| = \max\{|u(t_i)| : t_i \in \mathbb{T}\}.$$

This makes E into a Banach space. We define an operator $\mathcal{T} : E \to E$ by

$$(\mathcal{T}u)(t_m) = -\sum_{k=m}^{n+1}(t_{k+1}-t_k)\sum_{j=k}^{n}(t_{j+1}-t_j)\sum_{i=1}^{j}(t_i-t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big). \quad (9)$$

$\mathcal{T}$ is a continuous operator.

From the bounds placed on $\tilde{h}$ in Step 1 and from (9), if $r > (t_{n+1} - t_0)^3 M$, then $\mathcal{T}(\overline{B(r)}) \subset \overline{B(r)}$, where $B(r) = \{u \in E : \|u\| < r\}$. Therefore, by the Brouwer fixed point theorem [Zeidler 1986], there exists $u \in \overline{B(r)}$ such that $u = \mathcal{T}u$.

Step 3. We now show that u is a fixed point of $\mathcal{T}$ if and only if u is a solution of (8), (2).

First assume $u = \mathcal{T}u$. Then $u \in E$ and thus satisfies (2).

Further,

$$\begin{aligned}
u^{\Delta}(t_{m-2}) &= \frac{u(t_{m-1})-u(t_{m-2})}{t_{m-1}-t_{m-2}} \\
&= -\frac{\sum_{k=m-1}^{n+1}(t_{k+1}-t_k)\sum_{j=k}^{n}(t_{j+1}-t_j)\sum_{i=1}^{j}(t_i-t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big)}{t_m-t_{m-1}} \\
&\quad + \frac{\sum_{k=m-2}^{n+1}(t_{k+1}-t_k)\sum_{j=k}^{n}(t_{j+1}-t_j)\sum_{i=1}^{i}(t_i-t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big)}{t_{m-1}-t_{m-2}} \\
&= \frac{(t_{m-1}-t_{m-2})\sum_{j=m-2}^{n}(t_{j+1}-t_j)\sum_{i=1}^{j}(t_i-t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big)}{t_{m-1}-t_{m-2}} \\
&= \sum_{j=m-2}^{n}(t_{j+1}-t_j)\sum_{i=1}^{j}(t_i-t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big).
\end{aligned}$$

We also have

$$
\begin{aligned}
u^{\Delta\Delta}(t_{m-2}) &= \frac{u^{\Delta}(t_{m-1}) - u^{\Delta}(t_{m-2})}{t_{m-1}-t_{m-2}} \\
&= \frac{\sum_{j=m-1}^{n}(t_{j+1}-t_j)\sum_{i=1}^{j}(t_i-t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big)}{t_{m-1}-t_{m-2}} \\
&\quad - \frac{\sum_{j=m-2}^{n}(t_{j+1}-t_j)\sum_{i=1}^{j}(t_i-t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big)}{t_{m-1}-t_{m-2}} \\
&= -\frac{(t_{m-1}-t_{m-2})\sum_{i=1}^{m-2}(t_i-t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big)}{t_{m-1}-t_{m-2}} \\
&= -\sum_{i=1}^{m-2}(t_i-t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big)
\end{aligned}
$$

and

$$
\begin{aligned}
u^{\Delta\Delta\Delta}(t_{m-2}) &= \frac{u^{\Delta\Delta}(t_{m-1}) - u^{\Delta\Delta}(t_{m-2})}{t_{m-1}-t_{m-2}} \\
&= \frac{-\sum_{i=1}^{m-1}(t_i-t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big)}{t_{m-1}-t_{m-2}} \\
&\quad + \frac{\sum_{i=1}^{i-1}(t_i-t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big)}{t_{m-1}-t_{m-2}} \\
&= \frac{-(t_{m-1}-t_{m-2})\tilde{h}\big(t_m, u(t_m), u^{\Delta}(t_{m-1}), u^{\Delta\Delta}(t_{m-2})\big)}{t_{m-1}-t_{m-2}} \\
&= -\tilde{h}\big(t_m, u(t_m), u^{\Delta}(t_{m-1}), u^{\Delta\Delta}(t_{m-2})\big).
\end{aligned}
$$

This implies that $u^{\Delta\Delta\Delta}(t_{m-2}) + \tilde{h}(t_m, u(t_m), u^{\Delta}(t_{m-1}), u^{\Delta\Delta}(t_{m-2})) = 0$ and, thus, $u(t)$ solves problem (8), (2).

