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On duals of p-frames

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The concept of frames in a Banach space has been introduced by Gröchenig and developed by several authors. The main feature of a frame is to present every element of the underlying Banach space as a norm-convergent series. In this decomposition, the dual frame plays an essential role. The existence of a dual p-frame is not guaranteed in general. Some characterizations of duals of p-frames are given in this paper.

1. Introduction and preliminaries

A sequence $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is called a *frame* if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \le B\|f\|^2 \quad (f \in \mathcal{H}). \tag{1-1}$$

The numbers A and B are called *frame bounds*. A frame is called *tight* if $A = B$. In frame theory, the operator $T : l^2 \to \mathcal{H}$ given by $T\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i$ is useful in analyzing various properties of frames. It is called the *synthesis* or *preframe operator*. Its adjoint $T^* : \mathcal{H} \to l^2$; $f \mapsto \{\langle f, f_i\rangle\}_{i=1}^{\infty}$ is called the *analysis operator*. By composing T and T^*, we obtain the frame operator

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i=1}^{\infty} \langle f, f_i\rangle f_i \quad (f \in \mathcal{H}).$$

The frame operator S is invertible and the reconstruction formula

$$f = S^{-1}Sf = \sum_{i=1}^{\infty} \langle f, S^{-1} f_i\rangle f_i \quad (f \in \mathcal{H}) \tag{1-2}$$

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holds. The sequence $\{S^{-1}f_i\}_{i=1}^{\infty}$ which plays the same role as the dual in the theory of bases is also a frame. It is called the *canonical dual* of $\{f_i\}_{i=1}^{\infty}$. In general, the Bessel sequence $\{g_i\}_{i=1}^{\infty}$ is called a *dual* of $\{f_i\}_{i=1}^{\infty}$ if

$$f = \sum_{i=1}^{\infty} \langle f, g_i \rangle f_i \quad (f \in \mathcal{H}). \tag{1-3}$$

For general references on this theory, we refer the reader to [Christensen 2008, Section 5.1]. Recently, various generalizations of frames have been proposed: continuous frames [Ali et al. 1993; Askari-Hemmat et al. 2001; Gabardo and Han 2003], g-frames [Sun 2006], fusion frames [Casazza et al. 2008], von Neumann–Schatten frames [Sadeghi and Arefijamaal 2012], and so on. Frames for Banach spaces were first introduced in [Gröchenig 1991] and were developed in [Aldroubi et al. 2001; Cazassa and Christensen 1997; Casazza et al. 1999; 2005]. In particular, Christensen and Stoeva [2003] studied p-frames in Banach spaces and obtained a lot of interesting and important results.

In applications of frame theory the goal is to recognize the finer properties of functions by means of the magnitudes of the frame coefficients [Benedetto et al. 2006; Bolcskei et al. 1998; Candès and Donoho 2004; Heath and Paulraj 2002]. These properties, typically smoothness and decay properties or phase-space localization of functions, are measured by the Banach space norm. Dual frames have a key role in the decomposition of elements in the underlying space. Casazza et al. [2005] present some equivalent conditions for the existence of reconstruction formulas in Banach spaces. Moreover, sufficient conditions for the existence of dual frames are studied in [Aldroubi et al. 2001]. In this article, at the first, we review the definition and basic properties of p-frames, and then express some characterizations of duals of p-frames. The analogous results concerning frames in Hilbert spaces may be found in [Li 1995]. Finally, we discuss a stability theorem for duals of p-frames.

2. Elementary properties of p-frames

Throughout this paper, X is a separable Banach space with dual X^*, $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. A sequence $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is called a *p-frame* for X if there exist constants A, $B > 0$ such that

$$A\|x\| \le \left(\sum_{i=1}^{\infty} |g_i(x)|^p \right)^{\frac{1}{p}} \le B\|x\| \quad (x \in X). \tag{2-1}$$

The sequence $\{g_i\}_{i=1}^{\infty}$ is a *p-Bessel* sequence if at least the upper p-frame condition is satisfied. Analogous to frame theory in Hilbert spaces, one can define the synthesis operator as

$$T : l^q \to X^*, \quad T\{d_i\} := \sum_{i=1}^{\infty} d_i g_i.$$

A straightforward calculation shows that $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p-Bessel sequence with bound B if and only if T is well-defined and $\|T\| \le B$; see [Christensen and Stoeva 2003, Proposition 2.2].

