involve
a journal of mathematics

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2013 vol. 6, no. 2

INVOLVE 6:2 (2013)
dx.doi.org/10.2140/involve.2013.6.243

Spectral characterization for von Neumann's iterative algorithm in $\mathbb{R}^n$

Rudy Joly, Marco López, Douglas Mupasiri and Michael Newsome

(Communicated by Jim Haglund)

Our work is motivated by a theorem proved by von Neumann: Let S_1 and S_2 be subspaces of a closed Hilbert space X and let $x \in X$. Then

$$\lim_{k\to\infty} (P_{S_2} P_{S_1})^k(x) = P_{S_1\cap S_2}(x),$$

where P_S denotes the orthogonal projection of x onto the subspace S. We look at the linear algebra realization of the von Neumann theorem in $\mathbb{R}^n$. The matrix A that represents the composition $P_{S_2} P_{S_1}$ has a form simple enough that the calculation of $\lim_{k\to\infty} A^k x$ becomes easy. However, a more interesting result lies in the analysis of eigenvalues and eigenvectors of A and their geometrical interpretation. A characterization of such eigenvalues and eigenvectors is shown for subspaces with dimension $n-1$.

1. Introduction

In Euclidean n-space, we wish to find the point x_∞ in the intersection of two $(n-1)$-dimensional subspaces, S_1 and S_2, that is closest to an initial point x_0 in $\mathbb{R}^n$. That is, we want $x_\infty \in S_1 \cap S_2$ to be such that

$$\|x_0 - x_\infty\| \le \|x_0 - y\| \quad \text{for all } \ y \in S_1 \cap S_2.$$

We call x_∞ the orthogonal projection of x_0 onto $S_1 \cap S_2$. We start by stating von Neumann's theorem; see [Deutsch 2001], for example.

Theorem 1. *Let S_1 and S_2 be subspaces of a closed Hilbert space X and let $x \in X$. Then*

$$\lim_{k\to\infty} (P_{S_2} P_{S_1})^k(x) = P_{S_1\cap S_2}(x), \tag{1-1}$$

where P_S denotes the orthogonal projection onto the subspace S.

Von Neumann's theorem provides an iterative procedure (left-hand side of (1-1)) to find the orthogonal projection of x onto $S_1 \cap S_2$ (right-hand side of (1-1)).

MSC2010: primary 41A65; secondary 47N10.
Keywords: orthogonal projections, von Neumann, best approximations.

2. An example in $\mathbb{R}^2$

To illustrate von Neumann's theorem we consider the $\mathbb{R}^2$ case. Let $a_1, b_1, a_2, b_2 \in \mathbb{R}$ and let

$$S_1 = \{(x, y) \mid a_1 x + b_1 y = 0\} \quad \text{and} \quad S_2 = \{(x, y) \mid a_2 x + b_2 y = 0\}.$$

In order for S_1 and S_2 to be distinct 1-dimensional subspaces, we require that the a_i and b_i are not both zero[1] and that $a_1/b_1 \neq a_2/b_2$. Since the orthogonal projection onto a subspace is a linear transformation, we can represent such transformations by matrices. In the plane, the matrix that projects any point in $\mathbb{R}^2$ onto S_i is given by

$$A_i = \frac{1}{a_i^2 + b_i^2} \begin{pmatrix} b_i^2 & -a_i b_i \\ -a_i b_i & a_i^2 \end{pmatrix},$$

where $i = 1, 2$. Therefore, the matrix $A = A_2 A_1$ gives us the composition of the two projections.

$$A = \frac{a_1 a_2 + b_1 b_2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} \begin{pmatrix} b_1 b_2 & -a_1 b_2 \\ -a_2 b_1 & a_1 a_2 \end{pmatrix}$$

To compute iterations of the matrix A, we wish to express A in terms of a diagonal matrix D similar to A. This is possible, of course, if A is nondefective; that is, if the dimension of each of the eigenspaces of A is equal to the multiplicity of the corresponding eigenvalue. It is easily shown that A is nondefective in the $\mathbb{R}^2$ case. The matrix S of eigenvectors of A is then

$$S = \begin{pmatrix} a_1 & b_1 \\ b_1 & -a_2 \end{pmatrix},$$

with D being

$$D = S^{-1} A S.$$

Computing powers of the matrix A is then a matter of raising the eigenvalues of A to that power:

$$A^k = S D^k S^{-1}.$$

Applying von Neumann's theorem to this equation, we obtain

$$\lim_{k\to\infty} (A_2 A_1)^k = \lim_{k\to\infty} A^k = S\big(\lim_{k\to\infty} D^k\big) S^{-1} = A_\infty,$$

where A_∞ is the matrix representation of $P_{S_1 \cap S_2}$. Note that the limit exists if the eigenvalues of A have absolute value less than or equal to unity.