On the other hand, let $u(t)$ solve (8), (2).

Then, for $i = 1, 2, \dots, n$,

$$u^{\Delta\Delta}(t_i) - u^{\Delta\Delta}(t_{i-1}) = (t_i - t_{i-1})u^{\Delta\Delta\Delta}(t_{i-1}),$$

which means, for each $i = 1, 2, \dots, n$,

$$
\begin{aligned}
u^{\Delta\Delta}(t_i) - u^{\Delta\Delta}(t_i - 1) &= (t_i - t_{i-1})u^{\Delta\Delta\Delta}(t_{i-1}) \\
&= -(t_i - t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta}(t_{i-1})\big).
\end{aligned}
$$

By $u^{\Delta\Delta}(t_0) = 0$ and summing the above equations from $i = 1$ to $i = j$, where $j = 1, 2, \ldots, n$, we have

$$u^{\Delta\Delta}(t_j) = -\sum_{i=1}^{j}(t_i - t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big). \tag{10}$$

Also, for $j = 0, 1, \ldots, n$,

$$u^{\Delta}(t_{j+1}) - u^{\Delta}(t_j) = (t_{j+1} - t_j)u^{\Delta\Delta}(t_j).$$

Taking the sum of the above equations from $j = k$ to $j = n$, where $k = 0, 1, \ldots, n$, and by $u^{\Delta}(t_{n+1}) = 0$ and (10), we have

$$u^{\Delta}(t_k) = \sum_{j=k}^{n}(t_{j+1} - t_j)\sum_{i=1}^{j}(t_i - t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big). \tag{11}$$

Similarly, for $k = 0, 1, \ldots, n + 1$,

$$u(t_{k+1}) - u(t_k) = (t_{k+1} - t_k)u^{\Delta}(t_k).$$

Add the above equations from $k = m$ to $k = n + 1$, where $m = 0, 1, \ldots, n + 2$, and by (11) and $u(t_{n+2}) = 0$, we have

$$-\sum_{k=m}^{n+1}(t_{k+1} - (t_k)\sum_{j=k}^{n}(t_{j+1} - t_j)\sum_{i=1}^{j}(t_i - t_{i-1})\tilde{h}\big(t_{i+1}, u(t_{i+1}), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-1})\big).$$

Thus, $u = Tu$ and the claim holds.

Step 4. We now show that solutions $u(t)$ of (8), (2) satisfy

$$\alpha(t) \le u(t) \le \beta(t), \quad t \in \mathbb{T}.$$

Consider the case of obtaining $u(t) \le \beta(t)$. Let $v^{\Delta}(t) = \beta^{\Delta}(t) - u^{\Delta}(t)$. For the sake of establishing a contradiction, assume that

$$\max\{v^{\Delta}(t) : t \in \mathbb{T}\} := v^{\Delta}(l) > 0.$$

From the boundary conditions (2) and (7), we see that $l \equiv l_i \in \mathbb{T}^{\circ}$. Thus, $v^{\Delta}(l_{i+1}) \le v^{\Delta}(l_i)$ and $v^{\Delta}(l_{i-1}) \le v^{\Delta}(l_i)$. Therefore, $v^{\Delta\Delta}(l_i) \le 0$ and $v^{\Delta\Delta}(l_{i-1}) \ge 0$. This in turn implies that $v^{\Delta\Delta\Delta}(l_{i-1}) \le 0$. Consequently,

$$u^{\Delta\Delta\Delta}(l_{i-1}) \ge \beta^{\Delta\Delta\Delta}(l_{i-1}). \tag{12}$$