The following result shows other aspects of the synthesis operator:

Proposition 2.1 [Christensen and Stoeva 2003]. *Let $\{g_i\} \subseteq X^*$ be a p-frame. Then*

(i) *the adjoint of T given by $T^* : X \to l^p$; $f \mapsto \{g_i(f)\}_{i=1}^{\infty}$ has closed range*;

(ii) *X is reflexive*;

(iii) *T is onto.*

The next proposition deals with preservation of the p-frame property under the action of various operators. Its proof is straightforward and we omit it.

Proposition 2.2. *Let X and Y be two Banach spaces and $\Psi : Y \to X$ be a bounded operator. Then*

(i) *if $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p-Bessel sequence for X, then $\{\Psi^* g_i\}_{i=1}^{\infty}$ is a p-Bessel sequence for Y*;

(ii) *if $\{g_i\}_{i=1}^{\infty}$ is a p-frame for X, and Ψ is one-to-one with closed range, then $\{\Psi^* g_i\}_{i=1}^{\infty}$ is a p-frame for Y.*

Definition 2.3. Let X be a Banach space and $1 < p < \infty$. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq X$ is called a *p-Riesz basis* for X if the closed linear span of $\{f_i\}_{i=1}^{\infty}$ is X and there exist constants A and B such that, for any finite scalars $\{c_i\}$,

$$A\Big(\sum |c_i|^p\Big)^{\frac{1}{p}} \le \Big\|\sum c_i f_i\Big\| \le B\Big(\sum |c_i|^p\Big)^{\frac{1}{p}}. \tag{2-2}$$

Clearly, if $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p-Riesz basis for X^* then its synthesis operator has a bounded inverse. In particular, every p-Riesz basis for X^* is a q-frame for X with the same bounds.

If $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p-frame for X, then Proposition 2.1 shows that every $g \in X^*$ can be written as $g = \sum_{i=1}^{\infty} d_i g_i$ for some $\{d_i\}_{i=1}^{\infty} \in l^q$. Our aim is to find such a decomposition for the elements of X.

3. Main results

Let $\{f_i\}_{i=1}^{\infty}$ be a frame in a Hilbert space $\mathscr{H}$ with the synthesis operator T. The canonical dual $\{S^{-1} f_i\}_{i=1}^{\infty}$ deals with the frame operator S; see (1-2). It is not guaranteed that the canonical dual frame has the same structure as the frame itself

[Daubechies 1990]. Alternate duals are now presented as being a good candidate to apply the reconstruction formula (1-2).

Unfortunately, in p-frames, the frame operator cannot be defined. Hence, we first try to describe the canonical dual with respect to the synthesis operator. In fact, let $\{f_i\}_{i=1}^{\infty}$ be a frame in a Hilbert space $\mathcal{H}$ with the analysis operator T^*. Then the frame condition (1-1) implies that T^* is injective and has closed range [Christensen 2008, Corollary 5.4.3]. Hence, the operator $(T^*)^{-1} : \mathcal{R}(T^*) \to \mathcal{H}$ can be extended to a bounded operator $\Phi : l^2 \to \mathcal{H}$. Therefore,

$$S^{-1} f_i = S^{-1} T\delta_i = S^{-1} T T^* \Phi\delta_i = \Phi\delta_i,$$

where $\{\delta_i\}_{i=1}^{\infty}$ is the canonical orthonormal basis for l^2.

We summarize this fact in the following lemma.

Lemma 3.1. *Let $\{f_i\}_{i=1}^{\infty}$ be a frame in $\mathcal{H}$ with the analysis operator T^*. The canonical dual $\{f_i\}_{i=1}^{\infty}$ can be represented as $\{\Phi\delta_i\}_{i=1}^{\infty}$, where $\Phi : l^2 \to \mathcal{H}$ is the unique extension of $(T^*)^{-1}$ and $\{\delta_i\}_{i=1}^{\infty}$ is the canonical orthonormal basis of l^2.*

Let X be a Banach space with dual X^* and $1 < p < \infty$. The usual duality between X and X^* allows us to consider p-frames for X^*. In fact, a sequence $\{f_i\}_{i=1}^{\infty} \subseteq X$ is a p-frame if there exist constants A and B such that

$$A\|g\| \le \left(\sum_{i=1}^{\infty} |g(f_i)|^p \right)^{\frac{1}{p}} \le B\|g\| \quad (g \in X^*).$$

If the upper frame condition is satisfied we call $\{f_i\}_{i=1}^{\infty}$ a p-Bessel sequence.

