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The concept of frames in a Banach space has been introduced by Gröchenig and developed by several authors. The main feature of a frame is to present every element of the underlying Banach space as a norm-convergent series. In this decomposition, the dual frame plays an essential role. The existence of a dual p -frame is not guaranteed in general. Some characterizations of duals of p -frames are given in this paper.

1. Introduction and preliminaries

A sequence $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space \mathcal{H} is called a *frame* if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B\|f\|^2 \quad (f \in \mathcal{H}). \quad (1-1)$$

The numbers A and B are called *frame bounds*. A frame is called *tight* if $A = B$. In frame theory, the operator $T : l^2 \rightarrow \mathcal{H}$ given by $T\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i$ is useful in analyzing various properties of frames. It is called the *synthesis* or *preframe operator*. Its adjoint $T^* : \mathcal{H} \rightarrow l^2; f \mapsto \{\langle f, f_i \rangle\}_{i=1}^{\infty}$ is called the *analysis operator*. By composing T and T^* , we obtain the frame operator

$$S : \mathcal{H} \rightarrow \mathcal{H}, \quad Sf = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i \quad (f \in \mathcal{H}).$$

The frame operator S is invertible and the reconstruction formula

$$f = S^{-1}Sf = \sum_{i=1}^{\infty} \langle f, S^{-1}f_i \rangle f_i \quad (f \in \mathcal{H}) \quad (1-2)$$

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holds. The sequence $\{S^{-1}f_i\}_{i=1}^{\infty}$ which plays the same role as the dual in the theory of bases is also a frame. It is called the *canonical dual* of $\{f_i\}_{i=1}^{\infty}$. In general, the Bessel sequence $\{g_i\}_{i=1}^{\infty}$ is called a *dual* of $\{f_i\}_{i=1}^{\infty}$ if

$$f = \sum_{i=1}^{\infty} \langle f, g_i \rangle f_i \quad (f \in \mathcal{H}). \quad (1-3)$$

For general references on this theory, we refer the reader to [Christensen 2008, Section 5.1]. Recently, various generalizations of frames have been proposed: continuous frames [Ali et al. 1993; Askari-Hemmat et al. 2001; Gabardo and Han 2003], g -frames [Sun 2006], fusion frames [Casazza et al. 2008], von Neumann–Schatten frames [Sadeghi and Arefijamaal 2012], and so on. Frames for Banach spaces were first introduced in [Gröchenig 1991] and were developed in [Aldroubi et al. 2001; Cazassa and Christensen 1997; Casazza et al. 1999; 2005]. In particular, Christensen and Stoeva [2003] studied p -frames in Banach spaces and obtained a lot of interesting and important results.

In applications of frame theory the goal is to recognize the finer properties of functions by means of the magnitudes of the frame coefficients [Benedetto et al. 2006; Bolcskei et al. 1998; Candès and Donoho 2004; Heath and Paulraj 2002]. These properties, typically smoothness and decay properties or phase-space localization of functions, are measured by the Banach space norm. Dual frames have a key role in the decomposition of elements in the underlying space. Casazza et al. [2005] present some equivalent conditions for the existence of reconstruction formulas in Banach spaces. Moreover, sufficient conditions for the existence of dual frames are studied in [Aldroubi et al. 2001]. In this article, at the first, we review the definition and basic properties of p -frames, and then express some characterizations of duals of p -frames. The analogous results concerning frames in Hilbert spaces may be found in [Li 1995]. Finally, we discuss a stability theorem for duals of p -frames.

2. Elementary properties of p -frames

Throughout this paper, X is a separable Banach space with dual X^* , $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. A sequence $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is called a p -frame for X if there exist constants $A, B > 0$ such that

$$A\|x\| \leq \left(\sum_{i=1}^{\infty} |g_i(x)|^p \right)^{\frac{1}{p}} \leq B\|x\| \quad (x \in X). \quad (2-1)$$

The sequence $\{g_i\}_{i=1}^{\infty}$ is a p -Bessel sequence if at least the upper p -frame condition is satisfied. Analogous to frame theory in Hilbert spaces, one can define

the synthesis operator as

$$T : l^q \rightarrow X^*, \quad T\{d_i\} := \sum_{i=1}^{\infty} d_i g_i.$$

A straightforward calculation shows that $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p -Bessel sequence with bound B if and only if T is well-defined and $\|T\| \leq B$; see [Christensen and Stoeva 2003, Proposition 2.2].