[1] If, say, $a_1 = b_1 = 0$ then $S_1 = \mathbb{R}^2$.

3. Solution algorithm

It is possible to extend the solution method in the previous section to $\mathbb{R}^n$. Here we present a brief outline of the solution algorithm, as explained in [Hoffman and Kunze 1971].

(1) Choose bases for S_1 and S_2.

(2) Use the Gram–Schmidt procedure to produce orthonormal bases $\beta^{(1)}$ and $\beta^{(2)}$ for S_1 and S_2 respectively:

$$\beta^{(1)} = \{u_1^{(1)}, \dots, u_{n-1}^{(1)}\}, \quad \beta^{(2)} = \{u_1^{(2)}, \dots, u_{n-1}^{(2)}\}. \tag{3-1}$$

(3) Use the standard basis $\beta = \{e_1, \dots, e_n\}$ for the parent vector space $\mathbb{R}^n$.

(4) Use the following general formula to obtain the matrix representations A_i, with $i = 1, 2$, of the orthogonal projections $P_i : \mathbb{R}^n \to S_i$:

$$A_i = \left[\left(\sum_{j=1}^{n-1} \langle e_1, u_j^{(i)} \rangle u_j^{(i)} \right), \ \dots, \ \left(\sum_{j=1}^{n-1} \langle e_n, u_j^{(i)} \rangle u_j^{(i)} \right) \right].$$

(5) Compute $A = A_2 A_1$. Find the eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding independent eigenvectors $\boldsymbol{E}_1, \dots, \boldsymbol{E}_n$ of A. These give us the $n \times n$ matrices

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad S = (\boldsymbol{E}_1, \dots, \boldsymbol{E}_n).$$

(6) Compute S^{-1}.

(7) Iteration now proceeds as follows:

$$\begin{aligned} \boldsymbol{v}_k = A\boldsymbol{v}_{k-1} &= (SDS^{-1})\boldsymbol{v}_{k-1} = (SDS^{-1})(SDS^{-1})\boldsymbol{v}_{k-2} \\ &= \cdots = (SD^kS^{-1})\boldsymbol{v}_0 = A^k \boldsymbol{v}_0 \end{aligned} \tag{3-2}$$

for $k = 1, 2, 3, \dots$.

(8) Finally, we obtain $\boldsymbol{v}_\infty = [S(\lim_{k\to\infty} D^k)S^{-1}]\boldsymbol{v}_0$.

In step (5), we rely on the assumption that the matrix A is nondefective in order to find a similar diagonal matrix. We address this question in Section 5.

4. Eigenvalues in $\mathbb{R}^3$: geometric argument

If we consider two 2-dimensional subspaces in 3-space, S_1 and S_2, it is easy to illustrate geometrically the eigenvectors of the alternating projections. By examining a picture of two planes containing the origin in $\mathbb{R}^3$, we see three different types of eigenvectors; the first two are trivial, but the third is less so (refer to Figure 1).

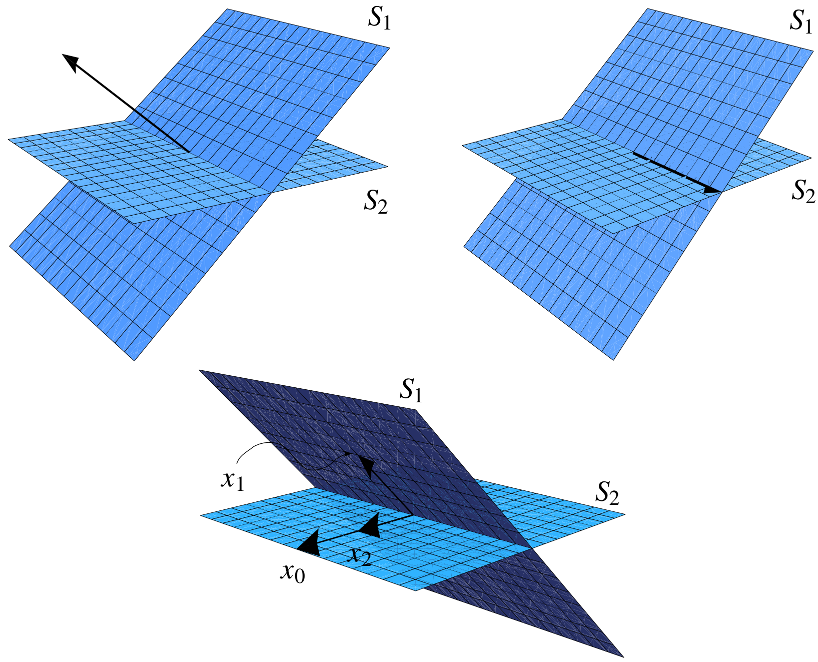

Figure 1. Top left: a vector orthogonal to S_1 gets projected to the origin (eigenvalue 0). Top right: a vector in $S_1 \cap S_2$ remains fixed (eigenvalue 1). Bottom: a vector in $(S_1 \cap S_2)^\perp$ gets projected to a collinear vector (eigenvalue in [0, 1]).