Definition 3.2. Let $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ be a p-Bessel sequence for X. A q-Bessel sequence $\{f_i\}_{i=1}^{\infty} \subseteq X$ for X^* is called a *dual* for $\{g_i\}_{i=1}^{\infty}$ if

$$g = \sum g(f_i) g_i \quad (g \in X^*) \quad \text{or} \quad f = \sum g_i(f) f_i \quad (f \in X). \tag{3-1}$$

If $\{g_i\}_{i=1}^{\infty}$ is p-frame, by using the Cauchy–Schwarz inequality, $\{f_i\}_{i=1}^{\infty}$ is automatically a q-frame for X^*, and vice versa. For more details see Theorem 2.10 of [Christensen and Stoeva 2003].

Denote the synthesis operators of $\{g_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$ by T and U, respectively. Also let X be reflexive. Then (3-1) holds if and only if $TU^* = I_{X^*}$ or $UT^* = I_X$. Although, for every $p \neq 2$, there exist a Banach space X and a p-frame for X without any dual [Casazza et al. 1999], Christensen and Stoeva [2003] showed that a p-frame $\{g_i\}_{i=1}^{\infty}$ has a dual if and only if $\mathcal{R}(T^*)$, the range of T^*, is complemented in l^p. Obviously, every p-Riesz basis for X^* has a unique dual.

Now we give a characterization of dual p-frames:

Proposition 3.3. *Let $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ be a p-frame for X with the synthesis operator T. Then there exists a one-to-one correspondence between duals of $\{g_i\}_{i=1}^{\infty}$ and bounded left inverses of T^*.*

Proof. Suppose that $\Phi : l^p \to X$ is a bounded left inverse of T^* and consider $\{\delta_i\}_{i=1}^{\infty}$ as the canonical basis of l^p. It is obvious that $\{f_i\}_{i=1}^{\infty} := \{\Phi\delta_i\}_{i=1}^{\infty}$ is a q-Bessel sequence and

$$f = \Phi T^* f = \Phi \sum_{i=1}^{\infty} g_i(f)\delta_i = \sum_{i=1}^{\infty} g_i(f) f_i \quad (f \in X).$$

Thus $\{f_i\}_{i=1}^{\infty}$ is a q-frame for X^*. Conversely, let $\{f_i\}_{i=1}^{\infty} \subseteq X$ be a dual for $\{g_i\}_{i=1}^{\infty}$. Consider $\Phi : l^p \to X$ as the synthesis operator of $\{f_i\}_{i=1}^{\infty}$. Then Φ is bounded and for each $f \in X$ we have

$$f = \sum_{i=1}^{\infty} g_i(f) f_i = \sum_{i=1}^{\infty} g_i(f)\Phi\delta_i = \Phi(\{g_i(f)\}_{i=1}^{\infty}) = \Phi T^* f. \qquad \square$$

As a consequence, we show that a p-frame $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ with a unique dual is a q-Riesz basis for X^*. In fact, by Proposition 3.3 there exists a one-to-one correspondence between the dual frames of $\{g_i\}_{i=1}^{\infty}$ and all bounded left inverse operators of T^*, in which T is the synthesis operator of $\{g_i\}_{i=1}^{\infty}$. Hence, $\{g_i\}_{i=1}^{\infty}$ has a unique dual if and only if T is injective. T is also surjective by Proposition 2.1. Thus T is invertible and $\|T^{-1}\| < \infty$. This implies that $\{g_i\}_{i=1}^{\infty}$ is a q-Riesz basis for X^*.