The following result shows other aspects of the synthesis operator:

Proposition 2.1 [Christensen and Stoeva 2003]. *Let $\{g_i\} \subseteq X^*$ be a p -frame. Then*

- (i) *the adjoint of T given by $T^* : X \rightarrow l^p; f \mapsto \{g_i(f)\}_{i=1}^{\infty}$ has closed range;*
- (ii) *X is reflexive;*
- (iii) *T is onto.*

The next proposition deals with preservation of the p -frame property under the action of various operators. Its proof is straightforward and we omit it.

Proposition 2.2. *Let X and Y be two Banach spaces and $\Psi : Y \rightarrow X$ be a bounded operator. Then*

- (i) *if $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p -Bessel sequence for X , then $\{\Psi^* g_i\}_{i=1}^{\infty}$ is a p -Bessel sequence for Y ;*
- (ii) *if $\{g_i\}_{i=1}^{\infty}$ is a p -frame for X , and Ψ is one-to-one with closed range, then $\{\Psi^* g_i\}_{i=1}^{\infty}$ is a p -frame for Y .*

Definition 2.3. Let X be a Banach space and $1 < p < \infty$. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq X$ is called a p -Riesz basis for X if the closed linear span of $\{f_i\}_{i=1}^{\infty}$ is X and there exist constants A and B such that, for any finite scalars $\{c_i\}$,

$$A\left(\sum |c_i|^p\right)^{\frac{1}{p}} \leq \left\| \sum c_i f_i \right\| \leq B\left(\sum |c_i|^p\right)^{\frac{1}{p}}. \tag{2-2}$$

Clearly, if $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p -Riesz basis for X^* then its synthesis operator has a bounded inverse. In particular, every p -Riesz basis for X^* is a q -frame for X with the same bounds.

If $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p -frame for X , then Proposition 2.1 shows that every $g \in X^*$ can be written as $g = \sum_{i=1}^{\infty} d_i g_i$ for some $\{d_i\}_{i=1}^{\infty} \in l^q$. Our aim is to find such a decomposition for the elements of X .

3. Main results

Let $\{f_i\}_{i=1}^{\infty}$ be a frame in a Hilbert space \mathcal{H} with the synthesis operator T . The canonical dual $\{S^{-1} f_i\}_{i=1}^{\infty}$ deals with the frame operator S ; see (1-2). It is not guaranteed that the canonical dual frame has the same structure as the frame itself

[Daubechies 1990]. Alternate duals are now presented as being a good candidate to apply the reconstruction formula (1-2).

Unfortunately, in p -frames, the frame operator cannot be defined. Hence, we first try to describe the canonical dual with respect to the synthesis operator. In fact, let $\{f_i\}_{i=1}^\infty$ be a frame in a Hilbert space \mathcal{H} with the analysis operator T^* . Then the frame condition (1-1) implies that T^* is injective and has closed range [Christensen 2008, Corollary 5.4.3]. Hence, the operator $(T^*)^{-1} : \mathcal{R}(T^*) \rightarrow \mathcal{H}$ can be extended to a bounded operator $\Phi : l^2 \rightarrow \mathcal{H}$. Therefore,

$$S^{-1} f_i = S^{-1} T \delta_i = S^{-1} T T^* \Phi \delta_i = \Phi \delta_i,$$

where $\{\delta_i\}_{i=1}^\infty$ is the canonical orthonormal basis for l^2 .

We summarize this fact in the following lemma.

Lemma 3.1. *Let $\{f_i\}_{i=1}^\infty$ be a frame in \mathcal{H} with the analysis operator T^* . The canonical dual $\{f_i\}_{i=1}^\infty$ can be represented as $\{\Phi \delta_i\}_{i=1}^\infty$, where $\Phi : l^2 \rightarrow \mathcal{H}$ is the unique extension of $(T^*)^{-1}$ and $\{\delta_i\}_{i=1}^\infty$ is the canonical orthonormal basis of l^2 .*

Let X be a Banach space with dual X^* and $1 < p < \infty$. The usual duality between X and X^* allows us to consider p -frames for X^* . In fact, a sequence $\{f_i\}_{i=1}^\infty \subseteq X$ is a p -frame if there exist constants A and B such that

$$A \|g\| \leq \left(\sum_{i=1}^{\infty} |g(f_i)|^p \right)^{\frac{1}{p}} \leq B \|g\| \quad (g \in X^*).$$

If the upper frame condition is satisfied we call $\{f_i\}_{i=1}^\infty$ a p -Bessel sequence.