On the other hand, since h is nonincreasing in its fourth variable, we have from (3) that

$$\begin{aligned}
&\beta^{\Delta\Delta\Delta}(l_{i-1}) - u^{\Delta\Delta\Delta}(l_{i-1})\\
&\quad = \tilde{h}\big(l_{i+1}, u(l_{i+1}), u^{\Delta}(l), u^{\Delta\Delta}(l_{i-1})\big) + \beta^{\Delta\Delta\Delta}(l_{i-1})\\
&\quad = h(l_{i+1}, \beta(l_{i+1}), \beta^{\Delta}(l_i), \frac{\beta^{\Delta}(l_i) - \sigma(l_{i-1}), u(l_{i-1})}{l_i - l_{i-1}}) + \frac{v^{\Delta}(l)}{v^{\Delta}(l)+1} + \beta^{\Delta\Delta\Delta}(l_{i-1})\\
&\quad \geq h(l_{i+1}, \beta(l_{i+1}), \beta^{\Delta}(l), \beta^{\Delta\Delta}(l_{i-1})) + \frac{v^{\Delta}(l)}{v^{\Delta}(l)+1} + \beta^{\Delta\Delta\Delta}(l_{i-1})\\
&\quad \geq -\beta^{\Delta\Delta\Delta}(l_{i-1}) + \frac{v^{\Delta}(l)}{v^{\Delta}(l)+1} + \beta^{\Delta\Delta\Delta}(l_{i-1})\\
&\quad = \frac{v^{\Delta}(l)}{v^{\Delta}(l)+1} > 0.
\end{aligned}$$

Hence, $u^{\Delta\Delta\Delta}(l_{i-1}) < \beta^{\Delta\Delta\Delta}(l_{i-1})$, but this contradicts (12). Therefore, $v^{\Delta}(l) \leq 0$. This implies that $u^{\Delta}(l) \geq \beta^{\Delta}(l)$, and hence

$$\sum_{l=t}^{t_{n+2}} (t_i - t_{i-1})\beta^{\Delta}(l) \leq \sum_{l=t}^{t_{n+2}} (t_i - t_{i-1})u^{\Delta}(l).$$

This, in turn, yields

$$\begin{aligned}
\beta(t_{n+2}) - \beta(t) &\leq u(t_{n+2}) - u(t), & u(t) &\leq \beta(t) - \beta(t_{n+2}),\\
\beta(t_{n+2}) - \beta(t) &\leq -u(t), & u(t) &\leq \beta(t).
\end{aligned}$$

A similar argument shows that $\alpha(t) \leq u(t)$, $t \in \mathbb{T}$.

Thus, the conclusion of the theorem holds and our proof is complete. □

3. Existence result

In this section, we make use of Theorem 2.1 to obtain positive solutions of the singular problem (1), (2). In particular, in applying Theorem 2.1, we deal with a sequence of regular perturbations of (1), (2). Ultimately, we obtain a desired solution of (1), (2) by passing to the limit on a sequence of solutions for the perturbations.

Theorem 3.1. *Assume conditions* (A), (B), *and* (C) *hold*, *along with the following*:

(D) *there exists* $c \in (0, \infty)$ *so that* $f(t_i, c, 0, 0) \leq 0$ *for all* $t \in \mathbb{T}^{\circ}$;

(E) $f(t_i, x, y, z)$ *is nonincreasing in its* z *variable for* $t_i \in \mathbb{T}^{\circ}$ *and* $x \in (0, c]$;

(F) $\lim_{x\to 0^+} f(t_i, x, y, z) = \infty$ *for* $t_i \in \mathbb{T}^{\circ}$, $y \in \left(-\frac{c}{r}, \frac{c}{r}\right)$, *where r is sufficiently large.*

Then (1), (2) *has a solution u satisfying*

$$0 < u(t) \le c, \quad t_i \in \mathbb{T}^\circ.$$

Proof. Again, for the proof, we proceed through a sequence of steps.