Proposition 3.4. *Assume that p-frame $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ has a dual. Then the q-Bessel sequence $\{f_i\}_{i=1}^{\infty} \subseteq X$ is a dual for $\{g_i\}_{i=1}^{\infty}$ if there exists a bounded operator $\Psi : X^* \to l^q$ such that $T\Psi = 0$. Conversely, all duals of $\{g_i\}_{i=1}^{\infty}$ (provided existence) can be described in this manner.*

Proof. Let $\{g_i\}_{i=1}^{\infty}$ be a p-frame. As a consequence of Proposition 2.1, the operator $(T^*)^{-1} : \mathcal{R}(T^*) \to X$ is well-defined. If $\{g_i\}_{i=1}^{\infty}$ has a dual, then $\mathcal{R}(T^*)$ is complemented and so this operator can be extended to a bounded linear operator $W : l^p \to X$. Now assume that $\{f_i\}_{i=1}^{\infty} \subseteq X$ is a dual for $\{g_i\}_{i=1}^{\infty}$ with the synthesis operator U. Then (3-1) immediately implies that $TU^* = I$. Define $\Psi : X^* \to l^q$ by $\Psi = U^* - W^*$. Clearly, Ψ is a bounded operator and

$$T\Psi = TU^* - TW^* = I - (WT^*)^* = 0.$$

Conversely, suppose that $\Psi : X^* \to l^q$ is a bounded operator via $T\Psi = 0$. Take $\Phi = W - \psi^*$. Then Φ is a bounded operator and $\Phi T^* = I$. Using Proposition 3.3 we conclude that $\{\Phi\delta_i\}_{i=1}^{\infty}$ is a dual for $\{g_i\}_{i=1}^{\infty}$. $\square$

Suppose that $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ is a p-frame with the synthesis operator T such that $\mathcal{R}(T^*) \subseteq l^p$ is complemented. Then $\{W\delta_i\}_{i=1}^{\infty}$ is called the *canonical dual* of $\{g_i\}_{i=1}^{\infty}$, where $W : l^p \to X$ is the extension of $(T^*)^{-1}$. Other duals, which are characterized by Proposition 3.3, are called *alternate duals*. In other words, the canonical dual is associated to the bounded inverse of T^* whereas alternate duals are in fact obtained by the left inverses of T^*.

Now we are ready to state a perturbation theorem about duals.

Theorem 3.5. *Let $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ be a p-frame for X with bounds A_1 and B_1. The p-frames sufficiently close to $\{g_i\}_{i=1}^{\infty}$ have a dual. More precisely, let $\{f_i\}_{i=1}^{\infty} \subseteq X$ be a dual for $\{g_i\}_{i=1}^{\infty}$ with bounds A_2 and B_2, and let $\{g_i'\}_{i=1}^{\infty}$ be another p-frame with bounds A' and B' such that $\{g_i - g_i'\}_{i=1}^{\infty}$ is a p-Bessel sequence with constant sufficiently small ϵ. Then there exists a dual q-frame $\{f_i'\}_{i=1}^{\infty}$ for $\{g_i'\}_{i=1}^{\infty}$ such that $\{f_i - f_i'\}_{i=1}^{\infty}$ is also a q-Bessel sequence with bound multiplied by ϵ.*

Proof. Denote by T_1 and T_2 the synthesis operators of $\{g_i\}_{i=1}^{\infty}$ and $\{g_i'\}_{i=1}^{\infty}$, respectively. Then $\|T_1 - T_2\| < \epsilon$ by Proposition 2.2 of [Christensen and Stoeva 2003]. Moreover, $(T_1^*)^{-1} : \mathcal{R}(T_1^*) \to X$ can be extended to a bounded operator $W : l^p \to X$ by the assumption. Hence

$$\|I - T_2^* W\| = \|(T_1 - T_2)^* W\| \leq \|W\| \|T_1 - T_2\| \leq \epsilon \|W\|.$$

Consequently $T_2^* W$ is invertible. It follows that T_2^* has a bounded right inverse. A similar argument shows that its left inverse also exists. Consider $U_1 : l^p \to X$ as the synthesis operator of $\{f_i\}_{i=1}^{\infty}$. Then

$$\|I - U_1 T_2^*\| = \|U_1 (T_1 - T_2)^*\| \leq \|U_1\| \|T_1 - T_2\| \leq \epsilon \|U_1\|.$$