Definition 3.2. Let $\{g_i\}_{i=1}^\infty \subseteq X^*$ be a p -Bessel sequence for X . A q -Bessel sequence $\{f_i\}_{i=1}^\infty \subseteq X$ for X^* is called a *dual* for $\{g_i\}_{i=1}^\infty$ if

$$g = \sum g(f_i) g_i \quad (g \in X^*) \quad \text{or} \quad f = \sum g_i(f) f_i \quad (f \in X). \quad (3-1)$$

If $\{g_i\}_{i=1}^\infty$ is p -frame, by using the Cauchy–Schwarz inequality, $\{f_i\}_{i=1}^\infty$ is automatically a q -frame for X^* , and vice versa. For more details see Theorem 2.10 of [Christensen and Stoeva 2003].

Denote the synthesis operators of $\{g_i\}_{i=1}^\infty$ and $\{f_i\}_{i=1}^\infty$ by T and U , respectively. Also let X be reflexive. Then (3-1) holds if and only if $TU^* = I_{X^*}$ or $UT^* = I_X$. Although, for every $p \neq 2$, there exist a Banach space X and a p -frame for X without any dual [Casazza et al. 1999], Christensen and Stoeva [2003] showed that a p -frame $\{g_i\}_{i=1}^\infty$ has a dual if and only if $\mathcal{R}(T^*)$, the range of T^* , is complemented in l^p . Obviously, every p -Riesz basis for X^* has a unique dual.

Now we give a characterization of dual p -frames:

Proposition 3.3. *Let $\{g_i\}_{i=1}^\infty \subseteq X^*$ be a p -frame for X with the synthesis operator T . Then there exists a one-to-one correspondence between duals of $\{g_i\}_{i=1}^\infty$ and bounded left inverses of T^* .*

Proof. Suppose that $\Phi : l^p \rightarrow X$ is a bounded left inverse of T^* and consider $\{\delta_i\}_{i=1}^\infty$ as the canonical basis of l^p . It is obvious that $\{f_i\}_{i=1}^\infty := \{\Phi\delta_i\}_{i=1}^\infty$ is a q -Bessel sequence and

$$f = \Phi T^* f = \Phi \sum_{i=1}^\infty g_i(f)\delta_i = \sum_{i=1}^\infty g_i(f)f_i \quad (f \in X).$$

Thus $\{f_i\}_{i=1}^\infty$ is a q -frame for X^* . Conversely, let $\{f_i\}_{i=1}^\infty \subseteq X$ be a dual for $\{g_i\}_{i=1}^\infty$. Consider $\Phi : l^p \rightarrow X$ as the synthesis operator of $\{f_i\}_{i=1}^\infty$. Then Φ is bounded and for each $f \in X$ we have

$$f = \sum_{i=1}^\infty g_i(f)f_i = \sum_{i=1}^\infty g_i(f)\Phi\delta_i = \Phi(\{g_i(f)\}_{i=1}^\infty) = \Phi T^* f. \quad \square$$

As a consequence, we show that a p -frame $\{g_i\}_{i=1}^\infty \subseteq X^*$ with a unique dual is a q -Riesz basis for X^* . In fact, by Proposition 3.3 there exists a one-to-one correspondence between the dual frames of $\{g_i\}_{i=1}^\infty$ and all bounded left inverse operators of T^* , in which T is the synthesis operator of $\{g_i\}_{i=1}^\infty$. Hence, $\{g_i\}_{i=1}^\infty$ has a unique dual if and only if T is injective. T is also surjective by Proposition 2.1. Thus T is invertible and $\|T^{-1}\| < \infty$. This implies that $\{g_i\}_{i=1}^\infty$ is a q -Riesz basis for X^* .

Proposition 3.4. *Assume that p -frame $\{g_i\}_{i=1}^\infty \subseteq X^*$ has a dual. Then the q -Bessel sequence $\{f_i\}_{i=1}^\infty \subseteq X$ is a dual for $\{g_i\}_{i=1}^\infty$ if there exists a bounded operator $\Psi : X^* \rightarrow l^q$ such that $T\Psi = 0$. Conversely, all duals of $\{g_i\}_{i=1}^\infty$ (provided existence) can be described in this manner.*