(1) A vector orthogonal to S_1 is in the kernel of P_{S_1}; therefore, it is an eigenvector of P_{S_1} with eigenvalue 0.

(2) A vector in $S_1 \cap S_2$ is an eigenvector of both P_{S_2} and P_{S_1} with eigenvalue 1.

(3) A vector in the orthogonal complement $(S_1 \cap S_2)^\perp$ will stay in $(S_1 \cap S_2)^\perp$ as it is projected orthogonally onto S_1 and S_2; i.e., $(S_1 \cap S_2)^\perp$ is invariant under both P_{S_1} and P_{S_2}. Therefore, a vector in $S_2 \cap (S_1 \cap S_2)^\perp$ is an eigenvector of $P_{S_2} P_{S_1}$. We claim that this eigenvector corresponds to an eigenvalue in the interval [0, 1].

It is easy to see from this geometric argument the characterization of eigenvalues in the case of $\mathbb{R}^3$. Next we address the question of whether this geometric intuition somehow generalizes to $\mathbb{R}^n$.

5. Characterization of eigenvalues in $\mathbb{R}^n$.

When we consider $(n-1)$-dimensional subspaces in $\mathbb{R}^n$, it is easy to see that the first two eigenvectors described in Section 4 generalize to higher dimensions. It is less trivial to show that the third type of eigenvector also generalizes to higher dimensions, and that these three types of vectors fully characterize the spectrum of $P_{S_2}P_{S_1}$.

Let S_1 and S_2 be $(n-1)$-dimensional subspaces of $\mathbb{R}^n$ with $S_1 \neq S_2$.

Lemma 2. *$S_1 \cap S_2$ is a proper subspace of $\mathbb{R}^n$ with* $\dim(S_1 \cap S_2) = n-2$.

Proof. The intersection of two subspaces is always a subspace. Note that for two distinct subspaces, we have

$$n = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2).$$

Therefore,

$$\begin{aligned}\dim(S_1 \cap S_2) &= \dim(S_1) + \dim(S_2) - n \\ &= n-1+n-1-n = n-2.\end{aligned}$$ □

Now, let $S_3 = (S_1 \cap S_2)^{\perp}$. Note that $n = \dim(S_1 \cap S_2) + \dim(S_3)$, which implies that $\dim(S_3) = 2$.

Lemma 3. $$\dim(S_3 \cap S_1) = \dim(S_3 \cap S_2) = 1.$$

Proof. We write $\dim(S_3 \cap S_1) = \dim(S_3) + \dim(S_1) - n = 2 + n - 1 - n = 1$. Similarly, $\dim(S_3 \cap S_2) = 1$. □

Lemma 4. *Let $T_1 : \mathbb{R}^n \to S_1$ and $T_2 : \mathbb{R}^n \to S_2$ be the orthogonal projections onto S_1 and S_2, respectively. Then S_3 is invariant under T_1 and T_2.*

Proof. Let $\{w, w^{\perp}\}$ be a basis for S_3 such that $w \in S_1$ and $w^{\perp} \in S_1^{\perp}$. If $v_0 \in S_3$, then $v_0 = c_1 w + c_2 w^{\perp}$ for some scalars c_1, c_2; therefore,

$$T_1(v_0) = c_1 T_1(w) + c_2 T_1(w^{\perp}) = c_1 w \in S_3.$$

Similarly, we can construct a basis $\{u, u^{\perp}\}$ for S_3 such that $u \in S_2$ and $u^{\perp} \in S_2^{\perp}$ to conclude that $T_2(v_0) \in S_3$. □

Now we are ready to prove the following theorem. Let θ be the angle between two hyperplanes defined as the angle between two vectors n_1 and n_2 normal to S_1 and S_2, respectively. Note that $n_1, n_2 \in S_3$.