Therefore, the p-frame $\{g_i'\}_{i=1}^{\infty}$ has a dual. We are looking for the desired dual. First by Proposition 3.4 there exists a bounded operator $\Psi : X^* \to l^q$ such that $T_1 \Psi = 0$. Assume that $W_2 : l^p \to X$ is an extension of $(T_2^*)^{-1}$. Put

$$h_i = (W_2 + \Psi^*)\delta_i.$$

Then $\{h_i\}_{i=1}^{\infty}$ is a q-Bessel sequence with the synthesis operator $U_2 := W_2 + \Psi^*$ by Proposition 2.2. Moreover, for each $f \in X$ we have

$$\begin{aligned} \|f - U_2 T_2^* f\| &= \|\Psi^* T_2^* f\| \\ &= \|\Psi^* T_1^* f - \Psi^* T_2^* f\| \\ &\leq \|T_1 - T_2\| \|\Psi\| \|f\| \leq \epsilon \|\Psi\| \|f\|. \end{aligned}$$

Therefore, $U_2 T_2^*$ is invertible for sufficiently small $\epsilon > 0$. In particular,

$$\|I - U_2 T_2^*\| \leq \epsilon \|\Psi\|.$$

It remains to show that the q-frame $\{f_i'\}_{i=1}^{\infty} := \{(U_2T_2^*)^{-1}f_i\}_{i=1}^{\infty}$ satisfies the theorem. It is easy to see that

$$U_2T_2^* = I + \Psi^*(T_1^* - T_2^*). \tag{3-2}$$

Hence,

$$1 - \epsilon\|\Psi\| \le \|U_2T_2^*\|. \tag{3-3}$$

For each sequence $\{d_i\}_{i=1}^{\infty}$ in l^p by using (3-2) and (3-3) we get

$$\begin{aligned}\left\|\sum_{i=1}^{\infty} d_i(f_i - f_i')\right\| &= \|U_1\{d_i\} - (U_2T_2^*)^{-1}U_1\{d_i\}\| \\ &\le \|U_1\|\|I - (U_2T_2^*)^{-1}\|\left(\sum_{i=1}^{\infty}|d_i|^p\right)^{\frac{1}{p}} \\ &\le \|U_1\|\|(U_2T_2^*)^{-1}\|\|U_2T_2^* - I\|\left(\sum_{i=1}^{\infty}|d_i|^p\right)^{\frac{1}{p}} \\ &\le \frac{\epsilon\|\Psi\|B_2}{1-\epsilon\|\Psi\|}\left(\sum_{i=1}^{\infty}|d_i|^p\right)^{\frac{1}{p}}.\end{aligned}$$

This means that $\{f_i - f_i'\}_{i=1}^{\infty}$ is a q-Bessel sequence and its bound is a multiple of ϵ. □

Let $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ be a p-Bessel sequence for X with the synthesis operator T. We say that a q-Bessel sequence $\{f_i\}_{i=1}^{\infty} \subseteq X$ with the synthesis operator U is an *approximately dual* of $\{g_i\}_{i=1}^{\infty}$ if

$$\|I - TU^*\| < 1 \quad \text{or} \quad \|I - UT^*\| < 1. \tag{3-4}$$

Obviously, $\{f_i\}_{i=1}^{\infty}$ is a dual of $\{g_i\}_{i=1}^{\infty}$ when $TU^* = I$ or $UT^* = I$. Approximate duals are studied in a Hilbert space setting in [Christensen and Laugesen 2010]. They are easier to construct than the classical dual frames. For p-frames, which don't have duals in general, it is natural to ask whether we can exploit the approximate duals instead of duals. Unfortunately, the answer is negative. In fact, if $\{g_i\}_{i=1}^{\infty}$ is a p-frame for X with an approximate dual $\{f_i\}_{i=1}^{\infty}$, then, with notation as above, the operator UT^* is invertible. Hence, $\{UT^*f_i\}_{i=1}^{\infty}$ is a p-frame by Proposition 2.2. Moreover,

$$f = (UT^*)^{-1}UT^*f = (UT^*)^{-1}\sum_{i=1}^{\infty} g_i(f)f_i = \sum_{i=1}^{\infty} g_i(f)(UT^*)^{-1}f_i.$$

Therefore, $\{(UT^*)^{-1}f_i\}$ is a dual of $\{g_i\}_{i=1}^{\infty}$.

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