Proof. Let $\{g_i\}_{i=1}^\infty$ be a p -frame. As a consequence of Proposition 2.1, the operator $(T^*)^{-1} : \mathcal{R}(T^*) \rightarrow X$ is well-defined. If $\{g_i\}_{i=1}^\infty$ has a dual, then $\mathcal{R}(T^*)$ is complemented and so this operator can be extended to a bounded linear operator $W : l^p \rightarrow X$. Now assume that $\{f_i\}_{i=1}^\infty \subseteq X$ is a dual for $\{g_i\}_{i=1}^\infty$ with the synthesis operator U . Then (3-1) immediately implies that $TU^* = I$. Define $\Psi : X^* \rightarrow l^q$ by $\Psi = U^* - W^*$. Clearly, Ψ is a bounded operator and

$$T\Psi = TU^* - TW^* = I - (WT^*)^* = 0.$$

Conversely, suppose that $\Psi : X^* \rightarrow l^q$ is a bounded operator via $T\Psi = 0$. Take $\Phi = W - \psi^*$. Then Φ is a bounded operator and $\Phi T^* = I$. Using Proposition 3.3 we conclude that $\{\Phi\delta_i\}_{i=1}^\infty$ is a dual for $\{g_i\}_{i=1}^\infty$. \square

Suppose that $\{g_i\}_{i=1}^\infty \subseteq X^*$ is a p -frame with the synthesis operator T such that $\mathcal{R}(T^*) \subseteq l^p$ is complemented. Then $\{W\delta_i\}_{i=1}^\infty$ is called the *canonical dual* of $\{g_i\}_{i=1}^\infty$, where $W : l^p \rightarrow X$ is the extension of $(T^*)^{-1}$. Other duals, which are characterized by Proposition 3.3, are called *alternate duals*. In other words, the canonical dual is associated to the bounded inverse of T^* whereas alternate duals are in fact obtained by the left inverses of T^* .

Now we are ready to state a perturbation theorem about duals.

Theorem 3.5. *Let $\{g_i\}_{i=1}^\infty \subseteq X^*$ be a p -frame for X with bounds A_1 and B_1 . The p -frames sufficiently close to $\{g_i\}_{i=1}^\infty$ have a dual. More precisely, let $\{f_i\}_{i=1}^\infty \subseteq X$ be a dual for $\{g_i\}_{i=1}^\infty$ with bounds A_2 and B_2 , and let $\{g'_i\}_{i=1}^\infty$ be another p -frame with bounds A' and B' such that $\{g_i - g'_i\}_{i=1}^\infty$ is a p -Bessel sequence with constant sufficiently small ϵ . Then there exists a dual q -frame $\{f'_i\}_{i=1}^\infty$ for $\{g'_i\}_{i=1}^\infty$ such that $\{f_i - f'_i\}_{i=1}^\infty$ is also a q -Bessel sequence with bound multiplied by ϵ .*

Proof. Denote by T_1 and T_2 the synthesis operators of $\{g_i\}_{i=1}^\infty$ and $\{g'_i\}_{i=1}^\infty$, respectively. Then $\|T_1 - T_2\| < \epsilon$ by Proposition 2.2 of [Christensen and Stoeva 2003]. Moreover, $(T_1^*)^{-1} : \mathcal{R}(T_1^*) \rightarrow X$ can be extended to a bounded operator $W : l^p \rightarrow X$ by the assumption. Hence

$$\|I - T_2^*W\| = \|(T_1 - T_2)^*W\| \leq \|W\|\|T_1 - T_2\| \leq \epsilon\|W\|.$$

Consequently T_2^*W is invertible. It follows that T_2^* has a bounded right inverse. A similar argument shows that its left inverse also exists. Consider $U_1 : l^p \rightarrow X$ as the synthesis operator of $\{f_i\}_{i=1}^\infty$. Then

$$\|I - U_1T_2^*\| = \|U_1(T_1 - T_2)^*\| \leq \|U_1\|\|T_1 - T_2\| \leq \epsilon\|U_1\|.$$

Therefore, the p -frame $\{g'_i\}_{i=1}^\infty$ has a dual. We are looking for the desired dual. First by Proposition 3.4 there exists a bounded operator $\Psi : X^* \rightarrow l^q$ such that $T_1\Psi = 0$. Assume that $W_2 : l^p \rightarrow X$ is an extension of $(T_2^*)^{-1}$. Put

$$h_i = (W_2 + \Psi^*)\delta_i.$$

Then $\{h_i\}_{i=1}^\infty$ is a q -Bessel sequence with the synthesis operator $U_2 := W_2 + \Psi^*$ by Proposition 2.2. Moreover, for each $f \in X$ we have

$$\begin{aligned} \|f - U_2T_2^*f\| &= \|\Psi^*T_2^*f\| \\ &= \|\Psi^*T_1^*f - \Psi^*T_2^*f\| \\ &\leq \|T_1 - T_2\|\|\Psi\|\|f\| \leq \epsilon\|\Psi\|\|f\|. \end{aligned}$$