Step 1. For $l \in \mathbb{N}$, $t_i \in \mathbb{T}^\circ$, $(x, y, z) \in \mathbb{R}^3$, define

$$f_l(t_i, x, y, z) = \begin{cases} f(t_i, |x|, y, z), & |x| \ge \frac{1}{l}, \\ f(t_i, \frac{1}{l}, y, z), & |x| < \frac{1}{l}. \end{cases}$$

Then f_l is continuous on $\mathbb{T}^\circ \times \mathbb{R}^3$ and nonincreasing for $t_i \in \mathbb{T}^\circ$, $x \in [-c, c]$.

Assumption (F) implies that there exists l_0 such that, for all $l \ge l_0$,

$$f_l(t_i, c, 0, 0) = f(t_i, c, 0, 0) > 0, \quad t_i \in \mathbb{T}^\circ.$$

Consider, for each $l \ge l_0$,

$$u^{\Delta\Delta\Delta}(t_{i-2}) + f_l(t_i, u(t_i), u^{\Delta}(t_{i-1}), u^{\Delta\Delta}(t_{i-2})) = 0, \quad t_i \in \mathbb{T}^\circ. \tag{13}$$

Define $\alpha(t) = 0$ and $\beta(t) = c$. Then α and β are lower and upper solutions for (13), (2) and $\alpha(t) \le \beta(t)$ on $\mathbb{T}^\circ$. Thus, by Theorem 2.1, there exists u_l a solution of (13), (2) satisfying $0 \le u_l(t) \le c$, $t_i \in \mathbb{T}$, $l \ge l_0$. Consequently,

$$|u_l^{\Delta}(t_i)| \le \frac{c}{(t_i - t_{i-1})}, \quad t_i \in \mathbb{T}^\circ. \tag{14}$$

Step 2. Let $l \in \mathbb{N}$, $l \ge l_0$. Since $u_l(t)$ solves (13), we get, from work similar to that exhibited in Theorem 2.1,

$$u_l^{\Delta}(t_m) = \sum_{j=1}^{n} (t_{j+1} - t_j) \sum_{i=1}^{j} (t_i - t_{i-1}) f_l\big(t_i, u_l(t_i), u_l^{\Delta}(t_{i-1}), u_l^{\Delta\Delta}(t_{i-2})\big) \tag{15}$$

for $t_m \in \mathbb{T}^\circ$. By assumption (F), there exists $\varepsilon_1 \in (0, 1/l_0)$ such that, if $l \ge 1/\varepsilon_1$,

$$f_l(t_2, x, y, z) > \frac{c}{t_2 - t_1}, \quad x \in (0, \varepsilon_1],\ y \in (-c, c). \tag{16}$$

For the sake of establishing a contradiction, assume that $u_l(t_1) < \varepsilon_1$ for $l \geq 1/\varepsilon_1$. Then, by (15) and (16),

$$\begin{aligned}
u_l^{\Delta}(t_1) &= -\sum_{j=1}^{n}(t_{j+1}-t_j)\sum_{i=1}^{j}(t_i-t_{i-1})f_l\big(t_i, u_k(t_i), u_k^{\Delta}(t_{i-1}), u_l^{\Delta\Delta}(t_{i-2})\big) \\
&\geq f_l\big(t_i, u_l(t_i), u_l^{\Delta}(t_{i-1}), u_l^{\Delta\Delta}(t_{i-2})\big) \\
&\qquad\qquad + \sum_{j=2}^{n}(t_{j+1}-t_j)\sum_{i=1}^{j}(t_i-t_{i-1})f_l\big(t_i, u_l(t_i), u_l^{\Delta}(t_{i-1}), u_l^{\Delta\Delta}(t_{i-2})\big) \\
&\geq f_l\big(t_i, u_l(t_i), u_l^{\Delta}(t_{i-1}), u_l^{\Delta\Delta}(t_{i-2})\big) \\
&\geq \frac{c}{t_2-t_1} = -\frac{c}{r}.
\end{aligned}$$

But this contradicts (14). Hence $u_l(t_1) \geq \varepsilon_1$ for all $l \geq 1/\varepsilon_1$.