Theorem 5. *Let S_1 and S_2 be distinct $(n-1)$-dimensional subspaces of $\mathbb{R}^n$, and let $T_1 : \mathbb{R}^n \to S_1$ and $T_2 : \mathbb{R}^n \to S_2$ be the orthogonal projections onto S_1 and S_2, respectively. Also, let $0 < \theta < \frac{\pi}{2}$ be the angle between the two hyperplanes. The spectrum of $T := T_2T_1$ is characterized by the following eigenvalues and multiplicities*:

$$\lambda_1 = 0,\ m_1 = 1, \quad \lambda_2 = 1,\ m_2 = n-2, \quad \lambda_3 = \cos^2\theta,\ m_3 = 1.$$

Proof. First, consider u_0 to be a vector orthogonal to S_1. Then $T(u_0) = 0$, and so $m_1 \geq 1$. Now let $\{w_1, \ldots, w_{n-2}\}$ be a basis for $S_1 \cap S_2$. Then $T(w_i) = w_i$ for all $1 \leq i \leq n-2$. Therefore, $\lambda_2 = 1$ is an eigenvalue. Since the basis vectors for $S_1 \cap S_2$ are linearly independent eigenvectors corresponding to λ_2, we have $m_2 \geq n-2$. Furthermore, consider $v_0 \in S_3 \cap S_2$. Then $T(v_0) \in S_3$ by Lemma 4, and $T(v_0) \in S_2$ since the range of T is S_2. Moreover,

$$\dim(S_3 \cap S_2) = 1;$$

therefore, $T(v_0) = \lambda v_0$ for some scalar λ. Furthermore, let $v_1 := T_1(v_0)$ and $v_2 := T_2(v_1) = T(v_0)$. For vectors n_1 and n_2 in the orthogonal complement of S_1 and S_2, respectively, we have that n_1, n_2, v_0, v_1, and v_2 are coplanar, since they are in the 2-dimensional subspace S_3. Thus

$$\angle(v_0, v_1) = \angle(v_1, v_2) = \angle(n_1, n_2) = \theta.$$

Hence, $\cos\theta = \dfrac{\langle v_0, v_1\rangle}{\|v_0\|\,\|v_1\|}$ and

$$\|v_2\|\,\|v_1\|\cos\theta = \frac{\|v_2\|}{\|v_0\|}\langle v_0, v_1\rangle = \lambda\langle v_0, v_1\rangle.$$

Note that $\langle v_1, (v_0 - v_1)\rangle = \langle v_2, (v_1 - v_2)\rangle = 0$, so

$$\|v_2\|\,\|v_1\|\cos\theta = \lambda\langle v_1 + (v_0 - v_1), v_1\rangle = \lambda\|v_1\|^2 :$$

thus $\dfrac{\|v_2\|}{\|v_1\|}\cos\theta = \lambda$. Moreover,

$$\|v_2\|\,\|v_1\|\cos\theta = \langle v_2, v_1\rangle = \langle v_2, v_2 + (v_1 - v_2)\rangle = \|v_2\|^2,$$

so $\cos\theta = \dfrac{\|v_2\|}{\|v_1\|}$. It follows that $\lambda = \cos^2\theta$. □

6. Conclusion

We have shown that for every finite-dimensional inner product space, the method of alternating orthogonal projections between two hyperplane subspaces S_1 and S_2 yields at most three distinct eigenvalues when we consider the composition of two orthogonal projections. Also, the eigenvectors of such a composition can be quickly identified to be in the subspaces $S_1^{\perp}$, $S_1 \cap S_2$, and $S_2 \cap (S_1 \cap S_2)^{\perp}$. We should mention the special, and somewhat trivial, cases where the angle between S_1 and S_2 is 0° or 90°. In the case where $\theta = 90°$, we have that $P_{S_2}P_{S_1} = P_{S_1 \cap S_2}$, and $P_{S_2}P_{S_1} = P_{S_1} = P_{S_2}$ when $\theta = 0°$. In these cases, there are two distinct eigenvalues: 0 and 1. For $\theta = 90°$, the respective multiplicities are 2 and $n-2$; for $\theta = 0°$, they are 1 and $n-1$. It is also noteworthy that the multiplicities obtained in Theorem 5 guarantee that $P_{S_2}P_{S_1}$ is nondefective, a necessary condition for the algorithm presented in Section 3.

Acknowledgements

This work was supported by a grant funded by the Division of Mathematical Sciences, National Science Foundation, award number 0502354. Special thanks go to Paul Barloon for his valuable support during this research project.

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Received: 2012-05-31 Accepted: 2013-06-02

rholy2@gmail.com *Johns Hopkins University, 29 Hillview Avenue, Boston, MA 02131, United States*

marcolopez@my.unt.edu *Department of Mathematics, University of North Texas, 1716 W. Hickory St. Apt 2, Denton, TX 76201, United States*

douglas.mupasiri@uni.edu *Department of Mathematics, University of Northern Iowa, 220 Wright Hall, Cedar Falls, IA 50614-0506, United States*

mln40@msstate.edu *Jackson State University, 8845 Hwy 12 West, Sallis, MS 39160, United States*

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2013 vol. 6 no. 2

1944-4176(2013)6:2;1-9