Therefore, $U_2T_2^*$ is invertible for sufficiently small $\epsilon > 0$. In particular,

$$\|I - U_2T_2^*\| \leq \epsilon\|\Psi\|.$$

It remains to show that the q -frame $\{f'_i\}_{i=1}^\infty := \{(U_2 T_2^*)^{-1} f_i\}_{i=1}^\infty$ satisfies the theorem. It is easy to see that

$$U_2 T_2^* = I + \Psi^*(T_1^* - T_2^*). \quad (3-2)$$

Hence,

$$1 - \epsilon \|\Psi\| \leq \|U_2 T_2^*\|. \quad (3-3)$$

For each sequence $\{d_i\}_{i=1}^\infty$ in l^p by using (3-2) and (3-3) we get

$$\begin{aligned} \left\| \sum_{i=1}^\infty d_i (f_i - f'_i) \right\| &= \|U_1 \{d_i\} - (U_2 T_2^*)^{-1} U_1 \{d_i\}\| \\ &\leq \|U_1\| \|I - (U_2 T_2^*)^{-1}\| \left(\sum_{i=1}^\infty |d_i|^p \right)^{\frac{1}{p}} \\ &\leq \|U_1\| \|(U_2 T_2^*)^{-1}\| \|U_2 T_2^* - I\| \left(\sum_{i=1}^\infty |d_i|^p \right)^{\frac{1}{p}} \\ &\leq \frac{\epsilon \|\Psi\| B_2}{1 - \epsilon \|\Psi\|} \left(\sum_{i=1}^\infty |d_i|^p \right)^{\frac{1}{p}}. \end{aligned}$$

This means that $\{f_i - f'_i\}_{i=1}^\infty$ is a q -Bessel sequence and its bound is a multiple of ϵ . \square

Let $\{g_i\}_{i=1}^\infty \subseteq X^*$ be a p -Bessel sequence for X with the synthesis operator T . We say that a q -Bessel sequence $\{f_i\}_{i=1}^\infty \subseteq X$ with the synthesis operator U is an *approximately dual* of $\{g_i\}_{i=1}^\infty$ if

$$\|I - T U^*\| < 1 \quad \text{or} \quad \|I - U T^*\| < 1. \quad (3-4)$$

Obviously, $\{f_i\}_{i=1}^\infty$ is a dual of $\{g_i\}_{i=1}^\infty$ when $T U^* = I$ or $U T^* = I$. Approximate duals are studied in a Hilbert space setting in [Christensen and Laugesen 2010]. They are easier to construct than the classical dual frames. For p -frames, which don't have duals in general, it is natural to ask whether we can exploit the approximate duals instead of duals. Unfortunately, the answer is negative. In fact, if $\{g_i\}_{i=1}^\infty$ is a p -frame for X with an approximate dual $\{f_i\}_{i=1}^\infty$, then, with notation as above, the operator $U T^*$ is invertible. Hence, $\{U T^* f_i\}_{i=1}^\infty$ is a p -frame by Proposition 2.2. Moreover,

$$f = (U T^*)^{-1} U T^* f = (U T^*)^{-1} \sum_{i=1}^\infty g_i(f) f_i = \sum_{i=1}^\infty g_i(f) (U T^*)^{-1} f_i.$$

Therefore, $\{(U T^*)^{-1} f_i\}$ is a dual of $\{g_i\}_{i=1}^\infty$.

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
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Potentially eventually exponentially positive sign patterns MARIE ARCHER, MINERVA CATRAL, CRAIG ERICKSON, RANA HABER, LESLIE HOGBEN, XAVIER MARTINEZ-RIVERA AND ANTONIO OCHOA	261
The surgery unknotting number of Legendrian links BIANCA BORANDA, LISA TRAYNOR AND SHUNING YAN	273
On duals of p -frames ALI AKBAR AREFIJAMAAL AND LEILI MOHAMMADKHANI	301
Shock profile for gas dynamics in thermal nonequilibrium WANG XIE	311
Expected conflicts in pairs of rooted binary trees TIMOTHY CHU AND SEAN CLEARY	323
Hyperbolic construction of Cantor sets ZAIR IBRAGIMOV AND JOHN SIMANYI	333
Extensions of the Euler–Satake characteristic for nonorientable 3-orbifolds and indistinguishable examples RYAN CARROLL AND CHRISTOPHER SEATON	345
Rank numbers of graphs that are combinations of paths and cycles BRIANNA BLAKE, ELIZABETH FIELD AND JOBBY JACOB	369



1944-4176(2013)6:3;1-8