Define $a_2 = \max\{|f_l(t_2, x, y, z)| : x \in [\varepsilon_1, c], y \in (-c, c)\}$. By assumption (F), there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that, if $l \geq 1/\varepsilon_2$ and $u_l < \varepsilon_2$, then

$$f_l(t_3, x, y, z) > \frac{c}{t_3-t_2} - T(a_2), \quad x \in (0, \varepsilon_2], \; y \in (-c, c). \tag{17}$$

For the sake of establishing a contradiction, assume that, for $l \geq 1/\varepsilon_2$, we have $u_l(t_2) < \varepsilon_2$. Then, by (15) and (17), we have

$$\begin{aligned}
u_l^{\Delta}(t_2) &= \sum_{j=1}^{n}(t_{j+1}-t_j)\sum_{i=1}^{j}(t_i-t_{i-1})f_l\big(t_i, u_l(t_i), u_l^{\Delta}(t_{i-1}), u_l^{\Delta\Delta}(t_{i-2})\big) \\
&= \sum_{j=2}^{n}(t_{j+1}-t_j)\sum_{i=2}^{j}(t_i-t_{i-1})f_l\big(t_i, u_l(t_i), u_l^{\Delta}(t_{i-1})u_l^{\Delta\Delta}(t_{i-2})\big) \\
&\qquad + Tf_l\big(t_2, u_l(t_2), u_l^{\Delta}(t_1), u_l^{\Delta\Delta}(t_0)\big) \\
&= \sum_{j=3}^{n}(t_{j+1}-t_j)\sum_{i=2}^{j}(t_i-t_{i-1})f_l\big(t_i, u_k(t_i), u_l^{\Delta}(t_{i-1}), u_l^{\Delta\Delta}(t_{i-2})\big) \\
&\qquad + f_l\big(t_3, u_l(t_3), u_l^{\Delta}(t_2), u_l^{\Delta\Delta}(t_1)\big) + Tf_l\big(t_2, u_l(t_2), u_l^{\Delta}(t_1), u_l^{\Delta\Delta}(t_0)\big) \\
&\geq \sum_{j=3}^{n}(t_{j+1}-t_j)\sum_{i=2}^{j}(t_i-t_{i-1})f_l\big(t_i, u_k(t_i), u_l^{\Delta}(t_{i-1}), u_l^{\Delta\Delta}(t_{i-2})\big) \\
&\qquad + f_k\big(t_2, u_k(t_2), u_k^{\Delta}(t_1)\big)f_l\big(t_3, u_l(t_3), u_l^{\Delta}(t_2), u_l^{\Delta\Delta}(t_1)\big) + Ta_2 \\
&> \sum_{j=3}^{n}(t_{j+1}-t_j)\sum_{i=2}^{j}(t_i-t_{i-1})f_l\big(t_i, u_k(t_i), u_l^{\Delta}(t_{i-1}), u_l^{\Delta\Delta}(t_{i-2})\big) + \frac{c}{t_3-t_2} \\
&> \frac{c}{t_3-t_2}.
\end{aligned}$$

But this contradicts (14). Hence $u_l(t_2) \geq \varepsilon_2$ for all $l \geq 1/\varepsilon_2$.

Continuing similarly for $t = 3, 4, \dots, nT$, we get $0 < \varepsilon_T < \cdots < \varepsilon_2 < \varepsilon_1$ such that $u_l(t_i) \geq \varepsilon_T$ for $t_i \in T$.

For $2 \leq i \leq n-1$, set

$$m_i = \max\bigl\{|f_l(t_i, x, y, z)| : x \in [\varepsilon_i, c],\, y \in (-c, c)\bigr\}.$$

By assumption (F), there exists $\varepsilon_n \in (0, \varepsilon_{n-1}]$ such that, if $l \geq 1/\varepsilon_n$ and $u_l(t_n) < \varepsilon_n$, then

$$f_l(t_n, x, y, z) > \frac{c}{t_n - t_{n-1}} - \sum_{i=2}^{n-1} m_i. \tag{18}$$

For the sake of establishing a contradiction, assume that, for $l \geq 1/\varepsilon_n$, we have $u_l(t_n) < \varepsilon_n$. Then, by (15) and (18), we have

$$\begin{aligned}
u_l^{\Delta}(t_n) &= \sum_{j=n+1}^{n+1} (t_{j+1} - t_j) \sum_{i=2}^{j} (t_i - t_{i-1}) f_l\bigl(t_i, u(t_i), u^{\Delta}(t_i), u^{\Delta\Delta}(t_{i-2})\bigr) \\
&= (t_{n+2} - t_{n+1}) \sum_{i=2}^{n+1} (t_i - t_{i-1}) f_l\bigl(t_i, u(t_i), u^{\Delta}(t_{i-1}), u^{\Delta\Delta}(t_{i-2})\bigr) \\
&= (t_{n+2} - t_{n+1}) \sum_{i=2}^{n} (t_i - t_{i-1}) f_l\bigl(t_i, u(t_i), u^{\Delta}(t_{i-1}), u^{\Delta\Delta}(t_{i-2})\bigr) \\
&\qquad + f_l\bigl(t_{n+1}, u(t_{n+1}), u^{\Delta}(t_n), u^{\Delta\Delta}(t_{n-1})\bigr) \\
&> \sum_{i=2}^{n-1} (m_i) + \frac{c}{t_n - t_{n-1}} - \sum_{i=2}^{n-1} (m_i) \\
&= \frac{c}{t_n - t_{n-1}}.
\end{aligned}$$

But this contradicts (14). Hence $u_l(t_n) \geq \varepsilon_n$ for all $l \geq 1/\varepsilon_n$. Therefore, by letting $\varepsilon = \varepsilon_n$, we get

$$0 < \varepsilon \leq u_l(t_i) \leq c, \quad t \in \mathbb{T}^{\circ},\ l \geq \frac{1}{\varepsilon}. \tag{19}$$

Since $u_l(t_i)$ satisfies (19) and (2), we can choose a subsequence $\{u_{l_k}(t)\} \subset \{u_l(t_i)\}$ such that $\lim_{k\to\infty} u_{l_k}(t) = u(t_i)$, $t \in \mathbb{T}^{\circ}$, $u(t_i) \in E$, where E is as defined in Step 2 of Theorem 2.1. Moreover, (15) yields, for each sufficiently large k,

$$u_{l_k}^{\Delta}(t_i) = \sum_{j=t_i+1}^{n} (t_{j+1} - t_j) \sum_{i=2}^{j} (t_i - t_{i-1}) f\bigl(t_i, u_{l_k}(t_i), u_{l_k}^{\Delta}(t_{i-1}), u_{l_k}^{\Delta\Delta}(t_{i-2})\bigr),$$

and so, letting $l \to \infty$ and from the continuity of f, we get

$$u^{\Delta}(t_i) = \sum_{t_i+1}^{n} (t_{j+1} - t_j) \sum_{i=2}^{j} (t_i - t_{i-1}) f\big(t_i, u(t_i), u^{\Delta}(t_{i-1}), u^{\Delta\Delta}(t_{i-2})\big).$$

Consequently,

$$u^{\Delta\Delta}(t_{i-1}) = \sum_{i=2}^{j} (t_i - t_{i-1}) f\big(t_i, u(t_i), u^{\Delta}(t_{i-1}), u^{\Delta\Delta}(t_{i-2})\big).$$

Thus,

$$u^{\Delta\Delta\Delta}(t_{i-2}) = -f\big(t_i, u(t_i), u^{\Delta}(t_{i-1}), u^{\Delta\Delta}(t_{i-2})\big).$$

Therefore, u solves (1), and, by (19), our theorem holds. □

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